

Quantum entanglement enhances the capacity of bosonic channels with memory

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The bosonic quantum channels have recently attracted a growing interest, motivated by the hope that they open a tractable approach to the generally hard problem of evaluating quantum channel capacities. These studies, however, have always been restricted to memoryless channels. Here, it is shown that the classical capacity of a bosonic Gaussian channel with memory can be significantly enhanced if entangled symbols are used instead of product symbols. For example, the capacity of a photonic channel with 70%-correlated thermal noise of one-third the shot noise is enhanced by about 11% when using 3.8-dB entangled light with a modulation variance equal to the shot noise.

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I. INTRODUCTION

A main goal of quantum information theory is to evaluate the information capacities of quantum communication channels. In particular, an important question is to determine how much *classical* information can be processed asymptotically via a quantum channel. This problem has been solved, today, only for a few quantum channels, and it has been addressed only recently for bosonic channels, i.e., continuous-variable quantum channels acting on a bosonic field such as the electromagnetic field [1]. The classical capacity of a purely *lossy* bosonic channel was solved exactly only very recently [2], while the case of a *noisy* bosonic channel is still not completely solved. More precisely, the classical capacity of the Gaussian bosonic channel (a continuous-variable quantum channel undergoing a Gaussian-distributed thermal noise) has been derived in [3] but this result only holds provided the optimal input ensemble is a tensor product of Gaussian pure states, as conjectured by several authors but not rigorously proven today (see, e.g., [4] for recent progress on this problem). All these studies, however, have been restricted to memoryless bosonic channels.

In this paper, we investigate the capacity of a bosonic Gaussian channel that exhibits memory. This study is motivated by the recent finding that, for some extension of the depolarizing channel with correlated noise, sending entangled qubit pairs can enhance the classical capacity [5]. Such an effect, which has even been experimentally demonstrated in optical fibers exhibiting slowly fluctuating birefringence [6], contrasts with the common knowledge that entanglement is of no use for information transfer via a memoryless quantum channel. Here, we consider channels with a thermal noise that has a *finite* bandwidth. The resulting memory effect is modeled by assuming that the noise affecting two subsequent uses of the channel follows a bivariate Gaussian distribution with a nonvanishing correlation coefficient, measuring the degree of memory of the channel. We prove that if the memory is nonzero and if the input

energy is constrained, then the transmission rate can be significantly enhanced by using entangled symbols instead of product symbols. The relation between the degree of memory and the resulting optimal input entanglement is analyzed.

II. BOSONIC GAUSSIAN CHANNELS

Let us define a memoryless bosonic Gaussian channel T acting on a mode of the electromagnetic field associated with the annihilation and creation operators a and a^\dagger , or, equivalently, the quadrature components $q=(a+a^\dagger)/\sqrt{2}$ and $p=i(a^\dagger-a)/\sqrt{2}$, satisfying the commutation relation $[q,p]=i$. If the input of the channel is initially in state ρ , we have

$$\rho \mapsto T[\rho] = \int d^2\beta q(\beta) D(\beta) \rho D^\dagger(\beta), \quad (1)$$

where $d^2\beta = d[\text{Re}(\beta)]d[\text{Im}(\beta)]$, while $D(\beta) = e^{\beta a^\dagger - \beta^* a}$ denotes the displacement operator (such that $|\alpha\rangle = D(\alpha)|0\rangle$ with $|0\rangle$ being the vacuum state and $|\alpha\rangle$ being a coherent state of mean value α). For a Gaussian channel, the kernel is a bivariate Gaussian distribution with variance N , that is, $q(\beta) = (\pi N)^{-1} e^{-|\beta|^2/N}$. The channel then randomly displaces an input coherent state according to a Gaussian distribution, which results in a thermal state (N is the variance of the added noise on the quadrature components q and p , or, equivalently the number of thermal photons added by the channel). The Gaussian CP map effected by this channel can also be characterized via the covariance matrix. Restricting to Gaussian states with a vanishing mean value, a complete state characterization is provided by the covariance matrix

$$\gamma = \begin{pmatrix} \langle q^2 \rangle & \frac{1}{2} \langle qp + pq \rangle \\ \frac{1}{2} \langle qp + pq \rangle & \langle p^2 \rangle \end{pmatrix}. \quad (2)$$

