

# Single-copy entanglement in critical quantum spin chains

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We consider the single-copy entanglement as a quantity to assess quantum correlations in the ground state in quantum many-body systems. We show for a large class of models that already on the level of single specimens of spin chains, criticality is accompanied with the possibility of distilling a maximally entangled state of arbitrary dimension from a sufficiently large block deterministically, with local operations and classical communication. These analytical results—which refine previous results on the divergence of block entropy as the rate at which maximally entangled pairs can be distilled from many identically prepared chains—are made quantitative for general isotropic translationally invariant spin chains that can be mapped onto a quasifree fermionic system, and for the anisotropic *XY* model. For the *XX* model, we provide the asymptotic scaling of  $\sim(1/6)\log_2(L)$ , and contrast it with the block entropy.

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## I. INTRODUCTION

Quantum phase transitions of second order are accompanied by a divergent length scale: this is the classical correlation length, the characteristic length associated with the two-point correlation function [1]. Recently, it has increasingly become clear that one should expect additional insight in the scaling of quantum correlations present in the ground state of a many-body system at or close to a quantum phase transition by expressing them in terms of entanglement properties [2–13]. Entanglement, after all, plays a fundamental role in quantum phase transitions at zero temperature. The theory of entanglement in turn—developed in the quantum-information context—provides tools to characterize and quantify genuine quantum correlations in contrast to correlations that occur in states that can be prepared with mere local operations together with classical communication (LOCC). In particular, one finds that in one-dimensional noncritical harmonic [3–5] or quantum spin systems [5,6,8–11], the degree of entanglement of a block of  $L$  systems, quantified in terms of the block entropy, typically saturates for large block size, with higher-dimensional “entropy-area laws” [4]. In contrast, in critical spin systems or in fermionic systems, the block entropy has logarithmic corrections as  $L \rightarrow \infty$  [6,8–10,12]. These findings are consistent with expectations from conformal field theory [5]. Such a behavior of the block entropy has also been related to the performance of DMRG simulations of ground-state properties [7,8]. This von Neumann entropy of a block quantifies the rate at which one can asymptotically distill maximally entangled qubit pairs under LOCC, when one has infinitely many identically prepared many-body systems at hand and is able to perform joint operations on many pairs at once [14].

Yet, in several contexts, in particular for condensed-matter systems, this asymptotic notion of entanglement implicitly referring to joint operations on many identical systems may not always be the most appropriate one. Instead, one may ask: Does a single specimen of a critical infinite system already contain an infinite amount of entanglement

(in the sense that any desired amount of entanglement can be distilled from a single specimen)? This will be the central question of this paper. We introduce the single-copy entanglement to quantify the quantum correlations in critical and noncritical many-body systems. More specifically, compared to the divergence of the block entropy, we ask the stronger question whether a single spin chain already contains an arbitrary amount of entanglement, such that from a single specimen a maximally entangled state of arbitrary dimension can be distilled.

We will make the argument quantitative by analytically considering a general framework of translationally invariant quantum spin models. As examples in which criticality is in one-to-one correspondence with a divergent single-copy entanglement, we consider isotropic spin models, as well as the *XY* model. For the isotropic *XY* model we establish the exact asymptotic scaling behavior of  $\sim(1/6)\log_2(L)$ , and relate it to the block entropy [9,10]. The results can also be conceived as statements concerning the divergence of fine-grained entanglement [13].

## II. SINGLE-COPY ENTANGLEMENT

Let us consider a one-dimensional quantum spin system, associated with a Hilbert space  $\mathcal{H}=(\mathbb{C}^2)^{\otimes n}$ , with a translationally invariant Hamiltonian. We distinguish a block of length  $L$  of consecutive spin systems of the chain. So we have a bipartitioning  $n|L$ , the whole system being in a pure state  $\rho=|\psi\rangle\langle\psi|$ .

