Using weak nonlinearity under decoherence for macroscopic entanglement generation and quantum computation

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Recently, there have been several suggestions that weak Kerr nonlinearity can be used for generation of macroscopic superpositions and entanglement and for linear optics quantum computation. However, it is not immediately clear that this approach can overcome decoherence effects. Our numerical study shows that nonlinearity of weak strength could be useful for macroscopic entanglement generation and quantum gate operations in the presence of decoherence. We suggest specific values for real experiments based on our analysis. Our discussion shows that the generation of macroscopic entanglement using this approach is within the reach of current technology.

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Strong nonlinear effects, if available, could be very useful for the generation of macroscopic superpositions and entanglement and for quantum information processing. For example, it is well known that macroscopic superpositions and entanglement could be generated in strong Kerr nonlinear media [1]. An optical quantum computer could be realized if strong nonlinearity was available [2]. However, it is extremely hard to obtain such strong nonlinear effects using nonlinear media. Nonlinear effects in existing media are extremely weak compared with the required level for the generation of macroscopic superpositions and entanglement or for quantum information processing.

Recently, it was suggested that weak Kerr nonlinearity can still be used for the production of macroscopic superpositions and entanglement [3,4] and for linear optics quantum computation [5]. Nemoto and Munro showed that weak nonlinearity could be used to generate entangled states and used for linear optics quantum computation [5], which could also be used for the generation of macroscopic superposition [6,7]. A key element of their scheme is a cross-Kerr interaction between a coherent state and a single-photon qubit. The phase of the initial coherent state changes by a certain amount when the qubit is $|1\rangle$, while it remains the same when the qubit is $|0\rangle$. An important point of this scheme is that the amplitude of the initial coherent state should become arbitrarily large to make the required nonlinear strength arbitrarily weak. The initial coherent state can easily gain an adequate amount of phase shift by $|1\rangle$ when α becomes very large. This effect could be used to generate quantum entanglement with a very weak nonlinearity. The other schemes based on the single-mode Kerr effect by van Enk [3] and Jeong *et al.* [4] use implicitly the same principle of using a coherent state of a large amplitude: all these schemes [3–5] use initial coherent states of large amplitudes in order to make the required separation between the component coherent states in the phase space with weak nonlinearity [8].

However, decoherence effects *during* the entanglement generation process in nonlinear media have not been investigated in these references [3–5] in spite of the uncertainty of the validity of such techniques under a real dissipative environment. As was mentioned, the amplitude of the initial coherent state must increase at the cost of decreasing the required nonlinear strength (or required interaction time in a nonlinear medium). The requirement of a large initial amplitude might reduce the coherence time of the evolving state; i.e., an entangled system with a large initial amplitude might lose its quantum coherence more rapidly than an entangled system with a small initial amplitude did. It is thus unclear whether the decoherence effects (particularly dephasing) could be overcome by this approach when generating entanglement and operating quantum gates. In this Brief Report, to answer this question, we investigate the effect of decoherence when Nemoto and Munro's idea [5] is applied to Gerry's scheme [6] to generate a macroscopic superposition (so-called Schrödinger cat state) in a dissipative environment.

The interaction Hamiltonian of cross-Kerr nonlinearity between modes 1 and 2 is $H_K=\hbar\chi a_1^\dagger a_1 a_2^\dagger a_2$, where $a(a^\dagger)$ represents the annihilation (creation) operator. The interaction between a coherent state $|\alpha\rangle_2$ and a single-photon qubit—e.g., $|\psi\rangle_1=(|0\rangle_1+|1\rangle_1)/\sqrt{2}$—is described as

$$U_K(t)|\psi\rangle_1|\alpha\rangle_2 = e^{iH_K t/\hbar}\frac{1}{\sqrt{2}}(|0\rangle_1+|1\rangle_1)|\alpha\rangle_2 = \frac{1}{\sqrt{2}}(|0\rangle_1|\alpha\rangle_2+|1\rangle_1|\alpha e^{i\theta}\rangle_2), \quad (1)$$

where $|0\rangle$ ($|1\rangle$) is the vacuum (single-photon) state, α is the amplitude of the coherent state, and $\theta=\chi t$ with the interaction time t. If θ is π and one measures out mode 1 on a superposed basis $(|0\rangle\pm|1\rangle)/\sqrt{2}$, a macroscopic superposition state $|\Phi_\pm\rangle=(|\alpha\rangle\pm|-\alpha\rangle)/\sqrt{M_\pm}$ is created, where $M_\pm=2\pm2\exp[-2|\alpha|^2]$. A macroscopic entanglement can be simply generated at a beam splitter with such a state.

