g factor of an electron or muon bound by an arbitrary central potential

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We consider the *g* factor of a spin-1/2 particle (electron or muon) bound by an arbitrary central field. We present an approach that allows one to express the relativistic *g* factor in terms of the binding energy. We derive the general expression for the correction to the *g* factor caused by a deviation of the central potential from the Coulomb one. As the application of this method, we consider the corrections to the *g* factor due to the finite nuclear size, including vacuum polarization radiative correction. The effect of the anomalous magnetic moment is also taken into account.

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I. INTRODUCTION

The study of the energy levels of the atomic electron (muon) has a long history. Various relativistic and radiative corrections have been obtained in analytic or semianalytic form, taking into account a deviation of the potential from the Coulomb one. Recently interest in an accurate theory of the *g* factor increased. An accurate theoretical approach needs to take into account relativistic and radiative corrections. Here we consider a case when such corrections correspond to some modification of a central potential. Such a problem is actual because of the Uehling potential, which is responsible for a dominant QED correction in muonic atoms, finite-nuclear-size effects, and some others. The results are presented in a fully relativistic approach, i.e., exact in $Z\alpha$ for the Coulomb interaction and its modifications.

The methods of calculation of energy levels have been successfully developed over the decades. The energy levels can be calculated with high accuracy by solving certain equations analytically or numerically, or using the perturbation theory with respect to the corresponding correction to the Hamiltonian. Calculation of the *g* factor is a more complicated problem. In the traditional approach to the calculation of the correction to the *g* factor within perturbation theory (see, e.g., [1]), one considers both the magnetic field and δV as a perturbation, while at the calculation of the correction to the energy levels it is necessary to consider only δV as a perturbation. The numerical approaches are also not easy to apply because they usually are much more accurate in determination of the energy than the wave function. However, it is the latter that is needed to calculate the *g* factor.

In this paper, we develop a framework that allows one to express the *g* factor of a Dirac particle bound in arbitrary central potential via the binding energy. Within this approach it is easy to obtain new corrections to the *g* factor using

numerous results obtained for the corrections to the energy levels. For a small deviation of this potential from the Coulomb one, we derive the general expression for the first correction to the *g* factor due to this deviation. The results are valid both for muonic and electronic atoms.

II. GENERAL RELATIONS

In the weak homogenous magnetic field $\boldsymbol{B}$, the correction to energy levels of a Dirac particle bound by a central potential reads

$$\Delta E = e \int d^3 r \bar{\Psi}(\boldsymbol{r})(\boldsymbol{\gamma} \cdot \mathbf{A})\Psi(\boldsymbol{r}) = \frac{e\mathbf{B} \cdot \langle \mathbf{J} \rangle}{2m}g, \quad (1)$$

where

$$\mathbf{A} = \frac{1}{2}[\boldsymbol{B} \times \mathbf{r}]. \quad (2)$$

The relativistic units in which $\hbar = c = 1$ are applied throughout the paper; the charge of the bound particle is $-e$, e is a charge of a proton, $e^2 = \alpha = 1/137$ is the fine structure constant.

Writing the Dirac wave function $\psi(\boldsymbol{r})$ in the form

$$\psi(\boldsymbol{r}) = \begin{pmatrix} f_1(r)\Omega \\ if_2(r)\widetilde{\Omega} \end{pmatrix}, \quad (3)$$

where Ω is the spherical spinor [2] with the angular momentum *J* and orbital momentum *L*, $\widetilde{\Omega} = -(\boldsymbol{\sigma} \cdot \boldsymbol{n})\Omega$, we obtain for the *g* factor of a bound Dirac particle

$$g = \frac{2m\kappa}{j(j+1)} \int dr r^3 f_1(r)f_2(r), \quad (4)$$

$\kappa = j + 1/2$ for $l > j$ and $\kappa = -(j+1/2)$ for $l < j$. The integral in the right-hand side of Eq. (4) can be presented as follows (see, e.g., [3])

$$\int dr r^3 f_1 f_2 = -\frac{1}{4m}\left[1 - 2\kappa \int dr r^2 (f_1^2 - f_2^2)\right]. \quad (5)$$

This formula is valid for the Dirac equation with any central potential. Equation (4) has been known for a while (see,

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e.g., [3]), but, to the best of our knowledge, it was never applied to a non-Coulomb problems.