There are several meaningful definitions of single-copy entanglement. We will primarily be concerned with the following question: Running a physical device once, a maximally entangled state of what dimension can be distilled from a single specimen with certainty? Hence, the state has a single-copy entanglement  $E_1(\rho)=\log_2(M)$ , with respect to the bipartitioning  $n|L$ , if  $\rho$  can be deterministically transformed under LOCC into  $|\psi_M\rangle\langle\psi_M|$ , i.e., a maximally entangled state with state vector

$$|\psi_M\rangle = (|1,1\rangle + \dots + |M,M\rangle)/\sqrt{M}, \quad (1)$$

where  $\{|1\rangle, \dots, |M\rangle\}$  are elements of a local basis of the respective Hilbert spaces of the subsystems, so if

$$\rho \rightarrow |\psi_M\rangle\langle\psi_M| \quad \text{under LOCC.} \quad (2)$$

Equivalently, this can also be formulated as the maximal number of pairs, so maximally entangled qubit pairs, that can be distilled. This quantity is the nonasymptotic analog of the block entropy, so the entropy of entanglement of the reduction associated with a block of length  $L$ . Denoting by  $\alpha_1^\downarrow, \dots, \alpha_{2L}^\downarrow$  the nonincreasingly ordered eigenvalues of the reduced state with respect to a block of length  $L$ , then Eq. (2) holds true if and only if [15]

$$\sum_{k=1}^K \alpha_k^\downarrow \leq K/M \quad (3)$$

for all  $1 \leq K \leq M$ , so obviously if and only if  $\alpha_1^\downarrow \leq 1/M$ . In other words, the transformation is possible if the reduction is more mixed in the sense of majorization than the reduction of the maximally entangled state of dimension  $M \times M$ . Given  $\alpha_1^\downarrow$ , the single-copy entanglement is nothing but

$$E_1(\rho) = \log_2[(\alpha_1^\downarrow)^{-1}]. \quad (4)$$

A variant is one allowing for probabilistic protocols. For a state  $\rho$  we say that

$$E_p(\rho) = \sup \sum_{k=0}^{\infty} p_k \log_2(M_k), \quad (5)$$

such that  $\rho$  can be transformed under LOCC into the ensemble  $\{(p_k, |\psi_{M_k}\rangle\langle\psi_{M_k}|) : k=0, 1, \dots\}$ . This is the average entanglement that can be distilled, allowing for maximally entangled states of different dimension with certain probabilities [16]. This rate is then the solution of a linear program [17]. By definition, we have that  $E_1(\rho) \leq E_p(\rho) \leq S(\text{tr}_{n \setminus L}[\rho])$ , where the last inequality follows from the fact that the block entropy bounds the rate of any (asymptotic) distillation protocol.

Finally, note that for the single-copy entanglement, superselection rules (SSRs) play a crucial role, notably a SSRs with respect to particle number conservation. In the presence of SSRs,  $E_1^{\text{SSR}}$  has to be understood as referring to a probabilistic transformation distilling entanglement-EPR states with state vector  $|\psi^{\text{SSR}}\rangle = (|0,1\rangle|1,0\rangle + |1,0\rangle|0,1\rangle)/\sqrt{2}$  under LOCC and SSR. We will now consider the behavior of the single-copy entanglement in the limit of large  $L$  for critical and noncritical spin chains.

### III. SINGLE-COPY ENTANGLEMENT IN GENERAL QUANTUM SPIN CHAINS

We start from the general set of translationally invariant quantum spin systems that is as in Ref. [10] mapped onto a fermionic quadratic Hamiltonian under a Jordan-Wigner transformation. This model embodies a large class of spin models, including the anisotropic and isotropic  $XY$  models as

important special cases. Hence, the Hamiltonian is

$$H = \sum_{q=0,1} \sum_{k,j=1}^n \left( \frac{B_{j-k}}{2} - \frac{A_{j-k}}{4} \right) \hat{\sigma}_k^{q+1} \left( \prod_{\substack{i \leq k+q \\ l \leq j+1-q}} \hat{\sigma}_i^3 \hat{\sigma}_l^3 \right) \hat{\sigma}_j^{q+1} \quad (6)$$