Using the dual-rail logic, where the logical qubit basis is defined as $|0_L\rangle\equiv|1\rangle\otimes|0\rangle$ and $|1_L\rangle\equiv|0\rangle\otimes|1\rangle$, the above process can be efficiently realized. For example, as shown in Fig. 1, Gerry's scheme [6] can be directly linked to Nemoto and Munro's idea [5] so that weak cross-Kerr nonlinearity can be used with a single photon, a coherent state, two photodetectors, and two beam splitters to generate a macroscopic

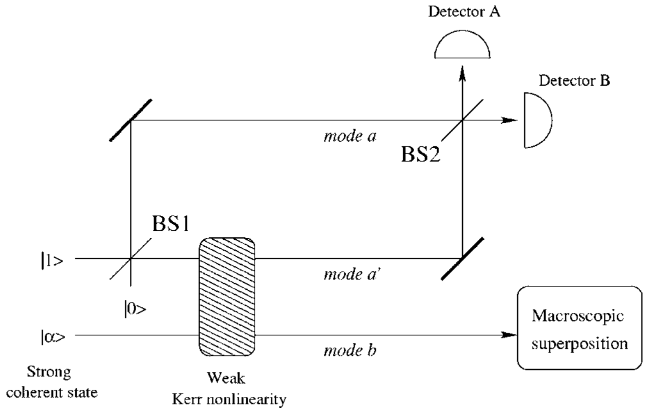

FIG. 1. A schematic of Gerry's scheme [6] combined with Nemoto and Munro's idea [5] using weak Kerr nonlinearity.

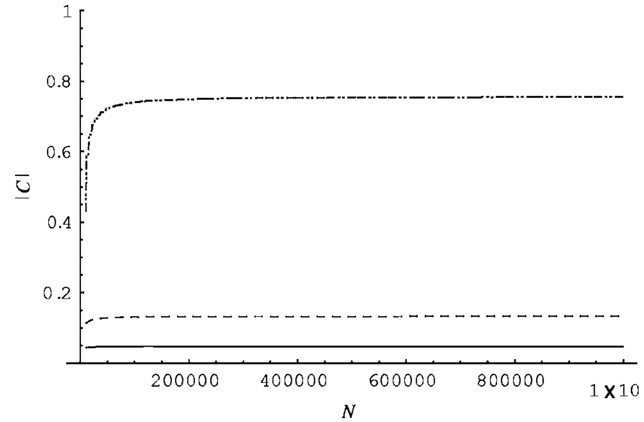

FIG. 2. This figure shows that our approach gives a very good approximation throughout the whole range of α that we consider in this paper. The coherence parameter $|C|$ for a separation $|\alpha e^{i\chi t}-\alpha| = 2\alpha_0 = 6$ has been plotted for $\alpha = 300$ (solid line), $\alpha = 1000$ (dashed line), and $\alpha = 10000$ (double-dot-dashed line).

superposition. If detector A (detector B) clicks, a macroscopic superposition state $|\Phi_-\rangle(|\Phi_+\rangle)$ is obtained at mode b in Fig. 1. Remarkably, it is clear from Fig. 1 that this approach is robust against the inefficiency of the single-photon source, loss of the single photon, and inefficiency of the photodetectors. Those factors will cause the photodetectors to be silent, and such cases can simply be discarded. Therefore, these will only make the deterministic property of the scheme nondeterministic but will not affect the quality of the obtained macroscopic superposition state.