Using the identity

$$\int dr r^2 (f_1^2 - f_2^2) = \langle \gamma_0 \rangle, \qquad (6)$$

we have

$$g = -\frac{\kappa[1 - 2\kappa\langle \gamma_0 \rangle]}{2j(j+1)}. \qquad (7)$$

Assuming that the potential $V(r)$ is independent of the mass of the bound particle, we find

$$\langle \gamma_0 \rangle = \left\langle \frac{\partial H}{\partial m} \right\rangle = \frac{\partial E}{\partial m}, \qquad (8)$$

where H is the Dirac Hamiltonian $H = \gamma_0(\boldsymbol{\gamma} \cdot \boldsymbol{p}) + \gamma_0 m + V(r)$.

Thus we arrive at an equation that expresses the g factor of the state via its binding energy,

$$g = -\frac{\kappa}{2j(j+1)}\left[1 - 2\kappa\frac{\partial E}{\partial m}\right]. \qquad (9)$$

This equation can be used even in the case when a potential substantially differs from the Coulomb one, as for heavy muonic atoms. In many cases, Eq. (9) essentially simplifies calculation of the corrections to the g factor of a Dirac particle. In particular, it can be applied to such problems as the finite-nuclear-size effect and vacuum polarization.

The equation (9) becomes essentially simpler if the potential is close to the Coulomb one, and the deviation $\delta V(r) = V(r) - V_C(r)$ can be treated as a perturbation. For the pure Coulomb case, when $\partial E_C/\partial m = E_C/m$, we immediately obtain the well-known result (see, e.g., [4,5])

$$\Delta g_C = -\frac{\kappa}{2j(j+1)}\left[1 - 2\kappa\frac{\gamma + n_r}{N}\right], \qquad (10)$$

where

$$N = \sqrt{(\gamma + n_r)^2 + (Z\alpha)^2}, \quad \gamma = \sqrt{\kappa^2 - (Z\alpha)^2},$$

and n_r is the radial quantum number. The correction to the energy is

$$\delta E = \int d^3 r \Psi^+(r) \delta V(r) \Psi(r). \qquad (11)$$

As follows from the dimensional reasons, the wave function in the Coulomb field can be presented in the form $\Psi(r) = m^{3/2}\widetilde{\Psi}(mr)$, where $\widetilde{\Psi}$ is dimensionless. Passing to the variable $\rho = mr$, we find

$$\delta E = \int d^3 \rho \widetilde{\Psi}^+(\rho) \delta V(\rho/m) \widetilde{\Psi}(\rho). \qquad (12)$$

Taking the derivative over m and returning to the variable r, we obtain

$$\frac{\partial \delta E}{\partial m} = -\frac{1}{m}\int d^3 r \Psi^+(r) r \frac{\partial \delta V(r)}{\partial r} \Psi(r), \qquad (13)$$

and

$$\delta g = -\frac{\kappa}{2j(j+1)}\left[1 + \frac{2\kappa}{m}\left\langle r\frac{\partial \delta V(r)}{\partial r}\right\rangle\right]. \qquad (14)$$

Equations (9) and (14) are the basis of our approach.

III. ANOMALOUS MAGNETIC MOMENT

It may be interesting to generalize our results to the case of a particle with a nonvanishing anomalous magnetic moment. In particular, it is necessary for antiprotonic atoms. The anomalous magnetic moment of the antiproton is big because of the complicated internal structure of the particle. Therefore, it makes sense to consider the modification of Eqs. (9) and (14) due to the anomalous magnetic moment separately from the radiative corrections.

As known, the modified Dirac Hamiltonian for a particle with the anomalous magnetic moment in an arbitrary electromagnetic field is of the form (see, e.g., [2])

$$H = \gamma_0[\boldsymbol{\gamma} \cdot (p + eA)] + \gamma_0 m - eA_0 + \frac{i\mu_a}{2}\gamma_0\sigma_{\mu\nu}F^{\mu\nu}, \qquad (15)$$

where $\sigma_{\mu\nu} = (1/2)[\gamma_\mu, \gamma_\nu]$, μ_a is the dimensional anomalous magnetic moment and $F^{\mu\nu}$ is the tensor of the electromagnetic field; we are reminded that the charge of the bound particle is $(-e)$.

