Entangling capacity and distinguishability of two-qubit unitary operators

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We prove that the entangling capacity of a two-qubit unitary operator without local ancillas, both with and without the restriction to initial product states, as quantified by the maximum attainable concurrence, is directly related to the distinguishability of a closely related pair of two-qubit unitary operators. These operators are the original operator transformed into its canonical form and the adjoint of this canonical form. The distinguishability of these operators is quantified by the minimum overlap of the output states over all possible input probe states. The entangling capacity of the original unitary operator is therefore directly related to the degree of non-Hermiticity of its canonical form, as quantified in an operationally satisfactory manner in terms of the extent to which it can be distinguished, by measurement, from its adjoint. Furthermore, the maximum entropy of entanglement, again without local ancillas, that a given two-qubit unitary operator can generate is found to be closely related to the classical capacities of certain quantum channels.

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I. INTRODUCTION

An interesting development that has recently taken place in quantum information science has been the widespread realization that the information-theoretic properties of quantum operations are of comparable importance to those of quantum states. We may view this to be a consequence of the fact that all properties of quantum states, including those most relevant to quantum communications and computation, such as entanglement and nonorthogonality, must somehow be created. It is quantum operations that are responsible for the creation of states with these properties. As such, there has been a considerable amount of activity devoted to establishing relationships between properties of quantum states, especially those that are useful in applications, and the properties of quantum operations that give rise to them.

Perhaps the most important resource in quantum communications and computation is entanglement. Entanglement is a resource that must be created by operations. Consequently, a considerable effort has been directed at characterizing the ability of quantum operations to generate entanglement [1–9]. Of particular importance is the maximum amount of entanglement that a quantum operation can generate for some set of possible initial states. Such a quantity is known as an entangling capacity.

The general nonorthogonality of quantum states is also one of their most pertinent and useful features. Nonorthogonal states cannot be perfectly distinguished. This fact forms the basis for many quantum cryptographic protocols. The distinguishability of operations is closely related to the distinguishability of the states they produce, since it is by discriminating among the output states of quantum operations that we distinguish among the operations themselves [10–13].

The aim of this paper is to prove and explore the consequences of a curiously simple relationship between the entangling capacity of and the distinguishability of bipartite unitary operations acting on two qubits. Specifically, we make use of the fact that any unitary operator U_{AB} on two qubits A and B can, using local operations on A and B, be put into a certain canonical form U_d. We find that the entangling capacity of U_{AB}, as quantified by the maximum concurrence of the output states it generates, with or without the restriction to an initial product state, but without ancillas is directly related to the distinguishability of U_d and $U_d^{\dagger}$. Here, distinguishability is quantified by the minimum, over all initial states, of the overlap of the output states they create. This result extends one previously obtained by Zhang *et al.* [5] which relates to perfect entanglers.

In Sec. II, we review the relevant background material on the entangling capacity and distinguishability of unitary operators. Section III is devoted to proving our main result relating to maximum concurrence and distinguishability and to exploring some of its consequences.

For bipartite pure states, the entropy of entanglement is often used as an entanglement measure. It is interesting to enquire as to whether or not the entangling capacity, in terms of the maximum entropy of entanglement, also has a direct operational connection with the distinguishability of states.

The entropy of entanglement draws its significance from its asymptotic properties, when there are many copies of a given entangled state available. It is natural to suspect that the entropic entangling capacity of a given quantum operation may relate to the distinguishability of quantum operations when the operation is performed many times. This leads us to consider classical information theory, which is of an intrinsically asymptotic nature.

In Sec. IV, we confirm this suspicion by showing that the entropic entangling capacity of U_{AB} is related to classical information transmission in two distinct ways. The first relationship concerns the so-called first-order classical capacity, where collective decoding of the individual signal carriers is forbidden. The second relates to the Hausladen-Jozsa-Schumacher-Westmoreland-Wootters (HJSWW) capacity [14], where arbitrary collective decoding of the pure state signals is permitted. The relationship we describe relates to

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the transmission of pure states. The generalization of the HJSWW capacity to mixed state signals is the well-known Holevo-Schumacher-Westmoreland capacity [15,16]. We therefore find that the relationship between the entangling capacity and distinguishability is not simply of a "one-shot" nature, but also has some interesting implications for the asymptotic limit. We conclude in Sec. V with a general discussion of our results.

II. ENTANGLING CAPACITIES AND DISTINGUISHABILITY OF UNITARY OPERATORS

A. Entangling capacities

Consider two quantum systems A and B. Associated with each system is a copy of the Hilbert space $\mathcal{H}$. The composite system has Hilbert space $\mathcal{H}_{AB}=\mathcal{H}^{\otimes 2}$. We shall take A and B to be qubits and so $\mathcal{H}$ is two-dimensional.

Let us consider a unitary operator U_{AB} on $\mathcal{H}_{AB}$. We would like to know how much entanglement such an operator can create between these two systems. To address this issue, we must choose a measure of entanglement. For a bipartite pure state of two qubits, two particular entanglement measures are in common use. These are the concurrence and the entropy of entanglement.

Let $|\Psi\rangle \in \mathcal{H}_{AB}$ be an arbitrary pure state of AB. Let ρ_A^{Ψ} (ρ_B^{Ψ}) be the corresponding reduced density operator of A (B) for this state. Both of these operators have the same eigenvalues, which we shall denote by q_j, where $j=1,2$. The concurrence of $|\Psi\rangle$ may be written as

$$C(|\Psi\rangle) = 2\sqrt{q_1 q_2} = 2\sqrt{\det(\rho_j^{\Psi})}. \tag{2.1}$$

The entropy of entanglement of $|\Psi\rangle$ is

$$E(|\Psi\rangle) = -\sum_{j=1}^{2} q_j \log q_j, \tag{2.2}$$

where here, as throughout this paper, the logarithm has base 2. Hill and Wootters [17] noted that the entropy of entanglement may be written in terms of the concurrence as

$$E(|\Psi\rangle) = h\left(\frac{1+\sqrt{1-[C(|\Psi\rangle)]^2}}{2}\right), \tag{2.3}$$

where $h(x)$ is the binary entropy function:

$$h(x) = -x\log x - (1-x)\log(1-x). \tag{2.4}$$

Both of these entanglement measures attain their maximum value, which in both cases is 1, iff $|\Psi\rangle$ is maximally entangled. They also take their minimum value, of 0, iff $|\Psi\rangle$ is a product state. These entanglement measures are also invariant under local unitary operations. Finally, they are monotonically increasing functions of each other, so the maximization of one is equivalent to the maximization of the other. As a consequence of this equivalence we shall, for the time being, concentrate on the concurrence as an entanglement measure.