where  $\hat{\sigma}_k^1, \hat{\sigma}_k^2, \hat{\sigma}_k^3$  denote the Pauli operators associated with sites  $k=1, \dots, n$ , equivalent to the fermionic Hamiltonian

$$H = \sum_{j,k=1}^n (\hat{a}_j^\dagger A_{j-k} \hat{a}_k + \hat{a}_j^\dagger B_{j-k} \hat{a}_k^\dagger - \hat{a}_j B_{j-k} \hat{a}_k). \quad (7)$$

The fermionic operators obey  $\{\hat{a}_j, \hat{a}_k\} = 0$  and  $\{\hat{a}_j^\dagger, \hat{a}_k\} = \delta_{j,k}$ . The Hamiltonians are related via a Jordan-Wigner transformation leading to the Hermitian Majorana operators,

$$\hat{m}_{2i-1} = \left( \prod_{j<i} \hat{\sigma}_j^3 \right) \hat{\sigma}_i^1, \quad \hat{m}_{2i} = \left( \prod_{j<i} \hat{\sigma}_j^3 \right) \hat{\sigma}_i^2, \quad (8)$$

where  $\hat{a}_j = (\hat{m}_{2j-1} - i\hat{m}_{2j})/2$ . Translational invariance, periodic boundary conditions, and Hermiticity are inherited by  $A_j, B_j \in \mathbb{R}$  satisfying  $A_j = A_{-j}$  and  $B_j = -B_{-j}$  for  $j=1-n, \dots, n-1$ . For simplicity, we assume that there exists a  $w \in \mathbb{N}$  such that  $A_j = B_j = 0$  for  $j > w$ . This model will be our starting point. For all isotropic instances, and also for the full  $XY$  model we will be able to identify when the single-copy entanglement is indeed logarithmically divergent. Then, one may distill a maximally entangled state of any dimension from a single specimen of the chain with certainty, containing in this sense an “infinite single-copy entanglement”.<sup>1</sup> We will make use of the powerful methods of Toeplitz determinants [9,10,19]. This path is in our instance complicated by the fact that we do not only consider isotropic models, and that in contrast to the block entropy the largest eigenvalue cannot straightforwardly be expressed as an integral of a Toeplitz determinant. The starting point, however, is the familiar one for assessing spin systems: The ground state of this system is a fermionic Gaussian, i.e., quasifree, state and is completely specified by the second moments of the Majorana operators. These operators satisfy  $\hat{m}_j = \hat{m}_j^\dagger$  and  $\{\hat{m}_j, \hat{m}_k\} = 2\delta_{j,k}$ . The second moments can be collected in a correlation matrix  $\gamma \in \mathbb{R}^{2n \times 2n}$ ,

$$\text{tr}[\rho \hat{m}_j \hat{m}_k] = \delta_{j,k} + i\gamma_{j,k}. \quad (9)$$

This matrix is skew symmetric. The entanglement properties of the block of length  $L$  can now be inferred from a principal submatrix  $\gamma_L \in \mathbb{R}^{2L \times 2L}$  of the correlation matrix  $\gamma$ . We consider the entries of  $\gamma_L$  in the limit of an infinite chain  $n \rightarrow \infty$ . Then,  $\gamma_L$  is a block Toeplitz matrix, the  $l$ th row,  $l=1, \dots, L$ , being given by  $(M_{l-1}, M_{l-2}, \dots, M_0, \dots, M_{l-L})$ , with  $2 \times 2$  blocks  $M_{L-1}, \dots, M_{1-L}$  that are found to be

<sup>1</sup>The notion of entanglement in systems with infinitely many degrees of freedom can also be grasped mathematically sharply in the language of von Neumann algebras [18].