As follows from Eq. (15), the correction to the energy of a bound particle in the homogenous magnetic field (2) is of the form [cf. Eq. (1)],

$$\Delta E = \int d^3 r \bar{\Psi}(r)\{e(\boldsymbol{\gamma} \cdot \mathbf{A}) - \mu_a(\boldsymbol{\Sigma} \cdot \mathbf{B})\}\Psi(r), \qquad (16)$$

where $\Sigma_j = (i/2)\epsilon_{jkl}\sigma_{kl}$ is the spin operator. The wave function $\Psi(r)$ is an eigenfunction of the Hamiltonian (15) with $-eA_0(r) = V(r)$ and $A_i = 0$. Since the potential $A_0(r)$ is spherically symmetric, the wave function still has the form (3), and the result for the g factor reads [cf. Eq. (4)],

$$g = \frac{2m}{j(j+1)}\left\{\kappa\int dr r^3 f_1 f_2 - \frac{\mu_a}{2e}\left[\int dr r^2(f_1^2 - f_2^2) - 2\kappa\right]\right\}. \qquad (17)$$

The radial wave functions f_1 and f_2 satisfy the system of equations (cf. [2]),

$$f_2' + (E - V - m)f_1 + \left(\frac{1-\kappa}{r} - \frac{\mu_a}{e}V'\right)f_2 = 0,$$

$$f_1' - (E - V + m)f_2 + \left(\frac{1+\kappa}{r} + \frac{\mu_a}{e}V'\right)f_1 = 0. \qquad (18)$$

Then we multiply the first equation by $r^3 f_2$ and the second one by $r^3 f_1$, sum up the results, and take the integral over r.

Integrating by parts the terms with the derivatives, we arrive at the following relation [cf. (5)]:

$$\int dr r^3 f_1 f_2 = -\frac{1}{4m}\left[1 - 2\kappa \int dr r^2 (f_1^2 - f_2^2)\right.$$
$$\left. -\frac{2\mu_a}{e}\int dr r^3 V'(f_1^2 - f_2^2)\right]. \qquad (19)$$

Using this formula and Eqs. (6) and (8), we present the result for the g factor in the form

$$g = \frac{1}{2j(j+1)}\left\{-\kappa(1+2a) + (2\kappa^2 + a)\frac{\partial E}{\partial m}\right.$$
$$\left. -a\frac{\kappa}{m}\int dr r^3 V'(f_1^2 - f_2^2)\right\}, \qquad (20)$$

where $a = -2m\mu_a/e$ is the dimensionless anomalous magnetic moment [for a free particle $g = 2(1+a)$]. This result is obtained for an arbitrary potential A_0 and is exact in the parameter a. We emphasize that the radial wave functions f_1 and f_2 and the binding energy E depend on the anomalous magnetic moment because of the equations (18).

In the nonrelativistic approximation, Eq. (20) turns into

$$g = \frac{1}{2j(j+1)}\left\{(1-2\kappa)(a-\kappa) - (2\kappa^2 + 2\kappa a + a)\right.$$
$$\left. \times \frac{1}{2m}\int dr r^3 V' f_{nr}^2\right\}, \qquad (21)$$

where f_{nr} is the radial part of the nonrelativistic wave function. This formula is valid even if $a \sim 1$.