The entangling capacity of a unitary operator is the maximum amount of entanglement that the operator can generate. In the most general situation we can contemplate, A and B can be entangled with local ancillary systems. At the time of writing, the most general solution to this problem is not known. We may consider instead the situation where the initial state is a pure state $|\Psi\rangle \in \mathcal{H}_{AB}$. Leifer *et al.* [4] solved this problem and obtained the general value of

$$C_{max}(U_{AB}) = \max_{|\Psi\rangle\in\mathcal{H}_{AB}} [C(U_{AB}|\Psi\rangle) - C(|\Psi\rangle)]. \tag{2.5}$$

In a prior work, Kraus and Cirac [1] addressed the more restricted problem where the set of possible initial states was taken to be the set of two-qubit product states in $\mathcal{H}_{AB}$. In this case, writing a typical product state $|\Psi\rangle \in \mathcal{H}_{AB}$ as $|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, the resulting product entangling capacity may be written as

$$C_{max}^{prod}(U_{AB}) = \max_{|\psi_A\rangle,|\psi_B\rangle\in\mathcal{H}} C(U_{AB}|\psi_A\rangle \otimes |\psi_B\rangle). \tag{2.6}$$

Kraus and Cirac obtained an explicit expression for $C_{max}^{prod}(U_{AB})$ for an arbitrary two-qubit unitary operator. The results of the above authors can be summarized as follows. Firstly, it is known [1,2] that any unitary operator U_{AB} acting on two qubits A and B can be written in the form

$$U_{AB} = (X_A \otimes X_B)U_d(Y_A \otimes Y_B), \tag{2.7}$$

where X_A, X_B, Y_A and Y_B are single-qubit unitary operators and the bipartite unitary operator U_d has the form

$$U_d = \exp[-i(\alpha_x\sigma_x \otimes \sigma_x + \alpha_y\sigma_y \otimes \sigma_y + \alpha_z\sigma_z \otimes \sigma_z)]. \tag{2.8}$$

Here, σ_x, σ_y and σ_z are the usual Pauli spin operators and the vector $d=(\alpha_x,\alpha_y,\alpha_z)$ has real components satisfying

$$0 \leqslant |\alpha_z| \leqslant \alpha_y \leqslant \alpha_x \leqslant \pi/4. \tag{2.9}$$

For the purposes of this paper, it is sufficient to confine our attention here to the case of $\alpha_z \geqslant 0$. We discuss this matter in more detail in Sec. III and in the Appendix. We then have

$$0 \leqslant \alpha_z \leqslant \alpha_y \leqslant \alpha_x \leqslant \pi/4. \tag{2.10}$$

We can write the eigenvalues of U_d in the form $e^{-i\lambda_j}$, where $j=1,\ldots,4$. The λ_j are given by

$$\lambda_4 = \alpha_x + \alpha_y - \alpha_z, \tag{2.11}$$

$$\lambda_3 = \alpha_x - \alpha_y + \alpha_z, \tag{2.12}$$

$$\lambda_2 = -\alpha_x + \alpha_y + \alpha_z, \tag{2.13}$$

$$\lambda_1 = -\alpha_x - \alpha_y - \alpha_z. \tag{2.14}$$

It follows readily from these equations and inequality (2.10) that the λ_j are ordered according to

$$\lambda_4 \geqslant \lambda_3 \geqslant \lambda_2 \geqslant \lambda_1. \tag{2.15}$$

In terms of the above definitions, Kraus and Cirac and Leifer *et al.* obtained the general forms of $C_{max}^{prod}(U_{AB})$ and $C_{max}(U_{AB})$ for two-qubit unitary operators. They found that when the following two inequalities are satisfied,

$$\alpha_x + \alpha_y \geqslant \pi/4, \tag{2.16}$$

$$\alpha_y + \alpha_z \leq \pi/4, \tag{2.17}$$

then these entangling capacities are

$$C_{max}^{prod}(U_{AB}) = C_{max}(U_{AB}) = 1. \tag{2.18}$$

This implies that U_{AB} can transform some product state into a maximally entangled state, in which case U_{AB} is said to be a perfect entangler. When either of these inequalities is violated, U_{AB} is not a perfect entangler and

$$C_{max}^{prod}(U_{AB}) = [C_{max}(U_{AB})]^2 = \max_{j,j'} |\sin(\lambda_j - \lambda_{j'})|. \tag{2.19}$$

Notice that these two inequalities cannot be violated simultaneously, as this would contradict the ordering of the α_j in Eq. (2.10).

It is clear from Eqs. (2.18) and (2.19) that the entangling capacity without the restriction to product states is always equal to the square root of the entangling capacity with this restriction. So, these capacities are very easily deduced from each other. For this reason and also because of the fact that, for the purposes of deriving the principal result of this paper, the product entangling capacity will be slightly easier to work with, we shall focus mainly on this quantity in subsequent sections.

We will show that there is an intriguing relationship between the product entangling capacity (2.19) and distinguishability of unitary operators. Prior to describing this relationship, we shall review some facts about the latter topic.

B. Distinguishability

Consider two unitary operators S and T referring to a possibly composite system with total Hilbert space $\mathcal{H}_{tot}$. This space is taken to have finite dimensionality D. We aim to distinguish as well as possible between these two operators using a probe state. The problem of optimally discriminating between two unitary operators was first addressed and solved by Childs *et al.* [10]. Useful further insights into this problem were obtained in the subsequent investigations of Acín [11] and D'Ariano *et al.* [12]. In particular, the latter authors established that for optimal discrimination, the initial probe state may be taken to be a pure state which is not entangled with any ancillary systems.

In the light of this, let the probe state be some pure state $|\Phi\rangle \in \mathcal{H}_{tot}$. To optimally distinguish between S and T, we require that the overlap between the states $S|\Phi\rangle$ and $T|\Phi\rangle$ be as small as possible. It is convenient to define the unitary operator $V = S^\dagger T$ and to consider the inner product between the final states:

$$\langle\Phi|S^\dagger T|\Phi\rangle = \langle\Phi|V|\Phi\rangle. \tag{2.20}$$

Since V is unitary, it can be spectrally decomposed as

$$V = \sum_{j=1}^{D} e^{i\theta_j}|v_j\rangle\langle v_j|, \tag{2.21}$$

where the angles θ_j are real and the $|v_j\rangle$ form an orthonormal basis for $\mathcal{H}_{tot}$. The state $|\Phi\rangle$ can be expanded in terms of this basis as

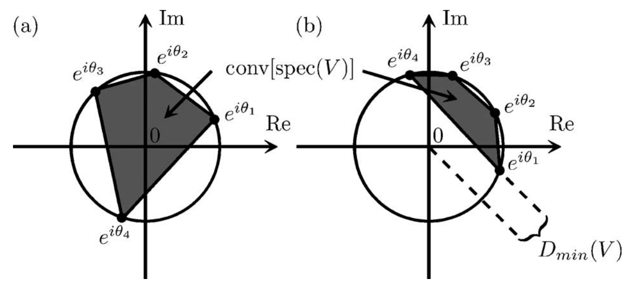

FIG. 1. Geometric depiction of (a) perfect and (b) imperfect distinguishability of two unitary operators S and T. The $e^{i\theta_j}$ are the eigenvalues of $V = S^\dagger T$. The set of possible values of the inner product between the final states $S|\Phi\rangle$ and $T|\Phi\rangle$ is, in each case, conv[spec(V)]. The minimum overlap between these final states is $D_{min}(V)$, the minimum distance from 0 to conv[spec(V)]. If, as is the case in (a), $0 \in$ conv[spec(V)], then $D_{min}(V) = 0$ and perfect discrimination between S and T is possible. If, on the other hand, $0 \notin$ conv[spec(V)], which is the case in (b), then S and T cannot be perfectly discriminated although their distinguishability continues to be governed by $D_{min}(V)$.