The Gaussian channel can then be written as

$$\gamma \mapsto \gamma + \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}. \quad (3)$$

III. CLASSICAL CAPACITY OF A QUANTUM CHANNEL

The coding theorem for quantum channels asserts that the one-shot classical capacity of a quantum channel T is given by

$$C_1(T) = \max \left[S \left(\sum_i p_i T[\rho_i] \right) - \sum_i p_i S(T[\rho_i]) \right], \quad (4)$$

where $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy of the density operator ρ . In Eq. (4), the maximum is taken over all probability distributions $\{p_i\}$ and collections of density operators $\{\rho_i\}$ satisfying the energy constraint

$$\sum_i p_i \text{Tr}(\rho_i a^\dagger a) \leq \bar{n}, \quad (5)$$

with \bar{n} being the maximum mean photon number at the input of the channel. For a monomodal bosonic Gaussian channel, it is conjectured that a Gaussian mixture of coherent states (i.e., a thermal state) achieves the channel capacity. In the rest of this paper, we will not question this generally admitted conjecture, so our analysis will be restricted to Gaussian states. The sum over i is replaced by an integral over α , where the input states $\rho_\alpha^{\text{in}} = |\alpha\rangle\langle\alpha|$ are drawn from the probability density $p(\alpha) = (\pi\bar{n})^{-1} e^{-|\alpha|^2/\bar{n}}$. Thus, the one-shot classical capacity of the channel becomes

$$C_1(T) = S(\bar{\rho}) - \int d^2\alpha p(\alpha) S(\rho_\alpha^{\text{out}}), \quad (6)$$

where we have defined the individual output states

$$\rho_\alpha^{\text{out}} = T[\rho_\alpha^{\text{in}}] = \frac{1}{\pi N} \int d^2\beta e^{-\frac{|\beta-\alpha|^2}{N}} |\beta\rangle\langle\beta| \quad (7)$$

and their mixture (saturating the energy constraint)

$$\bar{\rho} = \int d^2\alpha p(\alpha) \rho_\alpha^{\text{out}} = \frac{1}{\pi(\bar{n}+N)} \int d^2\beta e^{-\frac{|\beta|^2}{\bar{n}+N}} |\beta\rangle\langle\beta|. \quad (8)$$

In order to calculate the entropy of a state ρ , one computes the symplectic values of its covariance matrix γ , i.e., the solutions of the equation $|\gamma - \lambda J| = 0$, where

$$J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (9)$$

It can be shown that these values always come as a pair $\pm\lambda$, so that the entropy is given by $S(\rho) = g(|\lambda| - \frac{1}{2})$, where

$$g(x) = \begin{cases} (x+1)\log_2(x+1) - x\log_2 x, & x > 0, \\ 0, & x = 0, \end{cases} \quad (10)$$

is the entropy of a thermal state with a mean photon number of x . Since the input states ρ_α^{in} are coherent states with a covariance matrix

$$\gamma^{\text{in}} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (11)$$

the individual output states ρ_α^{out} and their mixture $\bar{\rho}$ are associated with the covariance matrices

$$\gamma^{\text{out}} = \frac{1}{2} \begin{pmatrix} 1+2N & 0 \\ 0 & 1+2N \end{pmatrix}, \quad (12)$$

$$\bar{\gamma} = \frac{1}{2} \begin{pmatrix} 1+2(\bar{n}+N) & 0 \\ 0 & 1+2(\bar{n}+N) \end{pmatrix}, \quad (13)$$

so that the one-shot capacity of the channel is

$$C_1(T) = g(\bar{n}+N) - g(N). \quad (14)$$

IV. BIMODAL CHANNEL

Consider two subsequent uses of a memoryless channel T , defining the bimodal channel