It is interesting also to consider an expansion of the g factor (20) in the parameter a for arbitrary field strength (when the nonrelativistic approximation is not valid). The linear in this parameter term reads

$$\delta g_a = \frac{a}{2j(j+1)}\left\{-2\kappa + \frac{\partial E}{\partial m} - \frac{2\kappa^2}{m}\frac{\partial}{\partial m}\int dr r^2 V' f_1 f_2\right.$$
$$\left. -\frac{\kappa}{m}\int dr r^3 V'(f_1^2 - f_2^2)\right\}. \qquad (22)$$

Here we used the formula for the linear in μ_a correction to the energy,

$$\delta E_a = \frac{2\mu_a}{e}\int dr r^2 V' f_1 f_2, \qquad (23)$$

and took into account that $\partial \mu_a/\partial m = 0$. Let us consider Eq. (22) for the pure Coulomb potential. Strictly speaking, the equations (18) have no sense for a pure Coulomb field because of the terms $\propto 1/r^2$, which lead to the phenomenon of falling to the center. It is a consequence of the pointlike source of the field and is absent if finite nuclear size is taken into account. However, for large quantum numbers that are mostly interesting for the experiments, the correction due to

the finite nuclear size does not change essentially the result obtained from Eq. (22) for a pure Coulomb field. Using the radial matrix element from Ref. [5], we obtain

$$\delta g_a = \frac{a}{2j(j+1)}\left\{-2\kappa + \frac{\gamma + n_r}{N} - \frac{\kappa(Z\alpha)^2}{N^2}\right.$$
$$\left. -\frac{4\kappa^2(Z\alpha)^4[2\kappa(\gamma + n_r) - N]}{\gamma(4\gamma^2 - 1)N^4}\right\}, \qquad (24)$$

where N and γ are defined after Eq. (10). We see that the correction to the g factor due to the anomalous magnetic moment has essentially more complicated dependence on quantum numbers than the leading term (10).

IV. FINITE-NUCLEAR-SIZE EFFECT

As an illustration of efficiency of our method, let us consider a finite-nuclear-size correction to the bound-electron g factor. For a $1s$ state, this problem was solved analitically in Ref. [6] in the nonrelativistic approximation ($Z\alpha \ll 1$), and numerically in Ref. [7] for an arbitrary value of the parameter $Z\alpha$. In Ref. [8], the result was obtained analytically in the next-to-leading approximation in the parameter $Z\alpha$, and for arbitrary states. The result of Ref. [8] allows one to describe well the correction to the g factor up to $Z=20$. On the other hand, in Ref. [9] the approximate formulas for the finite-nuclear-size correction to the energy levels, E_{fns}, were obtained, which give the result with a relative error of about 0.2% up to $Z=100$.

$$E_{fns}(ns_{1/2}) = m\frac{(Z\alpha)^2}{10n}[1 + (Z\alpha)^2 f_{ns_{1/2}}](2Z\alpha mR/n)^{2\gamma},$$

$$E_{fns}(np_{1/2}) = m\frac{(Z\alpha)^4(n^2-1)}{40n^3}[1 + (Z\alpha)^2 f_{np_{1/2}}](2Z\alpha mR/n)^{2\gamma},$$
$$(25)$$

where n is the principal quantum number, $n = 1, 2, \ldots$; $R \approx r_0$ is some effective radius (for details see Ref. [9]); and coefficients $f_i = f_i(Z\alpha)$ are

$$f_{1s_{1/2}}(Z\alpha) = 1.380 - 0.162(Z\alpha) + 1.612(Z\alpha)^2,$$

$$f_{2s_{1/2}}(Z\alpha) = 1.508 + 0.215(Z\alpha) + 1.332(Z\alpha)^2,$$

$$f_{2p_{1/2}}(Z\alpha) = 1.615 + 4.319(Z\alpha) - 9.152(Z\alpha)^2 + 11.87(Z\alpha)^3.$$
$$(26)$$

Using Eq. (9) and Eq. (25), we obtain

$$\delta g_{fns} = \frac{\kappa^2}{j(j+1)}\frac{\partial E_{fns}}{\partial m} = \frac{\kappa^2(2\gamma+1)}{j(j+1)}\frac{E_{fns}}{m}. \qquad (27)$$