$$|\Phi\rangle = \sum_{j=1}^{D} c_j|v_j\rangle. \tag{2.22}$$

For the sake of notational convenience, let us define

$$p_j = |c_j|^2. \tag{2.23}$$

It is clear that the p_j may take any non-negative values subject to the normalization of $|\Phi\rangle$, which is equivalent to

$$\sum_{j=1}^{D} p_j = 1. \tag{2.24}$$

Combining the above expressions, we obtain

$$\langle\Phi|V|\Phi\rangle = \sum_{j=1}^{D} p_j e^{i\theta_j}. \tag{2.25}$$

In discriminating between two pure states, it is their overlap, rather than their inner product, that is significant. If we wish to obtain the minimum value of the overlap of $S|\Phi\rangle$ and $T|\Phi\rangle$, we must evaluate

$$\min_{|\Phi\rangle:\||\Phi\rangle\|=1} |\langle\Phi|V|\Phi\rangle| = \min_{p_j:\sum_{j=1}^{D} p_j=1} \left|\sum_{j=1}^{D} p_j e^{i\theta_j}\right|. \tag{2.26}$$

The above expression has a simple geometrical interpretation that is depicted in Fig. 1. For each particular set of p_j, the number $\sum_{j=1}^{D} p_j e^{i\theta_j}$ represents a point in the complex plane. The set of all such points is easily recognizable as the convex hull of the eigenvalues $e^{i\theta_j}$ [18]. This is the smallest convex set which contains all of these eigenvalues. Here, the number of eigenvalues is finite, so their convex hull is simply a convex polygon with these eigenvalues at its vertices. For the sake of notational convenience, we shall write the convex hull of the spectrum of an operator V as conv[spec(V)].

The minimum overlap in Eq. (2.26) is then simply the minimum modulus of the complex numbers in conv[spec(V)]. In other words, it is the minimum distance from 0 to conv[spec(V)]. Let us denote this minimum distance by $D_{min}(V)$. Then we may write

$$\min_{|\Phi\rangle:\||\Phi\rangle\|=1} |\langle\Phi|S^{\dagger}T|\Phi\rangle| = D_{min}(V). \tag{2.27}$$

For the operators S and T to be perfectly distinguishable for some initial probe state $|\Phi\rangle$, we require that the final states be orthogonal. From the above considerations, we see that this will be the case iff $D_{min}(V)=0$, which is to say that $0\in \text{conv}[\text{spec}(V)]$.

III. MAXIMUM CONCURRENCE AND MINIMUM OVERLAP

We shall now show that for any two-qubit unitary operator U_{AB}, the product entangling capacity in Eqs. (2.18) and (2.19) and the distinguishability of U_d and its adjoint $U_d^{\dagger}$, as quantified by the minimum overlap, satisfy a curiously simple relationship. The distinguishability of U_d and its adjoint $U_d^{\dagger}$ is characterized using Eq. (2.27), where we make the identifications $S=U_d^{\dagger}$, $T=U_d$, $V=U_d^2$ and $\mathcal{H}_{tot}=\mathcal{H}_{AB}$. We are now in a position to prove our main result.

Theorem. For any two-qubit unitary operator U_{AB} with canonical form U_d defined by Eqs. (2.8) and (2.9),

$$[C_{max}^{prod}(U_{AB})]^2 + [D_{min}(U_d^2)]^2 = 1. \tag{3.1}$$

To prove this result, we shall treat separately the cases of U_{AB} being a perfect entangler and not being a perfect entangler.

Prior to proving Eq. (3.1) for perfect entanglers, we note that for such operators, this theorem has been effectively established in theorem 1 of [5], although without reference to the distinguishability of unitary operators. Our discovery of the general validity of Eq. (3.1) came about through the realization that the results of these authors relate to the distinguishability of unitary operators. This led us to enquire as to whether or not there is a general relationship between the distinguishability of U_d and $U_d^{\dagger}$ and the entangling capacity of U_{AB}. This enquiry led to our discovery of the general validity of Eq. (3.1). Also, for the case of perfect entanglers, our proof is possibly slightly simpler.

Another point is that, here, we explicitly prove Eq. (3.1) only for cases where $\alpha_z \geqslant 0$. We do this because the validity of Eq. (3.1) for negative α_z follows readily from its validity for $\alpha_z \geqslant 0$. We prove this in the Appendix.

Proof. Case (a): U_{AB} is a perfect entangler.

In this case, $C_{max}^{\text{prod}}(U_{AB})=1$. Proving Eq. (3.1) in this case then amounts to showing that $D_{min}(U_d^2)=0$, implying that U_d and $U_d^{\dagger}$ are perfectly distinguishable.

To begin, we know from the preceding section that $C_{max}^{prod}(U_{AB})=1$ is equivalent to inequalities (2.16) and (2.17) being satisfied. Also, to have $D_{min}(U_d^2)=0$, we require that $0\in \text{conv}[\text{spec}(U_d^2)]$. This latter condition can be understood in simple geometrical terms. Consider the angular separations of neighboring eigenvalues of U_d^2. To calculate these spacings, it is convenient to define $\tilde{\lambda}_j=\lambda_j-\lambda_1$, which all lie in the interval $[0,\pi]$. The spacings between neighboring angles $\tilde{\lambda}_j$ are identical to those of the original $\tilde{\lambda}_j$. In particular, we have

$$\pi \geqslant \tilde{\lambda}_4 \geqslant \tilde{\lambda}_3 \geqslant \tilde{\lambda}_2 \geqslant \tilde{\lambda}_1 = 0. \tag{3.2}$$

Using these transformed angles, it is easy to see that the four spacings are given by $2(\tilde{\lambda}_{j+1}-\tilde{\lambda}_j)$ for $j=1,\ldots,3$ and $2(\pi-\tilde{\lambda}_4)$ for $j=4$. Using Eqs. (2.11)–(2.14), we can write these spacings as

$$2(\tilde{\lambda}_2-\tilde{\lambda}_1) = 4(\alpha_y+\alpha_z) \leqslant \pi, \tag{3.3}$$

$$2(\tilde{\lambda}_3-\tilde{\lambda}_2) = 4(\alpha_x-\alpha_y) \leqslant \pi, \tag{3.4}$$

$$2(\tilde{\lambda}_4-\tilde{\lambda}_3) = 4(\alpha_y-\alpha_z) \leqslant \pi, \tag{3.5}$$

$$2(\pi-\tilde{\lambda}_4) = 2(\pi-2(\alpha_x+\alpha_y)) \leqslant \pi. \tag{3.6}$$

The inequalities in (3.3) and (3.6) are consequences of (2.17) and (2.16), respectively, while those in (3.4) and (3.5) follow from (2.10). Now, zero is an element of $\text{conv}[\text{spec}(U_d^2)]$ iff the eigenvalues of U_d^2 do not all lie in some arc of the unit circle which subtends an angle of less than π radians. This is equivalent to the four angular separations being no greater than π radians, which is what we have just demonstrated. So we have shown that whenever U_{AB} is a perfect entangler, U_d and $U_d^{\dagger}$ are perfectly distinguishable, and so Eq. (3.1) holds in this case.