The main problem here is that a very large nonlinear effect—i.e., a very large θ—is required to gain a large separation between two coherent component states. It was pointed out that an optical fiber of about 3000 km is required for $\theta = \pi$ for an optical frequency of $\omega \approx 5 \times 10^{14}$ rad/sec using currently available Kerr nonlinearity [9]. In such a case, the state after the nonlinear interaction will be completely decohered because of the significant losses in the fiber. In order to circumvent this problem, a large initial amplitude α can be used with a short interaction time t. If α is very large, the same amount of separation can be obtained in the phase space even though $\theta(=\chi t)$ is much smaller than π.

Now we consider the decoherence effects in the Kerr medium. The decoherence effects can be induced by solving the master equation [12]

$$\frac{\partial \rho}{\partial t} = \hat{J}\rho + \hat{L}\rho; \; \hat{J}\rho = \gamma a \rho a^\dagger, \quad \hat{L}\rho = -\frac{\gamma}{2}(a^\dagger a \rho + \rho a^\dagger a), \tag{2}$$

where γ is the energy decay rate. The formal solution of the master equation (2) can be written as $\rho(t) = \exp[(\hat{J}+\hat{L})t]\rho(0)$, which leads to the solution for the initial element $|\alpha\rangle\langle\beta|$:

$$\begin{aligned}\exp[(\hat{J}+\hat{L})t]|\alpha\rangle\langle\beta| &\equiv \tilde{\mathcal{D}}(t)|\alpha\rangle\langle\beta| \\ &= \exp\left[-\frac{1}{2}(1-e^{-\gamma t})(|\alpha|^2+|\beta|^2)+\alpha\beta^*\right] \\ &\quad \times |A\alpha\rangle\langle A\beta|, \end{aligned} \tag{3}$$

where $A = e^{-\gamma t/2}$.

We numerically assess the decoherence effects as follows. As we have explained, a photon loss at mode 1 (modes a and a' in Fig. 1) will simply cause the success probability to be less than 1. However, energy loss of the coherent state part (mode b in Fig. 1) in the nonlinear medium should be seriously considered since it will cause decoherence (dephasing) of the obtained state. Note that the average energy loss per time increases as the initial energy gets larger. This will cause a more rapid destruction of quantum coherence for a large α. The decoherence process $(\tilde{\mathcal{D}})$ will occur simultaneously with the unitary evolution by the Kerr effect $(\tilde{\mathcal{U}})$ of the input state by the interaction Hamiltonian H_K. This process may be modeled as follows. One may assume that $\tilde{\mathcal{U}}$ occurs for a short time Δt, and then $\tilde{\mathcal{D}}$ occurs for another Δt. In other words, $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{D}}$ continuously take turn for a short time in the nonlinear medium. By taking Δt arbitrarily small, one can obtain a good approximation of this process for a certain time $t(=N\Delta t)$, where N is an integer number. Let us set $\Delta\theta = \chi\Delta t = \pi/N$; then, a larger N will result in a better approximation.

We now use our model to analyze the behavior of the coherent state interacting with a logical qubit in the Kerr medium shown in Fig. 1. The first beam splitter BS1 and the single photon prepares the logical qubit state $(|0_L\rangle + |1_L\rangle)/\sqrt{2} \equiv (|1\rangle_a|0\rangle_{a'} + |0\rangle_a|1\rangle_{a'})/\sqrt{2}$. The total initial state after BS1 but before the Kerr interaction in Fig. 1 is

$$\rho(t=0) = \frac{1}{2}(|0_L\rangle\langle 0_L| + |0_L\rangle\langle 1_L| + |1_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|) \otimes |\alpha\rangle\langle\alpha|.$$

Let us fist consider the evolution of the second term $|0_L\rangle\langle 1_L| \otimes |\alpha\rangle\langle\alpha|$. After time $t(=N\Delta t)$ in the nonlinear medium, it evolves to

$$\{\tilde{\mathcal{D}}(\Delta t)\tilde{\mathcal{U}}(\Delta t)\}^N |0_L\rangle\langle 1_L| \otimes |\alpha\rangle\langle\alpha| = C|0_L\rangle\langle 1_L| \otimes |A\alpha\rangle\langle A\alpha e^{i\theta}|, \tag{4}$$

where $\tilde{\mathcal{U}}(\Delta t)\rho \equiv U_K(\Delta t)\rho U_K^\dagger(\Delta t)$ and