Moreover, using our method we can calculate the radiative correction to δg_{fns} for the $s_{1/2}$ state and the $p_{1/2}$ state (and arbitrary n_r) of electrons in a hydrogenlike atom coming from the effect of vacuum polarization. For these states, the finite-nuclear-size effect is the most significant. The radiative

correction δE_{fns} to the energy E_{fns} was considered in detail in Ref. [10]. The correction due to the vacuum polarization δE_{fns}^{VP} can be represented in the form

$$\delta E_{fns}^{VP} = E_{fns}\Delta^{VP}, \tag{28}$$

where E_{fns} was obtained in Ref. [9], and the explicit analytical form of Δ^{VP} for the $s_{1/2}$ state and the $p_{1/2}$ state is derived in Ref. [10]. Equation (28) with E_{fns} from Ref. [9] and Δ^{VP} from Ref. [10] provides the high accuracy of δE_{fns}^{VP} up to $Z = 100$. However, the correction Δ^{VP} was calculated in [10] for the mass of the bound particle m being equal to the mass M of the particle in the fermion loop. Therefore, in order to use Eq. (9) we should know the explicit dependence of Δ^{VP} on m/M (we can set $M=m$ only after the differentiation over m). The corresponding expression for Δ^{VP} can be easily obtained following the derivation of in [10]. For $Z\alpha \lesssim 1$, the main contribution to Δ^{VP} is given by the logarithmically enhanced term

$$\Delta^{VP} \approx \frac{2\alpha(Z\alpha)^2}{3\pi\gamma_1}\ln^2(mR), \tag{29}$$

where $\gamma_1 = \sqrt{1-(Z\alpha)^2}$, and R is the nuclear radius. We emphasize that the coefficient in front of the logarithm was obtained in [10] exactly in $Z\alpha$ but not in the perturbation theory. Note that the parameter $Z\alpha$ should also satisfy the condition $1-Z\alpha \gg 1/\ln^2(mR)$, which is fulfilled for all existing atoms. This term is the same for the $s_{1/2}$ and $p_{1/2}$ states and comes from the distances $1/m, 1/M \gg r \gg R$. Therefore, it contains only the logarithmic dependence on the mass of the bound particle, in contrast to the powerlike dependence of E_{fns}. Within the logarithmic accuracy, we have

$$\frac{\partial}{\partial m}\delta E_{fns}^{VP} \approx \Delta^{VP}\frac{\partial}{\partial m}E_{fns} = (2\gamma_1 + 1)\frac{\delta E_{fns}^{VP}}{m}. \tag{30}$$

Thus, for $Z\alpha \lesssim 1$, we obtain

$$\delta g_{fns}^{VP} = \frac{4}{3}(2\gamma_1 + 1)\frac{\delta E_{fns}^{VP}}{m}. \tag{31}$$

For $Z\alpha \ll 1$ the dependence of Δ^{VP} on m/M is more complicated and will be considered elsewhere.

V. CONCLUSION

We present an effective approach to obtain various corrections to the bound particle g factor using the corresponding corrections to the energy levels. It should be noted that this method is appropriate for the perturbations of potentiallike type. There are also corrections to the g factor that cannot be derived via this approach. The leading part of the nonpotential contribution is due to the virtual Delbrück scattering and it was calculated in Ref. [11]. On the other hand, there are corrections to the energy levels coming from the mass operator and vertex, which are not of potentiallike type. Thus they cannot be directly transformed to the corresponding corrections to the g factor.

As other applications of our method, we can suggest the Uehling potential contribution to the g factor of bound electrons and muons. In muonic atoms it is one of the dominant effects, while in electronic hydrogenlike atoms the accuracy is essentially higher (see, e.g., [12]).

For muonic atoms, the Uehling-potential correction to the energy levels was calculated in the nonrelativistic approximation for certain levels in [13]. The exact result in $Z\alpha$ was obtained in [1] for the ground state of hydrogenlike atoms and agrees with Eq. (9). The methods developed there can be easily applied to arbitrary levels and thus, using Eq. (9), one can find the correction to the g factor due to the Uehling potential.

Using the known corrections to the energy levels coming from the high-order terms of vacuum polarization, we can obtain the corresponding contribution to the g factor. One correction comes from the Wichmann-Kroll potential (which accounts for the higher-order in $Z\alpha$ terms in the induced charge density), see [14]. Others are the second-order correction with respect to the Uehling potential, and the first-order correction with respect to the radiative correction to the Uehling potential (of the order of α).

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