$$M_l = \begin{bmatrix} 0 & t_l \\ -t_l & 0 \end{bmatrix}, \quad t_l = \frac{1}{2\pi} \int_0^{2\pi} g(k) e^{-ilk} dk. \quad (10)$$

For no anisotropy, i.e.,  $B_j=0$  for all  $j$ , this matrix is a tensor product of a symmetric matrix and a unit skew-symmetric one. In generality, we have for this model,  $g(k) := \Lambda(k)/|\Lambda(k)|$ ,  $\Lambda(k) := A_0 + 2\sum_{j=1}^w A_j \cos(jk) - 4i\sum_{j=1}^w B_j \sin(jk)$ . This matrix  $\gamma_L$  can be brought into a standard normal form  $\Gamma_L$  of a skew-symmetric matrix with an  $O \in O(2L)$  preserving the anticommutation relations,

$$\Gamma_L = O \gamma_L O^T, \quad \Gamma_L = \oplus_{l=1}^L \begin{bmatrix} 0 & \mu_l \\ -\mu_l & 0 \end{bmatrix}. \quad (11)$$

This defines the quantities  $\mu_1, \dots, \mu_L \in [0, 1]$ . Such normal mode decompositions have been employed to evaluate both correlation functions [19] and the block entropy [6,9,10]. From now on we will be concerned with the largest eigenvalue of the reduction of a block of length  $L$ . All eigenvalues  $\alpha_1^\pm, \dots, \alpha_{2L}^\pm$  of the reduction are given by  $\{\alpha_1^\pm, \dots, \alpha_{2L}^\pm\} = \{\Pi_{l=1}^L (1 \pm \mu_l)/2\}$ . We will be looking at the behavior of the largest eigenvalue  $\alpha_1^+$  for large  $L$ . This largest eigenvalue is given by  $\alpha_1^+ = \Pi_{l=1}^L (1/2 + \mu_l/2)$ , or

$$\alpha_1^+ = \det[(1_L + |T_L|)/2], \quad (12)$$

$|T_L| = (T_L^T T_L)^{1/2}$ , where  $T_L$  is the  $L \times L$  Toeplitz matrix, with the  $l$ th row being given by

$$(t_{-l+1}, t_{-l+2}, \dots, t_0, \dots, t_{L-l}). \quad (13)$$

The numbers  $\mu_1, \dots, \mu_L$  are the singular values of  $T_L$ . This matrix  $T_L$ , satisfying  $|T_L| \leq 1_L$ , is generally not symmetric, as a consequence of the anisotropy of the model. Moreover, in contrast to the matrix  $T_L$  itself,  $1_L + |T_L|$  is not Toeplitz. In order to show that the single-copy entanglement is logarithmically divergent, we will make use of appropriate bounds that retain this property: whenever the  $A_0, \dots, A_w, B_0, \dots, B_w$  are such that one can prove that the sequence of  $L \times L$  Toeplitz matrices  $T_L$  satisfies

$$-\log|\det[T_L]| = \Omega[\log(L)] \quad (14)$$

(using Landau notation<sup>2</sup>) using a Fisher-Hartwig statement [9,10,19], then one can indeed conclude that  $E_1 = \Omega[\log(L)]$ , i.e., the single-copy entanglement diverges at least logarithmically with increasing block length  $L$ . This follows from the following chain:<sup>3</sup>

$$\begin{aligned} -\log \det[(1_L + |T_L|)/2] &\geq -\frac{1}{2} \log \det[(1_L + T_L^T T_L)/2] \\ &\geq -\frac{1}{4} \log \det[T_L^T T_L] \\ &= -\frac{1}{2} \log|\det[T_L]|, \end{aligned} \quad (15)$$

<sup>2</sup>For functions of  $L$ ,  $f = O(g)$  that  $f \leq cg$  for some  $c > 0$  and all  $L$ ,  $f = o(g)$  that  $f/g \rightarrow 0$  for  $L \rightarrow \infty$ , and  $f = \Omega(g)$  that  $f \geq Cg$  for some  $C > 0$  and all  $L$ .

