Thermal entanglement in a two-qubit Heisenberg *XXZ* spin chain under an inhomogeneous magnetic field

Guo-Feng Zhang[1,*] and Shu-Shen Li[1,2]

[1]*State Key Laboratory for Superlattices and Microstructures, Institute of Semiconductors, Chinese Academy of Sciences, P. O. Box 912, Beijing 100083, People's Republic of China*

[2]*China Center of Advanced Science and Technology (CCAST) (World Laboratory), P. O. Box 8730, Beijing 100080, People's Republic of China*

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The thermal entanglement in a two-qubit Heisenberg *XXZ* spin chain is investigated under an inhomogeneous magnetic field b. We show that the ground-state entanglement is independent of the interaction of z-component J_z. The thermal entanglement at the fixed temperature can be enhanced when J_z increases. We strictly show that for any temperature T and J_z, the entanglement is symmetric with respect to zero inhomogeneous magnetic field, and the critical inhomogeneous magnetic field b_c is independent of J_z. The critical magnetic field B_c increases with the increasing $|b|$ but the maximum entanglement value that the system can arrive at becomes smaller.

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I. INTRODUCTION

Entanglement is the most fascinating feature of quantum mechanics and plays a central role in quantum information processing. In recent years, there has been an ongoing effort to characterize qualitatively and quantitatively the entanglement properties of condensed matter systems and apply then in quantum communication and information. An important emerging field is the quantum entanglement in solid state systems such as spin chains [1–8]. Spin chains are the natural candidates for the realization of the entanglement compared with the other physics systems. The spin chains not only have useful applications such as the quantum state transfer, but also display rich entanglement features [9]. The Heisenberg chain, the simplest spin chain, has been used to construct a quantum computer and quantum dots [10]. By suitable coding, the Heisenberg interaction alone can be used for quantum computation [11–13]. The thermal entanglement, which differs from the other kinds of entanglements by its advantages of stability for the reduction in entanglement of an entangled state due to various sources of decoherence and in entanglement in time due to thermal interactions are absent as the entanglement at finite temperature takes thermal decoherence into account implicitly, requires neither measurement nor controlled switching of interactions in the preparing process, and hence becomes an important quantity of systems for the purpose of quantum computing. In the studies on the entanglement of the Heisenberg spin model, a lot of interesting work has been done [14–16]. It turns out that the critical magnetic field B_c is increased by introducing the interaction of the z-component of two neighboring spins in Ref. [17]. However, only the uniform field case is carefully studied in the above mentioned papers. The nonuniform case is rarely taken into account. We know that in any solid state construction of qubits, there is always the possibility of inhomogeneous Zeeman coupling [18,19]. Thus, it is necessary to consider the entanglement for a nonuniform field case. Asoudeh and Karimipour [20] studied the effect of inhomogeneous in the magnetic field on the thermal entanglement of an isotropic two-qubit *XXX* spin system. We noticed that the entanglement for a *XXZ* spin model in a nonuniform field has not been discussed. Although Asoudeh [20] states that the different types of anisotropic interactions may not be of much practical relevance to concrete physical realization of qubits, in the theoretical analysis we think it is very interesting and should be included in the studies of spin chain entanglement. This is the main motivation of this paper.

For a system in equilibrium at temperature T, the density matrix is $\rho=(1/Z)e^{-\beta H}$, where $\beta=1/kT$, k is the Boltzmann constant, and $Z=tre^{-\beta H}$ is the partition function. For simplicity, we write $k=1$. The entanglement of two qubits can be measured by the concurrence C which is written as $C=\max[0,2\max[\lambda_i]-\Sigma_i^4\lambda_i]$ [15], where λ_i are the square roots of the eigenvalues of the matrix $R=\rho S\rho^* S$, ρ is the density matrix, $S=\sigma_1^y\otimes\sigma_2^y$, and $*$ stands for the complex conjugate. The concurrence is available, no matter whether ρ is pure or mixed. In the case that the state is pure $\rho=|\psi\rangle\langle\psi|$ with

$$|\psi\rangle = a|0,0\rangle + b|0,1\rangle + c|1,0\rangle + d|1,1\rangle, \tag{1}$$

the concurrence is simplified to

$$C(\psi) = 2|ad - bc|. \tag{2}$$

II. THE MODEL AND THE GROUND-STATE ENTANGLEMENT

The Hamiltonian of the N-qubit anisotropic Heisenberg *XXZ* model under an inhomogeneous magnetic field is