$$\rho \mapsto T_{12}[\rho] = \int d^2\beta_1 d^2\beta_2 q(\beta_1, \beta_2) \times D(\beta_1) \otimes D(\beta_2) \rho D^\dagger(\beta_1) \otimes D^\dagger(\beta_2), \quad (15)$$

where $q(\beta_1, \beta_2) = (\pi N)^{-2} e^{-(|\beta_1|^2 + |\beta_2|^2)/N}$ since the noise affecting the two uses is uncorrelated. Ordering the quadrature components of the two modes in a column vector $R = [q_1, p_1, q_2, p_2]^T$, we define the covariance matrix γ_{12} of a bimodal state ρ_{12} as

$$\gamma_{12} = \text{Tr}(R \rho_{12} R^T) - \frac{1}{2} J_1 \oplus J_2, \quad (16)$$

where each J_j takes the form (9). We restrict ourselves to bimodal Gaussian states, characterized by

$$\gamma_{12} = \begin{pmatrix} \gamma_1 & \sigma_{12} \\ \sigma_{12}^T & \gamma_2 \end{pmatrix}, \quad (17)$$

where γ_1 is the covariance matrix associated with the reduced density operator $\rho_1 = \text{Tr}_2(\rho_{12})$ of mode 1 (and similarly for γ_2), while σ_{12} characterizes the correlation and/or entanglement between the two modes. For a memoryless channel, the optimal input states are simply products of coherent states, with a covariance matrix $\gamma_{12}^{\text{in}} = \gamma_1^{\text{in}} \oplus \gamma_2^{\text{in}}$ where γ_1^{in} and γ_2^{in} both take the form (11), while $\sigma_{12}^{\text{in}} = 0$. The optimal input modulation is a product of Gaussian distributions, $p(\alpha_1, \alpha_2) = (\pi\bar{n})^{-2} e^{-(|\alpha_1|^2 + |\alpha_2|^2)/\bar{n}}$. It follows that the classical capacity of this channel is additive, $\frac{1}{2} \times C_1(T_{12}) = C_1(T)$.

V. BOSONIC GAUSSIAN CHANNEL WITH MEMORY

Let us investigate what happens if the noise is correlated, for instance, when the two uses are closely separated in time and the channel has a finite bandwidth. We assume that the noise distribution takes the general form

$$q(\beta_1, \beta_2) = \frac{1}{\pi^2 \sqrt{|\gamma_N|}} e^{-\beta^\dagger \gamma_N^{-1} \beta}, \quad (18)$$

where $\beta = [\text{Re}(\beta_1), \text{Im}(\beta_1), \text{Re}(\beta_2), \text{Im}(\beta_2)]^T$ and γ_N is the covariance matrix of the noise quadratures, chosen to be

$$\gamma_N = \begin{pmatrix} N & 0 & -xN & 0 \\ 0 & N & 0 & xN \\ -xN & 0 & N & 0 \\ 0 & xN & 0 & N \end{pmatrix}. \quad (19)$$

Thus, the map T_{12} can be expressed by $\gamma_{12} \mapsto \gamma_{12} + \gamma_N$, so that the noise terms added on the p quadratures of modes 1 and 2 are correlated Gaussians with variance N (those added on the q quadratures are anticorrelated Gaussians with variance N [7]). The correlation coefficient x ranges from $x=0$ for a memoryless channel to $x=1$ for a channel with full memory.

We will now prove that for correlated thermal noise, the capacity is attained if some appropriate degree of entanglement is injected at the input of the channel [8]. Intuitively, if we take an Einstein-Podolsky-Rosen (EPR) state, i.e., the common eigenstate of $q_1 + q_2$ and $p_1 - p_2$ with respective eigenvalues q_+ and p_- , it is clear that the noise on q_+ and p_- effected by the channel is reduced as x increases. This suggests that using entangled input states may decrease the effective noise, hence increase the capacity. However, EPR states have infinite energy so they violate the energy constraint. Instead, we may inject (finite-energy) two-mode vacuum squeezed states, whose covariance matrix is given by