Case (b): U_{AB} is not a perfect entangler.

Let us now show that Eq. (3.1) also holds when U_{AB} is not a perfect entangler. We will be concerned here with the minimum distance from the origin to the convex hull of the complex numbers $e^{-2i\lambda_j}$. This distance is clearly identical to that between the origin and the convex hull of the $e^{-2i\tilde{\lambda}_j}$ and again it will be more convenient to work with the latter quantities.

In terms of these angles, we may write Eq. (2.19) as

$$C_{max}^{prod}(U_{AB}) = \max_{j,j'} |\sin(\tilde{\lambda}_j-\tilde{\lambda}_{j'})|. \tag{3.7}$$

We know that U_{AB} is not a perfect entangler when one of the two inequalities (2.16) and (2.17) is not satisfied. We shall treat the violation of each of these inequalities separately.

To prove Eq. (3.1) when (2.16) is violated, let us first show that

$$C_{max}^{prod}(U_{AB}) = \sin(\tilde{\lambda}_4). \tag{3.8}$$

To prove this, we see that when inequality (2.16) is not satisfied, we have

$$\tilde{\lambda}_4 = 2(\alpha_x+\alpha_y) < \frac{\pi}{2}. \tag{3.9}$$

We see from this inequality and the angular ordering in (3.2) that the angles $\tilde{\lambda}_j$ all lie in the half-open interval $[0,\pi/2)$. All differences between neighboring angles must clearly lie in this interval also. The arrangement of these angles is depicted in Fig. 2(i).

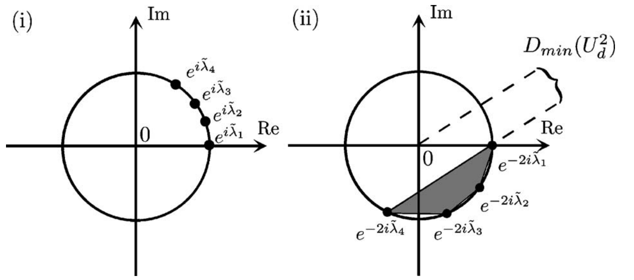

FIG. 2. Depiction of (i) the angles $\tilde{\lambda}_j$ and (ii) the rotated eigenvalues of U_d^2 when U_{AB} is not a perfect entangler and inequality (2.16) is violated. In (i), all four angles lie in the first quadrant, which implies that the product entangling capacity is given by Eq. (3.8). In (ii), the four eigenvalues $e^{-2i\tilde{\lambda}_j}$ lie in the lower half-plane, with the greatest separation being between $e^{-2i\tilde{\lambda}_4}$ and $e^{-2i\tilde{\lambda}_1}$. The minimum overlap is the distance from the origin to the midpoint of the chord joining $e^{-2i\tilde{\lambda}_1}$ and $e^{-2i\tilde{\lambda}_4}$, which gives Eq. (3.10) and thus Eq. (3.1).

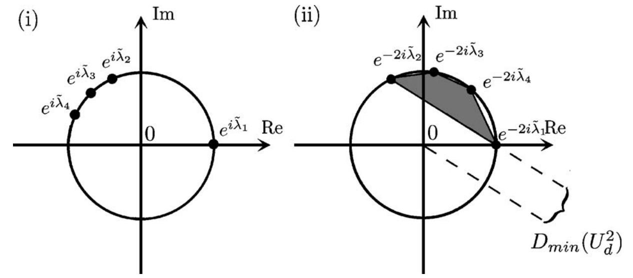

FIG. 3. Depiction of (i) the angles $\tilde{\lambda}_j$ and (ii) the rotated eigenvalues of U_d^2 when U_{AB} is not a perfect entangler and inequality (2.17) is violated. In (i), $e^{i\tilde{\lambda}_1}=1$ and the other $e^{i\tilde{\lambda}_j}$ lie in the second quadrant. From this, we find that the product entangling capacity is given by Eq. (3.11). In (ii), the four eigenvalues $e^{-2i\tilde{\lambda}_j}$ lie in the upper half-plane, with the greatest separation being between $e^{-2i\tilde{\lambda}_1}$ and $e^{-2i\tilde{\lambda}_2}$. The minimum overlap is the distance from the origin to the midpoint of the chord joining $e^{-2i\tilde{\lambda}_1}$ and $e^{-2i\tilde{\lambda}_2}$, which gives Eq. (3.14) and thus Eq. (3.1).

The sin function is monotonically increasing in the interval $[0,\pi/2]$. It follows that the two angles with the greatest separation attain the maximum in Eq. (3.7). Due to the angular ordering in (3.2), these are $\tilde{\lambda}_4$ and $\tilde{\lambda}_1$. However, $\tilde{\lambda}_1=0$, so we obtain Eq. (3.8).

Let us now calculate $D_{min}(U_d^2)$. Since all four angles $2\tilde{\lambda}_j$ lie in first and second quadrants, the complex numbers $e^{-2i\tilde{\lambda}_j}$ all lie in the lower half-plane. This can be seen in Fig. 2(ii). These complex numbers are the rotated eigenvalues of U_d^2, and so $D_{min}(U_d^2)$ is the distance from the origin to the convex hull of the $e^{-2i\tilde{\lambda}_j}$.

From the fact that these complex numbers all lie in the lower half-plane and from the angle ordering in (2.15), it follows that this minimum distance is equal to the distance from the origin to the midpoint of the chord joining $e^{-2i\tilde{\lambda}_4}$ and $e^{-2i\tilde{\lambda}_1}$. By elementary trigonometry, we find that

$$D_{min}(U_d^2)=\cos(\tilde{\lambda}_4). \tag{3.10}$$

Making use of both this and Eq. (3.8), we see that Eq. (3.1) is satisfied. This completes the proof of (3.1) when inequality (2.16) is not satisfied.