<sup>3</sup>From  $A \geq B \geq 0$  it follows that  $\det[A] \geq \det[B]$ .

where we also have made use of the concavity of the logarithm. So, whenever Eq. (14) holds, for appropriate length of the block  $L$ , a maximally entangled pair of any dimension can be distilled from a single specimen of the spin chain.

#### IV. ISOTROPIC MODELS

This case of  $B_0, \dots, B_w = 0$  is particularly transparent. Here, the asymptotics in  $L$  of the determinants  $\det[M_{x,L}]$  of the  $L \times L$  Toeplitz matrices

$$M_{x,L} := ix\mathbb{1} + (1-x^2)^{1/2} T_L \quad (16)$$

is known for all  $x \in (0, 1)$ , using a Fisher-Hartwig statement. This small detour to infer about the value of  $-\log|\det[T_L]|$ —corresponding to the case  $x=0$ —is needed as the Fisher-Hartwig conjecture has not been proven yet for this case. In general, one can identify the asymptotic behavior of determinants of Toeplitz matrices by investigating the so-called symbol.<sup>4</sup> The symbol associated with the Toeplitz matrices  $M_{x,L}$  is given by

$$G_x(k) = ix + (1-x^2)^{1/2} g(k), \quad (17)$$

with  $g$  as defined above. For this class of isotropic models, an explicit factorization of the symbol is known [10].<sup>5</sup> It follows hence from proven instances of the Fisher-Hartwig conjecture that there exists a  $c > 0$  and an  $x_0 \in (0, 1)$  such that (see footnote 2)

$$\log|\det[M_{x,L}]| = c_x \log(L) + o(\log(L)) \quad (21)$$

with  $c_x > c$  for all  $x \in (0, x_0)$ , whenever the function  $g$  is discontinuous in  $[0, 2\pi]$ , where the jumps reflect the Fermi surface. From this—and using that  $T_L$  has real eigenvalues—it follows that the system has a logarithmically divergent single-copy entanglement if the system is critical.<sup>6</sup> For example, for the  $XX$  model this analysis immediately delivers a logarithmically divergent single-copy entanglement, whenever the system is critical.

<sup>4</sup>For a square summable sequence  $\{s_k\}_k, s_k \in \mathbb{C}$ , the symbol  $G$  corresponding to a sequence of  $L \times L$  Toeplitz matrices with  $l$ th row  $(s_{l-1}, \dots, s_{L-l})$  is given by  $G(k) = \sum_{j=-\infty}^{\infty} e^{ijk} s_j$ .

<sup>5</sup>For this class of isotropic models, an explicit factorization of the symbol is given by

$$\begin{aligned} G_x(k) &= -(1-x^2)^{1/2} h(x) \prod_{r=1}^R (-1)^{r k / \pi} \\ &\quad \times \prod_{r=1}^R t_{(-1)^r f(x)}(k - k_r) t_{(-1)^r f(x)}(k + k_r), \end{aligned} \quad (18)$$

where

$$t_\beta(k) := \exp[-i\beta(\pi - k \bmod 2\pi)], \quad (19)$$

$$h(x) := [ix - (1-x^2)^{1/2}][ix + (1-x^2)^{1/2}], \quad (20)$$

and  $f(x) := (\log \circ h)(x)/(2\pi i)$ . The numbers  $k_r, r=1, \dots, R$ , are those  $k$  for which  $g$  is discontinuous.

<sup>6</sup>Here, we use critical in the sense that there is no gap between the energy of the ground and the first excited state.