$$\gamma_1^{\text{in}} = \gamma_2^{\text{in}} = \frac{1}{2} \begin{pmatrix} \cosh 2r & 0 \\ 0 & \cosh 2r \end{pmatrix}, \quad (20)$$

$$\sigma_{12}^{\text{in}} = \frac{1}{2} \begin{pmatrix} -\sinh 2r & 0 \\ 0 & \sinh 2r \end{pmatrix}, \quad (21)$$

with r being the squeezing parameter. Note that purely classical correlations between the quadratures in the input distribution $p(\alpha_1, \alpha_2)$ also help increase the capacity when $x > 0$, so we have to check that entanglement gives an extra enhancement in addition to this.

The mean photon number in each mode of the state characterized by Eqs. (20) and (21) is $\sinh^2 r$, so that the maximum allowed modulation (for a fixed maximum photon number \bar{n}) decreases as entanglement increases. Remarkably, there is a possible compromise between this reduction of modulation and the entanglement-induced noise reduction on q_+ and p_- . To show this, consider input states with $\sinh^2 r = \eta \bar{n}$, where η measures the *degree of entanglement* and is used to interpolate between a product of vacuum states ($\eta = 0$), which can be maximally modulated, and an entangled state ($\eta = 1$), for which the entire energy is due to entanglement and no modulation can be applied. At the output of the channel, we get states with a covariance matrix γ_{12}^{out} where

$$\gamma_{1,2}^{\text{out}} = \frac{1}{2} \begin{pmatrix} \cosh 2r + 2N & 0 \\ 0 & \cosh 2r + 2N \end{pmatrix}, \quad (22)$$

$$\sigma_{12}^{\text{out}} = \frac{1}{2} \begin{pmatrix} -\sinh 2r - 2xN & 0 \\ 0 & \sinh 2r + 2xN \end{pmatrix}, \quad (23)$$

while the mixture $\bar{\gamma}_{12}$ of these states is characterized by

$$\bar{\gamma}_{1,2} = \gamma_{1,2}^{\text{out}} + \begin{pmatrix} (1-\eta)\bar{n} & 0 \\ 0 & (1-\eta)\bar{n} \end{pmatrix}, \quad (24)$$

$$\bar{\sigma}_{12} = \sigma_{12}^{\text{out}} + \begin{pmatrix} y(1-\eta)\bar{n} & 0 \\ 0 & -y(1-\eta)\bar{n} \end{pmatrix}, \quad (25)$$

assuming that the energy constraint is saturated. Here, y stands for the classical input correlation coefficient (to compensate for the noise, the q displacements need to be correlated, and the p displacements anticorrelated).

VI. ENTANGLED-ENHANCED CAPACITY

In order to evaluate the transmission rate achieved by these states, we need first to compute the symplectic values $\lambda_{12}^{\text{out}}$ and $\bar{\lambda}_{12}$ of γ_{12}^{out} and $\bar{\gamma}_{12}$, respectively. The symplectic values $\pm\lambda_{12}$ of a covariance matrix γ_{12} of the generic form (17) are the solutions of the equation $|\gamma_{12} - \lambda_{12}(J_1 \oplus J_2)| = 0$, or, equivalently, the biquadratic equation

$$\lambda_{12}^4 - (|\gamma_1| + |\gamma_2| + 2|\sigma_{12}|)\lambda_{12}^2 + |\gamma_{12}| = 0. \quad (26)$$

Using Eqs. (22)–(25), we see that γ_{12}^{out} and $\bar{\gamma}_{12}$ admit each one pair of doubly degenerate symplectic values, namely, $\lambda_{12}^{\text{out}} = \pm\sqrt{u_{\text{out}}^2 - v_{\text{out}}^2}$ and $\bar{\lambda}_{12} = \pm\sqrt{\bar{u}^2 - \bar{v}^2}$, with

$$u_{\text{out}} = \frac{1}{2} + \eta\bar{n} + N, \quad v_{\text{out}} = \sqrt{\eta\bar{n}(1 + \eta\bar{n})} + xN,$$

$$\bar{u} = \frac{1}{2} + \bar{n} + N, \quad \bar{v} = \sqrt{\eta\bar{n}(1 + \eta\bar{n})} + xN - y(1 - \eta)\bar{n}.$$

