Unmodulated spin chains as universal quantum wires

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We study a quantum state transfer between two qubits interacting with the ends of a quantum wire consisting of linearly arranged spins coupled by an excitation conserving, time-independent Hamiltonian. We show that, if we control the coupling between the source and the destination qubits and the ends of the wire, the evolution of the system can lead to an almost perfect transfer even in the case in which all nearest-neighbour couplings between the internal spins of the wire are equal.

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The problem of designing quantum networks which enable efficient high-fidelity transfer of quantum states has recently been addressed by a number of authors (see Refs. [1–23]). Ideally, such a network should meet both the simplicity and the minimal control requirements. By simplicity, we mean that the network consists of typical elements coupled in a standard way so that a few networks can be combined together to form more complex systems. The minimal control requirement says that the transmission of a quantum state through the network should be possible without performing many control operations (such as switching interactions on and off, measuring, encoding and decoding, etc.). A 1D quantum network (quantum wire) which fulfills both the above requirements was proposed by Bose [1], who considered a spin chain with the nearest-neighbor Heisenberg Hamiltonian; here, the transmission of quantum state between the ends of the chain was achieved simply by a free evolution of the network. However, as was shown by Bose, if all neighbor couplings have the same strength, the fidelity of a transmission decreases with the chain length n. A similar model (with the Heisenberg Hamiltonian replaced by an XY one) was considered by Christandl *et al.* in Ref. [2]. They show that one can transfer quantum states through arbitrary long chains if spin couplings are carefully chosen in a way depending on the chain length n (see also Refs. [3–8]). Note, however, that this approach does not meet the simplicity requirement since one cannot merge several "modulated" quantum wires into a longer one.

Here, we study a transfer of quantum states between two qubits attached to the ends of a quantum wire consisting of n linearly arranged spins. In order to fulfill the requirement of simplicity, we assume that all couplings between neighboring spins forming the quantum wire are the same (and equal to 1), while the couplings between the source and the destination qubits and the ends of the wire are equal to a. We show that one can significantly improve the fidelity of the transfer between the source and the destination qubits by selecting the value of a appropriately. In particular, choosing a small enough, one can achieve a transfer whose fidelity can be arbitrarily close to 1, even for large n.

We assume that the Hamiltonian of the whole system of $n+2$ qubits conserves the number of excitation (e.g., it is an XY Hamiltonian), so the state $\otimes_{j=0}^{n+1}|0\rangle_j$ is its eigenstate. Let F_{ab} denote the fidelity of the transfer of an arbitrary state $|\phi\rangle=a|0\rangle+b|1\rangle$ from the source ($j=0$) to destination ($j=n+1$) qubit, and let $F=F_{01}$ stand for the fidelity of the state $|1\rangle$ transfer. It is easy to check that the average fidelity $\langle F_{ab}\rangle$, where the average is taken over all possible values of a and b, is uniquely determined by F, namely

$$\langle F_{ab}\rangle = \frac{1}{3} + \frac{1}{6}(1+F)^2.$$

Thus, from now on, we shall consider only the evolution of the system with the initial state given by $|\Psi(0)\rangle = |1\rangle_0|0\rangle_1\cdots|0\rangle_{n+1}$, i.e., the wire is in the polarized state. Then, the state space is spanned by vectors $|p\rangle=\otimes_{j=0}^{n+1}|\delta_{jp}\rangle_j$, $p=0,\dots,n+1$, and the Hamiltonian $H(a)$ of the system can be written as

$$H(a) = a(|0\rangle\langle 1| + |n\rangle\langle n+1|) + \sum_{p=1}^{n-1}|p\rangle\langle p+1| + \text{H.c.},$$

where, let us recall, both the source and the destination qubits are coupled to the quantum wire with strength a, while all couplings among spins of the quantum wire are taken to
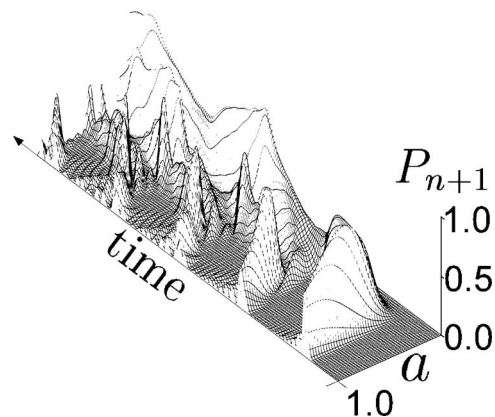

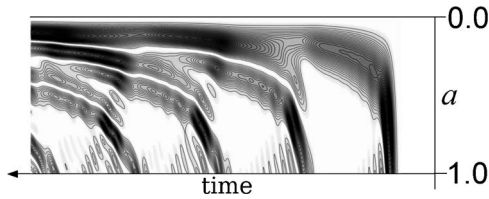FIG. 1. The evolution of $P_{n+1}(t)$ for the system of 30 spins for $t \in [0, 150]$.