*Corresponding author; electronic address: gf1978zhang2001@yahoo.com

$$H = \frac{1}{2}\sum_{i=1}^{N} [J\sigma_i^x\sigma_{i+1}^x + J\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z + (B+b)\sigma_i^z + (B-b)\sigma_{i+1}^z], \tag{3}$$

where J and J_z are the real coupling coefficients. The coupling constants $J>0$ and $J_z>0$ correspond to the antiferromagnetic case, and $J<0$ and $J_z<0$ the ferromagnetic case. $B\geqslant 0$ is restricted, and the magnetic fields on the two spins have been so parameterized that b controls the degree of inhomogeneity. Now we consider the Hamiltonian for $N=2$ case. Note that we are working in units so that B, b, J, and J_z are dimensionless.

In the standard basis $\{|1,1\rangle, |1,0\rangle, |0,1\rangle, |0,0\rangle\}$, the Hamiltonian can be expressed as

$$H = \begin{pmatrix} \frac{J_z+2B}{2} & 0 & 0 & 0 \\ 0 & \frac{-J_z+2b}{2} & J & 0 \\ 0 & J & \frac{-J_z-2b}{2} & 0 \\ 0 & 0 & 0 & \frac{J_z-2B}{2} \end{pmatrix}, \tag{4}$$

A straightforward calculation gives the following eigenstates:

$$|\phi_1\rangle = |0,0\rangle,$$

$$|\phi_2\rangle = |1,1\rangle,$$

$$|\phi_3\rangle = \frac{1}{\sqrt{1+\xi^2/J^2}}\left(\frac{\xi}{J}|1,0\rangle + |0,1\rangle\right),$$

$$|\phi_4\rangle = \frac{1}{\sqrt{1+\zeta^2/J^2}}\left(\frac{\zeta}{J}|1,0\rangle + |0,1\rangle\right), \tag{5}$$

with corresponding energies

$$E_1 = \frac{1}{2}(J_z - 2B),$$

$$E_2 = \frac{1}{2}(J_z + 2B),$$

$$E_3 = -\frac{J_z}{2} - \eta,$$

$$E_4 = -\frac{J_z}{2} + \eta, \tag{6}$$

where $\eta=\sqrt{b^2+J^2}$, $\xi=b-\eta$, and $\zeta=b+\eta$. Note that when $b\to 0$ and $J>0$, the two states $|\phi_3\rangle$ and $|\phi_4\rangle$, respectively, go to the maximally entangled state $(1/\sqrt{2})(|0,1\rangle-|1,0\rangle)$ and $(1/\sqrt{2})(|0,1\rangle+|1,0\rangle)$. For $J<0$, they, respectively, go to $(1/\sqrt{2})(|0,1\rangle+|1,0\rangle)$ and $(1/\sqrt{2})(|0,1\rangle-|1,0\rangle)$. We can also find that the eigenenergies are even function of the coupling constant J. Thus, we can think the ground-state entanglement exists for both antiferromagnetic and ferromagnetic cases and should be symmetric with respect to the coupling constant J. The ground state depends on the value of the magnetic field B, the coupling constant J_z, and η. It is readily found that the ground-state energy is equal to

$$E_1 = \frac{1}{2}(J_z - 2B), \quad \text{if } \eta < B - J_z;$$

$$E_3 = -J_z/2 - \eta, \quad \text{if } \eta > B - J_z. \tag{7}$$

Thus, when $\eta<B-J_z$, the ground state is the disentangled state $|\phi_1\rangle$ and when $\eta>B-J_z$, the ground state is the entangled state $|\phi_3\rangle$. For each value of the magnetic field B, there is a threshold parameter $J_z^f=B-\eta$ above which the ground state will become entangled. Accordingly, for each value of inhomogeneity η there is a value of magnetic field $B^f=\eta+J_z$ above which the ground state will loosen its entanglement. In the entangled phase, the entanglement of the ground state is found from Eqs. (2) and (5) to be

$$C(|\phi_3\rangle) = \frac{2|\lambda|}{1+\lambda^2}, \tag{8}$$

where $\lambda=\xi/J$. $\lambda=\pm 1$ (i.e., $b=0$), the system enters the maximally entangled phase $|\phi_3\rangle$ with entanglement $C(|\phi_3\rangle)=1$. This result accords with that in Ref. [20]. Here we can also know that the ground-state entanglement is independent of the interaction of z-component J_z.