The transmission rate per mode is then given by

$$R(y, \eta) = g\left(|\bar{\lambda}_{12}| - \frac{1}{2}\right) - g\left(|\lambda_{12}^{\text{out}}| - \frac{1}{2}\right). \quad (27)$$

When $x > 0$, the optimized rate R over y increases with the degree of entanglement η and attains a maximum at some optimal value $\eta^* > 0$ (see Fig. 1), so that the maximum is achieved by entangled input states as advertised.

It now suffices to maximize R with respect to both y and η in order to find the channel capacity C (assuming that the restriction to Gaussian states is justified and that no product but non-Gaussian states may outperform the Gaussian entangled states considered here). If we keep the signal-to-noise ratio \bar{n}/N constant, it is visible from Fig. 2 that the optimal degree of entanglement η^* is the highest at some particular value of the mean input photon number \bar{n} , and then decreases back to zero in the large- \bar{n} limit (except if $x=0$ or 1). Clearly, in this limit, the channel T tends to a couple of classical channels with Gaussian additive noise (one for each quadrature), so that entanglement cannot play a role any more [9]. Figure 3 shows the corresponding optimal value of the input correlation coefficient y^* for the same values of the other relevant parameters. Note that, even in the classical limit $\bar{n} \rightarrow \infty$, some nonzero input correlation is useful to enhance the capacity of a Gaussian channel with $x > 0$.

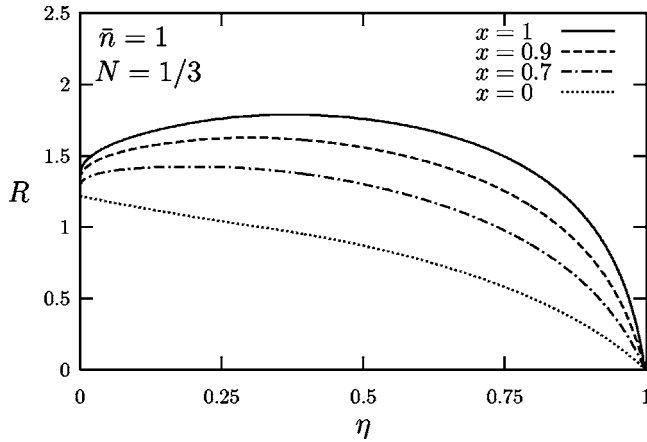


FIG. 1. Transmission rate R (maximized over the classical correlation coefficient y) as a function of the input entanglement η for a thermal channel with a degree of memory x . The mean number of photons is $\bar{n}=1$ at the input, while the number of added thermal photons is $N=1/3$.

VII. CONCLUSIONS

We have shown that entangled states are useful to enhance the classical capacity of a bosonic channel undergoing a thermal noise with memory. We determined the amount of entanglement that maximizes the information transmitted over the channel for a given input energy (mean photon number per mode) and noise level (mean number of thermal photons per mode). For example, the capacity of a channel with a mean number of thermal photons of $1/3$ and a correlation coefficient of 70% is enhanced by 10.8% if the mean photon number is 1 and the two-mode squeezing is 3.8 dB at the input. This capacity enhancement may seem paradoxical since using entangled signal states necessarily decreases the modulation variance for a fixed input energy, which seemingly lowers the capacity. However, due to the quantum correlations of entangled states, the noise affecting one mode can be partly compensated by the correlated noise affecting the second mode, which globally reduces the effective noise. Interestingly, there exists a regime in which this latter effect