FIG. 2. The density plot for the graph in Figure 1.

be one. If $a^2 \neq 2$, then all the eigenvalues of $H(a)$ are of the form $\lambda = 2\cos\gamma$, where γ is a solution of either of the two following equations ($\mu = \pm 1$):

$$\mu \cot(\gamma)\cot^{\mu}\left(\frac{n+1}{2}\gamma\right) = \frac{a^2}{2-a^2}.$$

The eigenvector $|v^{(\gamma)}\rangle$ corresponding to the eigenvalue $2\cos\gamma$ has coordinates $v_k^{(\gamma)} = \langle k | v^{(\gamma)}\rangle$ given by

$$v_0^{(\gamma)} = \frac{a}{c}\sin\gamma,$$

$$v_k^{(\gamma)} = \frac{1}{c}(\sin[(k+1)\gamma] + (1-a^2)\sin[(k-1)\gamma]),$$

for $k = 1, \ldots, n$, and

$$v_{n+1}^{(\gamma)} = \mu v_0^{(\gamma)},$$

where $c = c(a)$ is defined as

$$c^2 = (n+1)(2(1-a^2)\cos^2\gamma + a^4/2) + 2a^2 - a^4.$$

Let $P_j(t) = |\langle j | \Psi(t)\rangle|^2$ denote the probability of the excitation of the jth spin, if the initial state of the system is $|1\rangle_0 |0\rangle_1 \cdots |0\rangle_{n+1}$ [i.e., $P_0(0) = 1$]. Figure 1 shows the dependence of $P_{n+1}(t)$ on a for the first period of the evolution of a system of 30 spins. Figure 2 presents the density version of the same graph, while Fig. 3 shows the projection of the evolution along the time axis. As clearly seen in Fig. 3, decreasing a from 1 to about 0.6 significantly improves the fidelity of transfer; if we decrease a even further, then the fidelity drops down but for small a it grows again, approaching 1 as $a \to 0$.

In order to better understand Fig. 3, we graph the effect of decreasing a on the eigenvalue spectrum of the whole system as well as on the eigenvector population $|v_0^{(\gamma)}|^2$ (Fig. 4). For
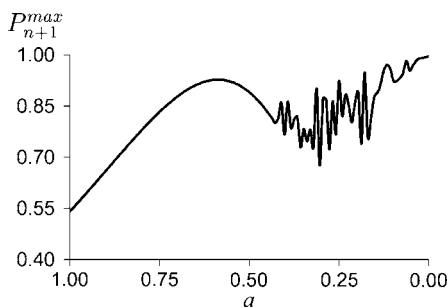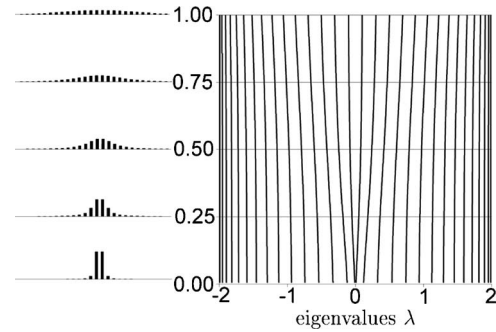

FIG. 4. The dependence of the eigenvalues of $H(a)$ on a (the right-hand side of the figure) and the eigenvector populations $\{|\langle v(\lambda)|\Psi(0)\rangle|^2\}_\lambda$ (on the left) for the system of 30 spins.

$a \sim 1$ the spectrum is nearly harmonic in the vicinity of 0, but the distribution of the eigenvectors is broad and the contribution from the anharmonic part of the spectrum spoils the transfer. If a decreases, the eigenvector distribution narrows so that the anharmonic contribution drops down; consequently, the transfer fidelity increases. Then, for small a, the harmonicity of the spectrum in the vicinity of zero breaks down, which reduces the fidelity of the transfer. If a is even smaller, the system is very sensitive for spontaneous resonances between different modes which, in turn, are greatly affected by small changes of the value of a. For very small values of a the behavior of the system is determined only by the two (or three when n is odd) central eigenvalues of $H(a)$, which enables an almost perfect transfer of states.
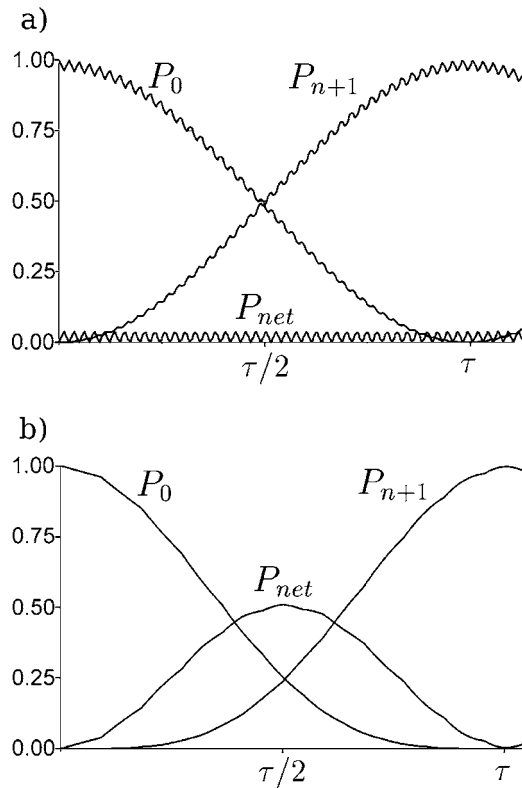

FIG. 3. The dependence of $P_{n+1}^{\max} = \max\{P_{n+1}(t) : t \in [0, 20\,000]\}$ on a for the system of 30 spins.