Let us now turn our attention to the situation where inequality (2.17) is violated. We will begin by showing that when this is the case, we have

$$C_{max}^{prod}(U_{AB})=\sin(\tilde{\lambda}_2). \tag{3.11}$$

To see this, we note that when (2.17) is not satisfied, we have

$$\tilde{\lambda}_2=2(\alpha_y+\alpha_z)>\frac{\pi}{2}. \tag{3.12}$$

Since $\tilde{\lambda}_j\in[0,\pi]$, we see that $\tilde{\lambda}_1=0$ and the remaining angles lie in the second quadrant. This is shown in Fig. 3(i). Let us now consider differences between these angle differences $\tilde{\lambda}_j-\tilde{\lambda}_{j'}$. Because, from Eq. (3.7), we wish to find the maximum absolute value of the sin, we may, without loss of generality, take $j>j'$. In the case where $j'=1$, Eq. (3.11) follows readily by combining the fact that the sin function is decreasing in the interval $[\pi/2,\pi]$ with the angular ordering in (3.2). For $j'=2$ or 3, let us write $\tilde{\lambda}_j-\tilde{\lambda}_{j'}=\Delta$. We shall now use the elementary sin difference rule in the following way:

$$\sin(\tilde{\lambda}_2)-\sin(\Delta)=2\cos\left(\frac{\tilde{\lambda}_2+\Delta}{2}\right)\sin\left(\frac{\tilde{\lambda}_2-\Delta}{2}\right)\geqslant 0. \tag{3.13}$$

The reason why we have the inequality here is as follows. Firstly, the cos factor must be non-negative. This is because $\tilde{\lambda}_2+\Delta$ is equal to either $\tilde{\lambda}_3$ or $\tilde{\lambda}_4$ which we know, from Eq. (3.2), lies in the interval $[0,\pi]$. Therefore, the argument of the cos factor lies in the first quadrant and the cos factor is therefore non-negative. As for the sin factor, we note that the angle $(\tilde{\lambda}_2-\Delta)/2$ also lies in the first quadrant, as a consequence of Eqs. (2.17) and (3.2), where the sin function is also non-negative. Finally, because both terms on the left-hand side of Eq. (3.13) are non-negative, and because $\tilde{\lambda}_2=\tilde{\lambda}_2-\tilde{\lambda}_1$ (by virtue of the fact that $\tilde{\lambda}_1=0$), we obtain Eq. (3.11).

Let us now calculate $D_{min}(U_d^2)$. From the above considerations it follows that the $e^{-2i\tilde{\lambda}_j}$, the rotated eigenvalues of U_d^2, all lie in the upper half-plane, as shown in Fig. 3(ii). The minimum distance from 0 to the convex hull of the $e^{-2i\tilde{\lambda}_j}$ is clearly equal to the distance from the origin to the midpoint of the chord joining $e^{-2i\tilde{\lambda}_2}$ and $e^{-2i\tilde{\lambda}_1}$. By elementary trigonometry we find that

$$D_{min}(U_d^2)=-\cos(\tilde{\lambda}_2). \tag{3.14}$$

Combining this with Eq. (3.11) shows that Eq. (3.1) is satisfied. This, together with the result in the Appendix, completes the proof of Eq. (3.1) for all two-qubit unitary operators. □

Let us now make some observations about this theorem. The first point to note is that in Kraus and Cirac's original presentation, the product entangling capacities of perfect and imperfect entanglers were treated separately. The theorem we have proven above unifies these two scenarios, expressing as it does the product entangling capacity of U_{AB} with the single equation (3.1) rather than the two equations (2.18) and (2.19). The two scenarios described by Kraus and Cirac, in which U_{AB} is a perfect or imperfect entangler, correspond directly to U_d and $U_d^\dagger$ being perfectly or imperfectly distinguishable.

The relationship in Eq. (3.1) between the product entangling capacity $C_{max}^{prod}(U_{AB})$ and the distinguishability of U_d and $U_d^\dagger$ suggests that $C_{max}^{prod}(U_{AB})$ may serve to quantify the non-Hermiticity of U_d. This idea is reinforced by the following observation: $D_{min}(U_d^2)$ reaches its maximum of 1, which corresponds to complete indistinguishability, iff U_d is Hermitian. This is easy to see. The distance from the origin to the midpoint of the chord joining two points on the unit circle attains the value 1 iff these two points are identical. Combining this with Eq. (3.1) and the fact that, for imperfect entanglers, the points in question are the (rotated) eigenvalues of U_d^2 that are furthest apart, we see that this scenario arises only when all the $\tilde{\lambda}_j$ are all equal, indeed equal to 0 since $\tilde{\lambda}_1=0$. This is equivalent to $U_d^2=1$ which implies that $U_d=U_d^\dagger$ as a consequence of unitarity. So, when the product entangling capacity of U_{AB} is zero, U_d is Hermitian. Also, the converse of this statement follows trivially from Eq. (3.1). So we are led to see that the product entangling capacity of U_{AB} is directly related to and increases with the non-Hermiticity of U_d, as quantified operationally by our practical ability to distinguish this operator from its adjoint with a probe state.

Finally, we note that combining Eq. (3.1) with Eqs. (2.18) and (2.19) gives the following relationship between $C_{max}(U_{AB})$, the entangling capacity without the restriction to initial product states, and the distinguishability of U_d and $U_d^\dagger$:

$$[C_{max}(U_{AB})]^4+[D_{min}(U_d^2)]^2=1. \qquad (3.15)$$

We are somewhat undecided about which of the two relationships in Eqs. (3.1) and (3.15) should be regarded as being more significant. What we will find, however, in the subsequent section, is that certain expressions involving entangling capacities, as quantified by the entropy of entanglement, are more naturally formulated in terms of the product state capacity rather that the corresponding capacity without this restriction.

IV. INFORMATION-THEORETIC INTERPRETATIONS OF THE PRODUCT ENTROPIC ENTANGLING CAPACITY

A. The first-order classical capacity

In the preceding section, we showed that the maximum concurrence that can be obtained by acting on an initial product state of two qubits $|\psi_A\rangle\otimes|\psi_B\rangle$ with a bipartite unitary operator U_{AB} is related, through Eq. (3.1), to the distinguishability of the operators U_d and $U_d^\dagger$.

The act of discriminating between these two operators is a "one-shot" procedure. However, in the theory of classical information transmission, one is typically concerned with asymptotic quantities that relate to very large strings of classical information. It is therefore interesting to enquire as to whether or not there is a link between such quantities and the maximum entropy of entanglement, which we may obtain from the maximum concurrence using Eq. (2.3), whose significance derives from its usefulness in asymptotic entanglement processing.

In this section, we shall see that relationships of this nature do indeed exist. We shall describe two of them. The first does not make use of Eq. (3.1) and relates to the so-called first-order classical capacity. This capacity is derived from the assumption that collective decoding of the signal states is impossible. Interestingly, in contrast with Eq. (3.1), this relationship expresses a certain connection between the product entropic entangling capacity of U_{AB} and the *in*distinguishability of states, in terms of a trade-off with the ability of certain quantum channels to faithfully transmit classical information, as quantified by the first-order classical capacity. This relationship depends upon the ability to conjugate states. Since complex conjugation cannot be carried out on an unknown state, the sender is required to know the state in advance of transmission if this relationship is to correspond to a physically realizable scenario. However, to our knowledge, it has not appeared in the literature so far so we include it for the sake of completeness, as its simplicity suggests that it may be of some formal interest.