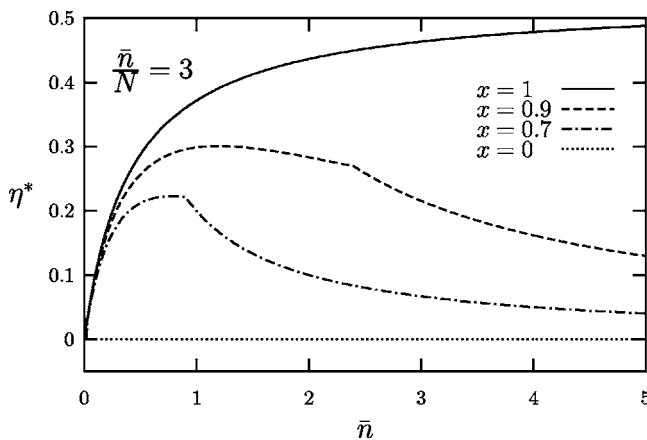


FIG. 2. Optimal degree of input entanglement η^* as a function of the mean input photon number \bar{n} for a fixed signal-to-noise ratio $\bar{n}/N=3$.

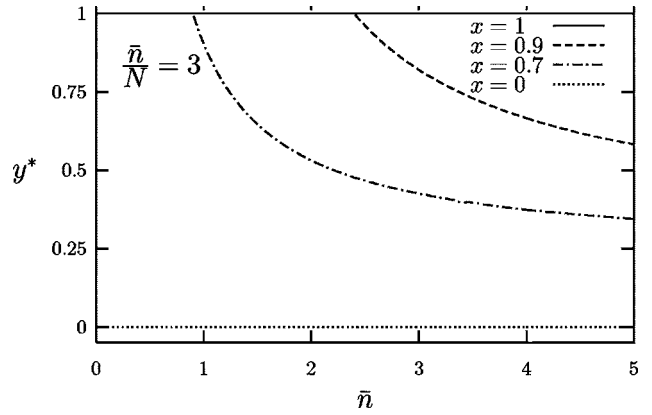


FIG. 3. Optimal degree of input correlation y^* as a function of the mean input photon number \bar{n} for a fixed signal-to-noise ratio $\bar{n}/N=3$.

dominates, resulting in a net enhancement of the amount of classical information transmitted per use of the channel. The capacity gain G , measuring the entanglement-induced capacity enhancement, is plotted in Fig. 4. It illustrates that a capacity enhancement of tens of percents is achievable by using entangled light beams with experimentally accessible levels of squeezing.

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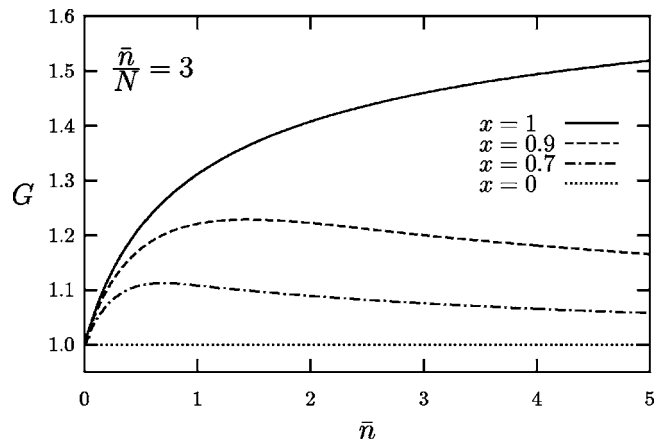


FIG. 4. Capacity gain $G = \max_{y, \eta} R(y, \eta) / \max_y R(y, 0)$ as a function of the mean input photon number \bar{n} for a fixed signal-to-noise ratio $\bar{n}/N=3$.

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- [8] Our channel with correlated thermal noise can be mapped, via a capacity-preserving beam-splitter transformation, onto a pair of channels with independent but phase-sensitive noise. The capacity of the latter channels being attained with product squeezed states, which are mapped back onto entangled states by this transformation, this confirms the role of entanglement for channels with memory (correlated noise).
- [9] Although the fraction of the input mean photon number that is due to entanglement $\eta^* \rightarrow 0$, its absolute value $\eta^* \bar{n}$ tends to a constant as $\bar{n} \rightarrow \infty$ (except for $x=1$).
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