FIG. 5. The evolution of $P_0(t)$, $P_{n+1}(t)$, $P_{net}(t)$, for $a = 0.01$ and (a) $n = 198$; (b) $n = 199$. If $n = 198$, then $\lambda_a \sim 10^{-4}$ and $\tau \sim 16\,000$; for $n = 201$ we have $\lambda_a = 1.41 \times 10^{-3}$ and $\tau \sim 2200$.

In this phase the behavior of the system depends strongly on the parity of n. If $a \to 0+$, then the initial state $|\Psi(0)\rangle$ is concentrated basically on two eigenvectors for n even (see Fig. 4); if n is odd, then the evolution of the system takes place in a three-dimensional space. More specifically, let $\hat{\lambda}_a$ be the smallest positive eigenvalue of $H(a)$. If n is even and $a \to 0+$, then for the two eigenvectors $|x\rangle = |v(\hat{\lambda}_a)\rangle$, $|y\rangle = |v(-\hat{\lambda}_a)\rangle$ corresponding to the eigenvalues $\pm\hat{\lambda}_a$, we have $x_0 = y_0 = x_{n+1} = -y_{n+1} \sim 1/\sqrt{2}$. Consequently, an almost perfect transfer of states between qubits 0 and $n+1$ occurs after time $\tau \sim \pi/(2\hat{\lambda}_a)$. For odd n's, the spectrum of $|\Psi(0)\rangle$ is concentrated on three eigenvectors: besides $|x\rangle = |v(\hat{\lambda}_a)\rangle$ and $|y\rangle = |v(-\hat{\lambda}_a)\rangle$, we need to take into account the eigenvector $|z\rangle = |v(0)\rangle$ corresponding to the eigenvalue zero. Then, $z_0 = |z_{n+1}| \sim 1/\sqrt{2}$, $x_0 = y_0 = |x_{n+1}| = |y_{n+1}| \sim 1/2$, and the sign of z_{n+1} is opposite to the signs of both x_{n+1} and y_{n+1}. In this case an almost perfect transfer from the source to the destination qubits occurs after time $\tau \sim \pi/\hat{\lambda}_a$. Let us mention that a similar phenomenon of transferring the states between two qubits weakly interacting with a network has been recently observed by Li *et al.* [9] and Plenio and Semiao [10], who consider transferring qubit states through a ladder and a cycle, respectively.

Straightforward computations show that the fidelity of the transfer scales as $1 - O(a^2 n)$, and it grows to 1 only when $a \ll 1/\sqrt{n}$. Moreover, we have $\hat{\lambda}_a \sim a^2$ for an even n, and $\hat{\lambda}_a \sim 2a/\sqrt{n}$ when n is odd. Hence, the time of the transfer is $\Theta(a^{-2})$ for even n, and $\Theta(\sqrt{n}/a)$ for odd n. Thus, for $a = \delta/\sqrt{n}$, an almost perfect ($F \approx 1 - \delta^2$) transfer is possible for $\delta \ll 1$ in a time which scales linearly with the quantum wire length n. The speed of the transfer n/τ is approximated by $2\delta/\pi$ and δ^2/π when n is odd and even, respectively. Thus, for small a, a quantum wire with odd n ensures more effective transfer. In Fig. 5 the evolution of the excitation probabilities of the source qubit $P_0(t)$, the destination qubit $P_{n+1}(t)$, and the total probability of the excitation of the wire qubits $P_{net}(t) = \Sigma_{j=1}^{n} P_j(t)$, are presented for $a = 0.01$ and $n = 198, 199$. Note that if n is even, the spins of the wire remain almost unexcited during the evolution, i.e., the source and destination qubits evolve like two isolated and directly coupled qubits. Thus, the evolution of a system resembles Rabi oscillations between the source and destination qubits, despite the fact that they lie a significant distance from each other.

Finally, we remark that if n is even and $a \to 0+$, after time $t \sim \tau/2$ the state of the system is close to $(1/\sqrt{2})(|0\rangle_0|1\rangle_{n+1} + |1\rangle_0|0\rangle_{n+1})|0\rangle_1 \cdots |0\rangle_n$. Thus, such an unmodulated quantum wire can also be used to generate entanglement between the source and destination qubits.

In conclusion, we have shown that one can significantly improve a transfer of the qubit states between the ends of a quantum wire by controlling the coupling on its ends. In particular, an unmodulated spin chain of arbitrary length n can be used as a universal quantum wire to transfer of quantum states with arbitrary high fidelity in a time which scales linearly with n.

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