$$C = \exp\left[-\alpha^2(1 - e^{-\gamma(t/N)})\sum_{n=1}^{N} \exp[-\gamma(t/N)]^{(n-1)} \times\{1 - \exp[I\chi n(t/N)]\}\right], \tag{5}$$

where $\Gamma=\chi/\gamma$ and α is assumed to be real without losing generality. Here we have defined C as the "coherence parameter" since it determines the degree of decoherence (dephasing) for the resulting macroscopic superposition state. The amplitude parameter A has been defined to quantify the average energy loss. The second beam splitter BS2 and detectors A and B in Fig. 1 perform a measurement onto the superposed basis states $(|0_L\rangle \pm |1_L\rangle)/\sqrt{2}$. The macroscopic superposition state obtained by the measurement is

$$\rho_{\pm}(t) = \mathcal{N}_{\pm}(|A\alpha\rangle\langle A\alpha| \pm C|A\alpha\rangle\langle A\alpha e^{i\theta}| \pm C^*|A\alpha e^{i\theta}\rangle\langle A\alpha| + |A\alpha e^{i\theta}\rangle\langle A\alpha e^{i\theta}|), \tag{6}$$

where $\mathcal{N}_{\pm}$ are the normalization factors. State $\rho_+(t)[\rho_-(t)]$ is obtained when detector B [detector A] clicks in Fig. 1. If $|C|=1$ (and $A\neq 0$), the state $\rho_{\pm}(t)$ is a pure superposition of coherent states, while if $|C|=0$, it is simply a statistical mixture of two coherent states. Let us assume that one wishes to gain $\theta=\pi$ when the amplitude of the initial coherent state is α_0 so that a macroscopic superposition state $|\alpha_0\rangle \pm |-\alpha_0\rangle$ (unnormalized) could be obtained (without decoherence). In this case, an interaction time $t=\pi/\chi$ is required. However, if the amplitude of the initial coherent state is $\alpha(>\alpha_0)$, the required interaction time t is obtained from the equation, $|\alpha e^{i\chi t}-\alpha| = 2\alpha_0$, which can be derived from a simple geometric analysis. The required interaction time is then $t(\alpha,\alpha_0,\chi) = \cos^{-1}[1-2\alpha_0^2/\alpha^2]/\chi$.

Note that the state (6) can be changed into a symmetric form in the phase space—i.e., $A\alpha \to \gamma$ and $A\alpha e^{i\theta} \to -\gamma$. This can be done by applying the displacement operator $D(x) = \exp(xa^\dagger + x^* a)$, where a and $a^\dagger$ are annihilation and creation operators. The displacement operation can be performed using a strong coherent field (an additional local oscillator) and a biased beam splitter. This may be required for quantum information processing but it does not make any essential difference in our discussions.

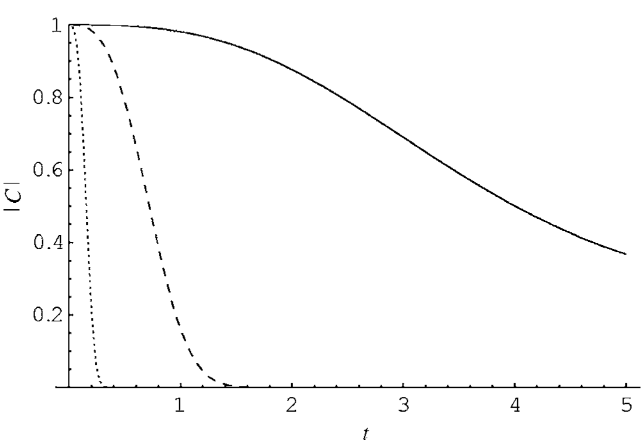

FIG. 3. The decrease of coherence parameter $|C|$ against time t. $\alpha=3$ (solid line), $\alpha=30$ (dashed line), and $\alpha=300$ (dotted line). Decoherence occurs faster as the initial amplitude gets larger. $\chi/\gamma = 0.0125$.