The second relationship we shall describe, however, does not suffer from this weakness. Moreover, this second relationship, unlike the previous one, does make explicit use of Eq. (3.1) and the connection between entangling capacity and distinguishability it expresses. It relates to the HJSWW classical capacity [14]. Unlike the first-order capacity, the HJSWW capacity refers to situations where arbitrary collective decoding of the signal states is permitted. Our second relationship establishes that the product entropic entangling capacity of U_{AB} is precisely equal to the maximum of the HJSWW capacities of a certain set of quantum channels constructed using U_d and $U_d^\dagger$.

To describe the first of these relationships, let us recall Eq. (2.3), which expresses the entropy of entanglement in terms of the concurrence. Since these functions are monotonically increasing with one another, we easily conclude that the maximum entropy of entanglement that U_{AB} can generate from an initial product state is

$$E_{max}^{prod}(U_{AB})=h\left(\frac{1+\sqrt{1-[C_{max}^{prod}(U_{AB})]^2}}{2}\right). \qquad (4.1)$$

Consider one party Alice who wishes to send classical information to her distant colleague Bob using nonorthogonal quantum states. Alice sends him a stream of qubits over a noiseless quantum channel. Each of these systems acts as a signal carrier and they are all prepared in one of the N quantum states ρ_j, where $j=1,\ldots,N$. For a given signal carrier, the probability of her preparing the state ρ_j is p_j. On receiving the state ρ_j, Bob's task is to determine, using a measurement, which of these states Alice sent. The possible states may be nonorthogonal so, in general, Bob will not be able to distinguish among them reliably.

We assume here that Bob measures each of the signal carriers individually. That is to say, collective measurements on strings of signal carriers are not permitted. Subject to this restriction, the most general measurement he can perform will have K possible outcomes, for some positive integer K. Corresponding to outcome $k \in [1, \ldots, K]$ is a positive operator Π_k acting on the Hilbert space of a single signal carrier. These operators, known as positive, operator-valued measure (POVM) elements, must sum to the identity operator on the single signal carrier Hilbert space.

These operators serve to describe the statistical properties of Bob's measurement. The communication channel is characterized by the channel matrix. The elements of this matrix are the probabilities of obtaining each of the K possible measurement results for each of the N possible initial states. These probabilities are

$$P(k|\rho_j) = \mathrm{Tr}(\Pi_k \rho_j). \tag{4.2}$$

For nonorthogonal signal states, the channel matrix cannot be the identity matrix. It will have nonzero off-diagonal elements and so the channel behaves in many respects like a noisy classical channel, where the noise is an unavoidable consequence of the indistinguishability of nonorthogonal quantum states. The similarity is sufficiently strong to enable us to employ Shannon's noisy coding theorem. By redundant coding of the classical messages at Alice's end and classical error correction of the measurement results at Bob's, Alice can send Bob asymptotically error-free classical information at a nonzero rate. The maximum rate at which she can do this is characterized by the so-called first-order classical capacity. This quantity is defined in terms of the mutual information

$$I = \sum_{k=1}^{K} \sum_{j=1}^{N} p_j P(k|\rho_j) \log\left(\frac{P(k|\rho_j)}{\sum_{j'=1}^{N} p_{j'} P(k|\rho_{j'})} \right). \tag{4.3}$$

The mutual information clearly depends not only on the states ρ_j and their probabilities p_j, but also, through the POVM elements in Eq. (4.2), on Bob's measurement.

The significance of this quantity is as follows: the maximum rate at which Alice can send Bob asymptotically error-free classical information using the quantum states ρ_j, over all possible probability distributions p_j and all noncollective measurements that Bob may perform, is given by $\mathcal{C}_1(\{\rho_j\})$ bits per signal carrier. This quantity is the first-order classical capacity, and it is given by

$$\mathcal{C}_1(\{\rho_j\}) = \max_{\{p_j\}} \max_{\{\Pi_k\}} I; \tag{4.4}$$

that is, it is the maximum of the mutual information with respect to the probabilities p_j and the measurement Bob performs. We should mention that implicit in the maximization over the set of possible measurements is a maximization over K, the number of measurement outcomes.

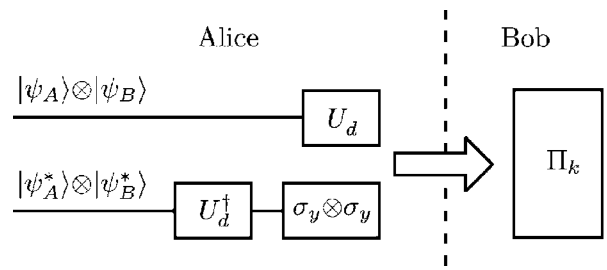

FIG. 4. Illustration of a scenario that leads to an interpretation of the product entropic entangling capacity of U_{AB} in terms of the first-order classical capacity. A classical description of the product state $|\psi_A\rangle \otimes |\psi_B\rangle$ is used to manufacture, at will, either this state or $|\psi_A^*\rangle \otimes |\psi_B^*\rangle$, its complex conjugate in the computational basis. These states are then acted on by U_d and $(\sigma_y \otimes \sigma_y)U_d^\dagger$, respectively. The resulting signals are then received by Bob who measures them individually. The maximum rate at which Alice can send classical information to Bob this way, over all product states $|\psi_A\rangle \otimes |\psi_B\rangle$, is related to the product entropic entangling capacity of U_{AB} through Eq. (4.9).

This quantity is difficult to calculate for most sets of quantum states. However, for a pair of pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, it can be calculated exactly. Sasaki *et al.* [19,20] have shown that the explicit expression is

$$\mathcal{C}_1(\{|\psi_j\rangle\}) = 1 - h\left(\frac{1 + \sqrt{1 - |\langle \psi_1|\psi_2\rangle|^2}}{2} \right). \tag{4.5}$$

Notice the similarity between the second term in this expression and that we gave for the product entropic entangling capacity in Eq. (4.1). Clearly, this similarity would be even greater if we could interpret the concurrence as an overlap between two pure states. Fortunately, we are able to do this. Using the results of Wootters [21], it was observed by Leifer *et al.* [4] that the concurrence of a bipartite pure state $|\Psi\rangle$ of two qubits can be written as

$$C(|\Psi\rangle) = |\langle \Psi|\sigma_y \otimes \sigma_y|\Psi^*\rangle|. \tag{4.6}$$

denotes complex conjugation in the computational basis—i.e., the basis of product eigenstates of $\sigma_z \otimes \sigma_z$. We note that, in this basis, $U_d^* = U_d^\dagger$.