**V. ANISOTROPIC XY MODEL**

For the *XY* model we can conclude that the single-copy entanglement is logarithmically divergent if and only if the system is critical. For this model, we have that  $A_0=-1, A_1=a/2$ , and  $B_{-1}=-B_1=\gamma a/4$ , and 0 elsewhere. For  $\gamma=0$ , we obtain the *XX* model (the isotropic *XY* model), for  $a=1, \gamma=1$ , the critical Ising model. Along the line  $\gamma \in [-1, 1], a=1$ , the anisotropic model is critical. Then, we encounter a generally nonsymmetric matrix  $T_L$ . The associated symbol is given by

$$g(k) = \frac{a \cos(k) - 1 + ia\gamma \sin(k)}{[a \cos(k) - 1]^2 + \gamma^2 a^2 \sin^2(k)}^{1/2}. \quad (22)$$

For  $\gamma \neq 0$  and  $1/a \in (0, 1)$ , the symbol is continuous, and one finds a saturating block entropy [9] (and hence a saturating single-copy entanglement). Along the critical line  $a=1, \gamma \in (-1, 1)$ , in turn, we can identify the explicit factorization of the discontinuous symbol. There is a single discontinuity at  $k_1=0$ <sup>7</sup>, and in the terms of footnote <sup>5</sup> we find  $\beta_1=1/2$ , so that  $g(k)$  can be decomposed as

$$g(k) = \phi(k)t_{1/2}(k), \quad (23)$$

where  $\phi$  is a continuously differentiable function. For the case of a single discontinuity and  $\alpha_1=0$ , the Fisher-Hartwig conjecture has been proven for any  $\beta_1 \in \mathbb{C}$  with  $\text{Re}(\beta_1) < 5/2$  [20], including our case at hand. Hence, we find

$$-\log|\det[T_L]| = \Omega[\log(L)], \quad (24)$$

and hence  $E_1=\Omega[\log(L)]$ . Together with the result of the subsequent section this shows that the single-copy entanglement of the *XY* model is logarithmically divergent exactly if the model is critical. Note that this implies also a less technical alternative proof of the logarithmic divergence of the block entropy in the critical *XY* model.

**VI. SCALING OF SINGLE-COPY ENTANGLEMENT IN THE XX MODEL**

In the light of these findings, it is interesting to see how the exact asymptotic behavior is compared to that of the block entropy, including prefactors. We make this specific for the isotropic *XX* model, where now  $T_L=T_L^T$ . The technicality when evaluating  $\alpha_1^\dagger=\det[(1_L+|T_L|)/2]$  that we encounter here is that the function  $f:\mathbb{C}\rightarrow\mathbb{C}$ ,

$$f(x) := \log_2[(1+|x|)/2], \quad (25)$$

is not analytic. So before we can exploit Fisher-Hartwig-type results, we have to approximate  $\alpha_1^\dagger$  with sequences based on functions with appropriate continuity properties. We can take any functions  $f_*:\mathbb{C}\times\mathbb{R}_+\rightarrow\mathbb{C}$  which are analytic on  $\{z \in \mathbb{C}:\text{Im}(z)<\delta\}$  for a  $\delta>0$ , such that on the real axis  $\lim_{\delta\searrow 0} f_*(x, \delta)=f(x, 0)$  for  $x \in \mathbb{R}$ . Take, e.g.,

$\delta\searrow 0$

<sup>7</sup>Here,  $k_1=0$ , which is at the boundary of the domain of  $g$ . Then the integral has to be understood as in the range from  $-\varepsilon, 2\pi-\varepsilon, \varepsilon>0$ ; see Refs. [20].