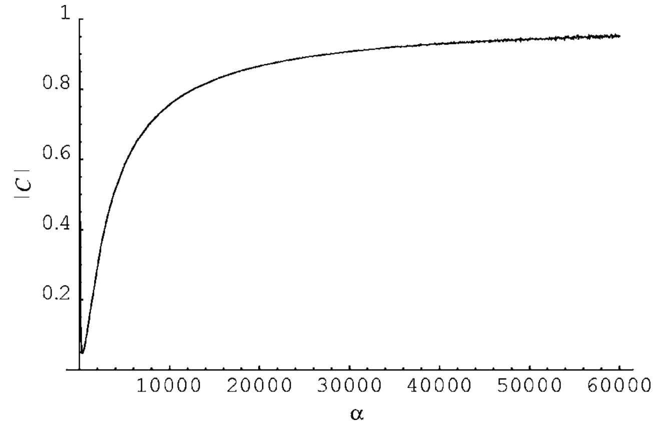

FIG. 4. The coherence parameter $|C|$ against the initial amplitude α for the same separation ($\alpha_0=3$). The decoherence effect diminishes as α gets larger. See text for details.

In our numerical calculation, we have chosen $N=10^6$—i.e., $\Delta\theta=\pi/10^6$. It is clear from Fig. 2 that this value gives a very good approximation for the whole range of α in our study because $|C|$ rapidly converges as N increases. Our numerical results can be summarized as follows. The first effect is that the same amount of separation between the coherent states is gained in a shorter time for a larger initial amplitude, which we already learned from Nemoto and Munro [5]. The second effect is that decoherence occurs faster as the initial amplitude gets larger as shown in Fig. 3, which could be expected intuitively. Our observation is that the first effect overcomes the second one so that decoherence effects diminish as the initial amplitude gets larger for the same amount of separation: Figure 4 shows that the coherence parameter $|C|$ increases as the initial amplitude α increases for the same separation. In what follows we present the detailed values obtained from our numerical study. One may need an optical fiber of about 3000 km for $\theta=\pi$ using currently available Kerr nonlinearity [9]. We choose $\Gamma = 0.0125$ that the amplitude will reduce as $A\approx 0.533$ for 15 km while $\theta=\pi$ is obtained for 3000 km. This corresponds 0.364 dB/km of signal loss, which is a typical value for commercial fibers used for telecommunication and easily achieved using current technology [10,11]. Note that signal losses in some pure silica core fibers are even less than 0.15 dB/km [11]. If the required amplitude for the obtained cat state is $\alpha_0=3$ and the initial amplitude is also $\alpha=3$ (so that $\theta=\pi$), the amplitude parameter is $A\approx 2.7\times 10^{-55}$; i.e., the resulting state will be virtually the vacuum.[1] If $\alpha=300$, the amplitude parameter is $A=0.45$ so that the "effective" amplitude $|A\alpha - A\alpha^{i\theta}|/2$ of the cat state calculated from the separation between the two coherent states is ≈ 1.35. In this case the coherence parameter is $|C|\approx 0.047$; i.e., the state is almost completely decohered. If $\alpha=3000$, the effective am-

[1]In this case ($\alpha=\alpha_0=3$) a nonlinearity $\approx 10^4$ times larger than the currently existing value is required for $A>0.9$ and $|C|\approx 0.8$.

plitude is ≈ 2.76 and the coherence parameter is $|C|\approx 0.43$. If $\alpha=30\,000$, the effective amplitude is ≈ 2.97 and the coherence parameter is $|C|\approx 0.91$; i.e., the resulting state will be close to a pure macroscopic superposition state. Therefore, in order to obtain $|C|>0.9$ for the (effective) amplitude $|\alpha|\approx 3$ of the cat state, one needs the initial coherent state of $\alpha=30\,000$. In this case, an optical fiber of only about 190 m will be required.