So let us make the identifications

$$|\psi_1\rangle = |\Psi\rangle = U_d|\psi_A\rangle \otimes |\psi_B\rangle, \tag{4.7}$$

$$|\psi_2\rangle = (\sigma_y \otimes \sigma_y)U_d^\dagger|\psi_A^*\rangle \otimes |\psi_B^*\rangle, \tag{4.8}$$

for some product state $|\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}^{\otimes 2}$. Using these expressions, we now describe an information transmission procedure that links the product entropic entangling capacity of U_{AB} with the first-order classical capacity of certain quantum channels. The situation we shall consider is depicted in Fig. 4.

Each of Alice's transmissions begins with a classical description of some product state $|\psi_A\rangle \otimes |\psi_B\rangle$. To prepare the state $|\psi_1\rangle$, she creates $|\psi_A\rangle \otimes |\psi_B\rangle$ and then applies the unitary operator U_d. To prepare the state $|\psi_2\rangle$, she creates the state $|\psi_A^*\rangle \otimes |\psi_B^*\rangle$ and then applies the unitary operator $(\sigma_y \otimes \sigma_y)U_d^\dagger$. Notice the necessity of her having a classical description of the state $|\psi_A\rangle \otimes |\psi_B\rangle$ here. To produce the state $|\psi_2\rangle$, she cannot begin with an unknown product state $|\psi_A\rangle \otimes |\psi_B\rangle$ and then apply a quantum operation to it, as no quantum operation can conjugate an unknown state.

Following Alice's preparation of her desired state $|\psi_1\rangle$ or $|\psi_2\rangle$, she sends it to Bob, who proceeds to extract classical information. We find, upon combining Eqs. (4.1) and (4.5) with the fact that product entropic entangling capacities of U_{AB} and U_d are equal, that the product entropic entangling capacity of U_{AB} and the first-order classical capacity of the channel are related by

$$E_{max}^{prod}(U_{AB}) + \max_{|\psi_A\rangle\otimes|\psi_B\rangle\in\mathcal{H}^{\otimes 2}} \mathcal{C}_1(\{|\psi_j\rangle\}) = 1, \qquad (4.9)$$

where the $|\psi_j\rangle$ are defined by Eqs. (4.7) and (4.8). This relationship clearly expresses a trade-off between the product entropic entangling capacity of U_{AB} and the maximum first-order classical capacity over the channels we have described.

B. The HJSWW classical capacity

The relationship we have just described depends upon the conjugation of states. As such, a classical description of the product state $|\psi_A\rangle \otimes |\psi_B\rangle$ is required. Despite this, the simple nature of this relationship is something that we believe is of some formal interest.

Here we shall describe a relationship between the product entropic entangling capacity and classical capacities of quantum channels that does not suffer from this weakness. Also, this relationship does make use of Eq. (3.1), unlike the previous one. Finally, this relationship involves the HJSWW classical capacity of quantum channels. This capacity, which allows for collective measurements, is at least as high as the first-order classical capacity.

Let us once again consider Eq. (3.1). We see from this equation and Eq. (4.1) that the product entropic entangling capacity can be expressed as

$$E_{max}^{prod}(U_{AB}) = h\left(\frac{1 + D_{min}(U_d^2)}{2}\right). \qquad (4.10)$$

The relationship we shall describe makes use of the fact that if Alice wishes to send classical information to Bob, but unlike in the previous scenario, he is able to perform collective measurements on the signal carriers he receives, then an enhanced rate of classical information transmission can be achieved. This effect, known as the superadditive quantum coding gain, has recently been demonstrated for the first time in the laboratory by Sasaki and collaborators [22,23].

Suppose Alice wishes to send classical information to Bob by sending, over a noiseless quantum channel, states drawn from a source of N pure states $|\psi_j\rangle$ with respective probabilities p_j. In the limit when Bob can perform measurements on arbitrarily long strings of signal carriers, the classical capacity is given by [14]

$$\mathcal{C}_\infty(\{|\psi_j\rangle\}) = \max_{\{p_j\}} S(\rho), \qquad (4.11)$$

where S is the von Neumann entropy and $\rho = \sum_{j=1}^{N} p_j |\psi_j\rangle\langle\psi_j|$. For two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, the explicit form of the capacity $\mathcal{C}_\infty(\{|\psi_j\rangle\})$ is easily determined. We find

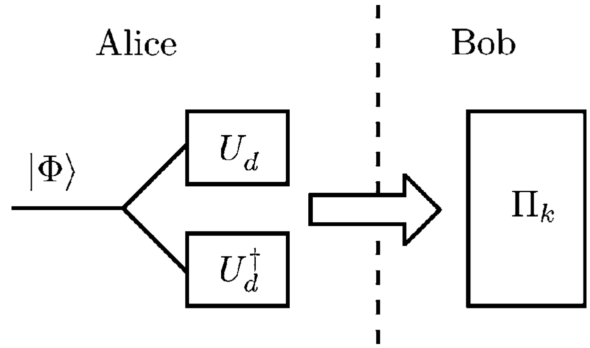

FIG. 5. Illustration of a scenario that leads to an interpretation of the product entropic entangling capacity of U_{AB} in terms of the HJSWW classical capacity. Alice produces a number of copies of a bipartite pure state $|\Phi\rangle$. Upon each one, she acts with either U_d or $U_d^\dagger$. The resulting modified states are sent to Bob, who is able to perform collective measurements on several of them together. The maximum rate at which Alice can send classical information to Bob this way, over all possible initial states $|\Phi\rangle$, is equal to the product entropic entangling capacity of U_{AB}, as expressed by Eq. (4.16).

$$S(\rho) = h\left(\frac{1 + \sqrt{(p_1 - p_2)^2 + 4p_1p_2|\langle\psi_1|\psi_2\rangle|^2}}{2}\right). \qquad (4.12)$$

Maximizing this expression with respect to the probabilities p_j and taking into account the normalization constraint $p_1 + p_2 = 1$ leads to the conclusion that the maximum is obtained when both probabilities are equal to $1/2$. This gives

$$\mathcal{C}_\infty(\{|\psi_j\rangle\}) = h\left(\frac{1 + |\langle\psi_1|\psi_2\rangle|}{2}\right). \qquad (4.13)$$

This expression is highly reminiscent of Eq. (4.10), especially in view of the fact that $D_{min}(U_d^2)$ is an overlap.

To see how the similarity of these two expressions can lead to a concrete physical relationship between the product entropic entangling and HJSWW classical capacities, consider the scenario depicted in Fig. 5. In Alice's laboratory there is a machine that produces many copies of some two-qubit pure state $|\Phi\rangle$. This machine can be set to produce any such state. However, it cannot be reset; once one particular state has been set to be mass produced, it is impossible to reprogram the machine to produce a different state.

On each copy of the state $|\Phi\rangle$, Alice acts with one of the operators U_d and $U_d^\dagger$. For each copy, she is completely free in her choice of which operator to use. She then transmits the modified state to Bob, who is able to perform arbitrary collective measurements on arbitrarily long strings of signal states.