$$f_*(z, \delta) := \log[1/2 + (z^2 + \delta^2)^{1/2}/2]. \quad (26)$$

We are then in the position to identify the asymptotic behavior of the single-copy entanglement. This can be done similarly to Ref. [9] using the characteristic polynomial  $F:\mathbb{C}\rightarrow\mathbb{C}$  of  $T_L$  defined as  $F(\lambda):=\det[\lambda 1_L-T_L]$ : the function  $F$  is meromorphic, and all zeros are in the interval  $[-1, 1]$ . One can hence write

$$d_* = \lim_{\delta\searrow 0} \lim_{\varepsilon\searrow 0} \frac{1}{2\pi i} \int dz f_*(z, \delta) \frac{F'(z)}{F(z)} \quad (27)$$

where the integration path is chosen to enclose the interval  $[-1, 1]$ , with path from  $(-1-\delta+i\varepsilon, 1+\delta+i\varepsilon)$ , toward the negative real numbers along a circle segment with radius  $\delta/2$ , then  $(1+\delta-i\varepsilon, -1-\delta-i\varepsilon)$ , and again along a circle segment to  $-1-\delta+i\varepsilon$ , such that  $\lim_{\delta\searrow 0} d_*=d$ . The symbol of  $\lambda 1 - T_L$  with factorization as in Eq. (18) for the *XX* model is known [9]; see footnote 5. Using a Fisher-Hartwig statement, we find that the linear terms in  $L$  do not contribute, using Cauchy's theorem and using that  $\lim_{\delta\searrow 0} f_*(\pm 1, \delta)=0$ , and finally arrive at<sup>8</sup>

$$d = \log(L) - \frac{2}{\pi^2} \int_{-1}^1 \frac{\log_2[(1+|x|)/2]}{1-x^2} dx + o(\log(L)). \quad (28)$$

This in turn finally implies that whenever  $1/a \in [-1, 1]$  and the *XX* model is critical, we observe the scaling behavior

$$E_1 = \frac{1}{6} \log_2(L) + o(\log(L)), \quad (29)$$

independent of  $a$ ; it saturates in the noncritical case. This result is astonishing: the single-copy entanglement not only diverges, but has up to a factor of 2 the same leading asymptotic behavior as the block entropy scaling as  $S=(1/3)\log_2(L)+o(\log(L))$ . Half of the asymptotically distillable entanglement is hence already available on the single-shot level. In turn, this relationship between the largest eigenvalue on the one hand and the Shannon entropy of the full spectrum on the other hand appears to be an intriguing property of the spectra of the reductions. Note that this argument can be applied in the same manner to all those spin chains that can be mapped onto isotropic fermionic models as considered in Sec. IV. This quantitative connection will be explored in more detail elsewhere.

**VII. OUTLOOK AND SUMMARY**

Finally, let us comment on the crucial role of SSRs for the single-copy entanglement. This is relevant, e.g., when assessing the single-copy entanglement in the hard-core limit of the Bose-Hubbard model (infinite repulsion energy) [1,21]. There, the Hamiltonian is isomorphic to the *XX* model, via the mapping  $\hat{\sigma}_j^1=\hat{b}_j+\hat{b}_j^\dagger, \hat{\sigma}_j^2=-i(\hat{b}_j-\hat{b}_j^\dagger)$ , and  $\hat{\sigma}_j^3=1-2\hat{b}_j^\dagger\hat{b}_j$  for each site  $j$ . Yet, the concept of entanglement is different due to the presence of a particle number conservation SSR in the

<sup>8</sup>Here, for  $x=|x|e^{i\varphi} \in \mathbb{C}\setminus\{z \in \mathbb{C}:\text{Im}(z)=0, \text{Re}(z)<0\}$  we take  $\log(x)=\log|x|+i\varphi$ .

former case: Transformations under LOCC have to be replaced by those under LOCC+SSR [21]. The single-copy entanglement in the above sense can, however, still be efficiently evaluated in  $L$ ; and these superselection rules must be respected when assessing single-copy entanglement.

In this paper, we have fleshed out the notion of single-copy entanglement in quantum spin chains. Such a notion is the appropriate one when one is not interested in the entanglement properties of an asymptotic supply of identically prepared many-body systems, but those of single specimens.

It is the hope that these findings also serve as a guideline when assessing entanglement in actual experimental situations, whether in condensed-matter systems or in systems of ultracold atoms in optical lattices.

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