One can simply produce macroscopic entanglement using an additional 50:50 beam splitter on the state produced in Fig. 1 [13]. The state (6) after this additional beam splitter becomes

$$\rho_{\pm}^{E}(t) = \mathcal{N}_{\pm}(|\beta,\beta\rangle\langle\beta,\beta| \pm C|\beta,\beta\rangle\langle\beta',\beta'| \pm C^{*}|\beta',\beta'\rangle\langle\beta,\beta| + |\beta',\beta'\rangle\langle\beta',\beta'|), \tag{7}$$

where $\beta=A\alpha/\sqrt{2}$, $\beta'=A\alpha e^{i\theta}/\sqrt{2}$, and $|\beta,\beta\rangle=|\beta\rangle|\beta\rangle$. Applying displacement operators $D_1(x)D_2(x)$, this state can be transformed to a symmetric form as

$$\rho_{\pm}^{E'}(t) = \mathcal{N}_{\pm}(|\gamma,\gamma\rangle\langle\gamma,\gamma| \pm C'|\gamma,\gamma\rangle\langle-\gamma,-\gamma| \pm C'^{*}|-\gamma,-\gamma\rangle \times\langle\gamma,\gamma| + |-\gamma,-\gamma\rangle\langle-\gamma,-\gamma|), \tag{8}$$

where $x=-(\beta+\beta')/2$, $\gamma=(\beta-\beta')/2$, and $C'=C\exp[2A^2\alpha^2 i \sin\theta]$. Note that the states (7) and (8) contain the same amount of entanglement because local unitary transformations do not increase or decrease the amount of entanglement. The state (8) can be represented in a 2×2 Hilbert space by defining the basis as $\{|\Phi_+\rangle,|\Phi_-\rangle\}$. The state $\rho_+^{E'}$ in Eq. (8) in the 2×2 Hilbert space spanned by the new basis is then

$$\rho_{+}^{E'} = \frac{1}{K+2R+Z}\begin{pmatrix} K & V & V & D \\ -V & R & R & W \\ -V & R & R & W \\ D & -W & -W & Z \end{pmatrix}, \tag{9}$$

where $K=M_+^2(2+C'+C'^*)$, $V=-M_+\sqrt{M_+M_-}(C'-C'^*)$, $D=M_+M_-(2+C'+C'^*)$, $R=M_+M_-(2-C'-C'^*)$, $W=M_-\sqrt{M_+M_-}(C'-C'^*)$, and $Z=M_-^2(2+C'+C'^*)$. We have

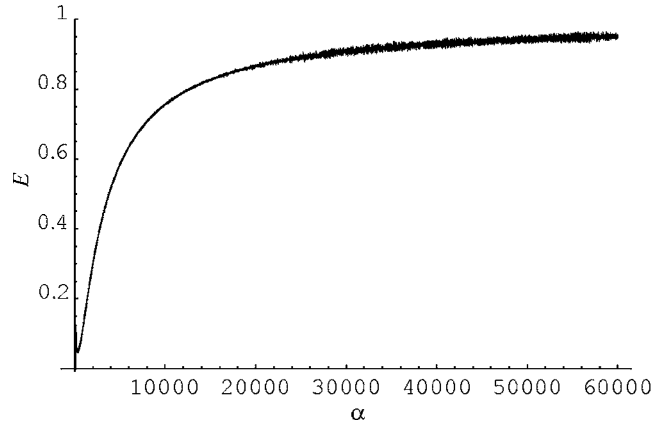

FIG. 5. The measure of entanglement E of the state obtained by the process described in Fig. 1 against the initial amplitude α.

numerically calculated the negative eigenvalue λ_- of the partial transpose for $\rho_+^{E'}$ as a measure of entanglement [14]. In Fig. 5, one can clearly see that the degree of entanglement $E(=-2\lambda_-)$ increases as the initial amplitude becomes larger.

Our discussion clarifies that an inefficient single-photon source, two inefficient detectors, and weak nonlinearity, beam splitters, and a coherent state source are required resources for generation of macroscopic entanglement, and such a method can overcome the decoherence effect. In the same manner, weak nonlinearity can also be used for quantum gate operations [5] in the presence of decoherence. However, it should be noted that some effects in the nonlinear media such as phase noise may not be negligible in real experiments. It is also a separate problem to investigate the decoherence effect in *nonlinear media* for the other schemes [3,4] with weak nonlinearity.

Note added. Recently, an update [15] to Ref. [5] was published, where the authors discussed decoherence of qubits for their computation scheme.

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