The HJSWW capacity of this channel is given by Eq. (4.13), where

$$|\psi_1\rangle = U_d^\dagger|\Phi\rangle, \qquad (4.14)$$

$$|\psi_2\rangle = U_d|\Phi\rangle. \qquad (4.15)$$

Clearly, the only variable is the initial state $|\Phi\rangle$ and it is interesting to maximize the HJSWW capacity with respect to this state. It is easy to see that the maximum is, essentially by definition, given by Eq. (4.10). We therefore obtain

$$E_{max}^{prod}(U_{AB}) = \max_{|\Phi\rangle \in \mathcal{H}^{\otimes 2}} \mathcal{C}_\infty(\{|\psi_j\rangle\}), \tag{4.16}$$

where the $|\psi_j\rangle$ are defined by Eqs. (4.14) and (4.15).

We regard this relationship as being the asymptotic analog of Eq. (3.1). In Eq. (3.1), we saw that the more distinguishable U_d and $U_d^\dagger$ are, the higher is the product entangling capacity of U_{AB} in terms of the concurrence. It is natural then that, in the asymptotic limit, the entangling capacity, which is quantified by the maximum entropy of entanglement, is related to the extent to which we can send classical information by modulating an initial state with U_d and $U_d^\dagger$, which is precisely what is expressed by Eq. (4.16).

V. DISCUSSION

The principal aim of the present paper has been to describe an interesting relationship between the entangling capacity of a two-qubit unitary operator U_{AB} and the distinguishability of its canonical form U_d and its adjoint $U_d^\dagger$. We saw, in Eq. (3.1), that the product entangling capacity of U_{AB} quantifies the distinguishability of U_d and $U_d^\dagger$. This implies that it may also be viewed as a measure of the non-Hermiticity of U_d. As a measure of non-Hermiticity, it is highly significant from a practical point of view, since it quantifies the extent to which we can operationally distinguish U_d from $U_d^\dagger$ with a probe state.

In the asymptotic limit, the entanglement properties of pure bipartite states are naturally quantified using the entropy of entanglement. It is interesting to enquire as to whether or not our relationship between the maximum concurrence and minimum overlap has any implications for this limit. Indeed it does. We found two relationships between the product entropic entangling capacity and the classical capacities of quantum channels. The first does not actually involve Eq. (3.1) but we included it for the sake of completeness as, to our knowledge, this relationship was not previously known. It also suffers from a weakness; it corresponds to a classical information transmission scenario where the sender, Alice, must know the state in advance.

Our second relationship, however, is more satisfactory in a number of respects. Firstly, it makes explicit use of Eq. (3.1). Secondly, it applies to an unknown initial state. Finally, it corresponds to a more general scenario where collective decoding of the signal carriers is permitted, thus making greater use of the ability of quantum states to carry classical information.

The results in this paper relate to bipartite unitary operators where the subsystems are qubits. It is natural to enquire as to whether or not they can be extended to higher dimensional subsystems. What is interesting is the fact that the theory of distinguishing between a pair of unitary operators outlined in Sec. II applies very generally, certainly to all unitary operators on finite dimensional Hilbert spaces. However, the theory of the entangling capacity of bipartite unitary operators is not similarly well-developed. Exact results are only known for qubit subsystems. It should be said in this context that the entangling power, which is the *average* amount of entanglement that an arbitrary bipartite unitary operator can generate, has been established by Zanardi and co-workers [7–9] for higher-dimensional subsystems. However, here we are interested in the entangling capacity, which is the *maximum* amount of entanglement that can be created.

One of the reasons why the entangling capacity of a general $D_1 \times D_2$ unitary operator, for $D_1, D_2 > 2$, has not yet been established is the fact that a suitable canonical form U_d is not known for such operators. Also, for higher-dimensional subsystems, there are difficulties in defining a single quantity which generalizes the concurrence [24,25]. This is related to fact that for higher-dimensional bipartite pure entangled states, there is no single quantity which completely characterizes the entanglement. In general, we have a set of entanglement monotones which can vary with a certain degree of independence from one state to another [26]. For one state to be unambiguously more entangled than another, the values of all entanglement monotones must be at least as high for one of the states as they are for the other. Otherwise, the states are incomparable.

These considerations lead us to conclude this paper with a question. Might it be the case that for $D_1 \times D_2$ unitary operators with $D_1, D_2 > 2$, there is, in general, no unique entangling capacity at the one-shot level, even if we restrict ourselves to initial product states? This would be the case if, for all $D_1, D_2 > 2$, we can find a unitary operator U_{AB} which has the following property: there is no initial product state $|\psi_A\rangle \otimes |\psi_B\rangle$ such that $U_{AB}|\psi_A\rangle \otimes |\psi_B\rangle$ simultaneously maximizes all entanglement monotones. Due to the fact that these monotones exhibit a certain degree of mutual independence, this seems possible.

ACKNOWLEDGMENTS

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APPENDIX: PROOF OF EQ. (3.1) FOR NEGATIVE α_z

We prove here that Eq. (3.1) holds when α_z is negative. This can be done rather easily if it also holds for nonnegative α_z, and we established that it does in Sec. III. To proceed, we note that the validity of Eq. (3.1) for nonnegative α_z implies that

$$[C_{max}^{prod}(U_{d'})]^2 + [D_{min}(U_{d'}^2)]^2 = 1 \tag{A1}$$

for all $d' = (\alpha_x', \alpha_y', \alpha_z')$ which satisfy

$$0 \leqslant \alpha_z' \leqslant \alpha_y' \leqslant \alpha_x' \leqslant \pi/4. \tag{A2}$$

Now consider U_{AB} and U_d related by Eq. (2.7) where $d = (\alpha_x, \alpha_y, \alpha_z)$, $\alpha_z < 0$ and inequality (2.9) holds. We make the following observation: if $\alpha_x' = \alpha_x$, $\alpha_y' = \alpha_y$ and $\alpha_z' = -\alpha_z$, then

$$(\sigma_z \otimes 1) U_d (\sigma_z \otimes 1) = U_{d'}^\dagger. \tag{A3}$$

This implies that U_{AB} is locally equivalent to and therefore has the same product entangling capacity as $U_{d'}^\dagger$. Indeed, it also has the same product entangling capacity as $U_{d'}$, because $U_{d'}^{\dagger}$ is simply the complex conjugate of $U_{d'}$ in the computational basis. It has been established [1] that complex conjugation in this basis does not change the entangling capacity. We may therefore write

$$C_{max}^{prod}(U_{AB}) = C_{max}^{prod}(U_{d'}). \tag{A4}$$

Also, the fact that U_d and $U_{d'}^{\dagger}$ are unitarily related through Eq. (A3) implies that $D_{min}(U_d^2)=D_{min}(U_{d'}^{\dagger 2})$, which is easily seen to be also equal to $D_{min}(U_{d'}^2)$. We may then write

$$D_{min}(U_d^2) = D_{min}(U_{d'}^2). \tag{A5}$$

Finally, substitution of Eqs. (A4) and (A5) into Eq. (A1) gives Eq. (3.1) as desired.

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