III. THE THERMAL ENTANGLEMENT

As the thermal fluctuation is introducing into the system, the entangled ground states will be mixed with the unentangled excited state. This effect will make the entanglement decrease. At the same time, the disentangled ground state mixes with entangled excited states. To see the change of the entanglement, we calculate the entanglement of the thermal state $\rho=(1/Z)e^{-\beta H}$. In the standard basis $\{|1,1\rangle, |1,0\rangle, |0,1\rangle, |0,0\rangle\}$, the density matrix of the system can be written as

$$\rho_{12} = \frac{1}{Z}\begin{pmatrix} e^{-E_2/kT} & 0 & 0 & 0 \\ 0 & e^{J_z/2kT}(m-n) & -s & 0 \\ 0 & -s & e^{J_z/2kT}(m+n) & 0 \\ 0 & 0 & 0 & e^{-E_1/kT} \end{pmatrix} \tag{9}$$

where $Z=e^{-E_2/kT}(1+e^{2B/kT})+2e^{(J_z+B)/kT}\cosh(\eta/kT)$, $m=\cosh(\eta/kT)$, $n=b\sin\mathrm{h}(\eta/kT)/\eta$, $s=e^{J_z/2kT}J\sin\mathrm{h}(\eta/kT)/\eta$. In the following calculation, we will write the Boltzmann constant $k=1$. From Eq. (9) and the definition of concurrence, we can obtain the concurrence at the finite temperature.

Case 1: $J_z=0$. Our model corresponds to a XX spin model. The eigenvalues and eigenvectors can be easily obtained. In Fig. 1, we give the results at different temperature for the nonuniform magnetic field ($B=0$) and the uniform magnetic

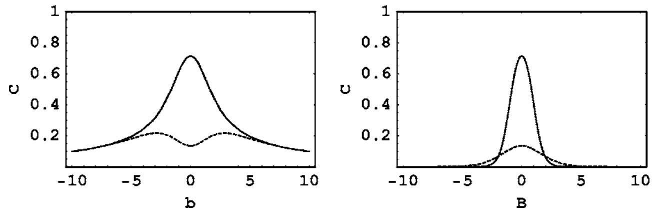

FIG. 1. The concurrence for $J_z=0$ and $J=1$ case. $T=0.4$ (solid curve) and $T=1.0$ (dotted curve). The left panel corresponds to the nonuniform case and the right panel corresponds to the uniform case. T is plotted in units of the Boltzmann's constant k. We work in units where B and b are dimensionless.

($b=0$). From the figure, we can know that the entanglement is symmetric with respect to zero magnetic field, the nonuniform magnetic field can lead to higher entanglement and double-peak structure. This results accord with those seen from Ref. [5].

Case 2: $J_z=J$. In order to compare with the results in Ref. [20], we make the substitutions $J\to 2J$, $B\to 2B$, and $b\to 2b$. The eigenvalues and eigenvectors can be easily obtained as

$$|\varphi_1\rangle = |0,0\rangle,$$

$$|\varphi_2\rangle = |1,1\rangle,$$

$$|\varphi_3\rangle = \frac{1}{\sqrt{1+x^2/J^2}}\left(\frac{x}{J}|1,0\rangle + |0,1\rangle\right),$$

$$|\varphi_4\rangle = \frac{1}{\sqrt{1+y^2/J^2}}\left(\frac{y}{J}|1,0\rangle + |0,1\rangle\right), \tag{10}$$

with corresponding energies

$$e_1 = J - 2B,$$

$$e_2 = J - 2B,$$

$$e_3 = -J - 2|J|\sqrt{1+\delta^2},$$

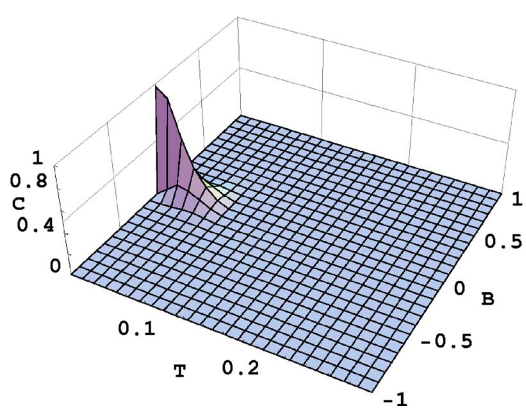

FIG. 2. (Color online) The thermal concurrence for $J_z=J=-1$ case. The inhomogeneous magnetic field $b=0.458$. T is plotted in units of the Boltzmann's constant k. We work in units where B and b are dimensionless.

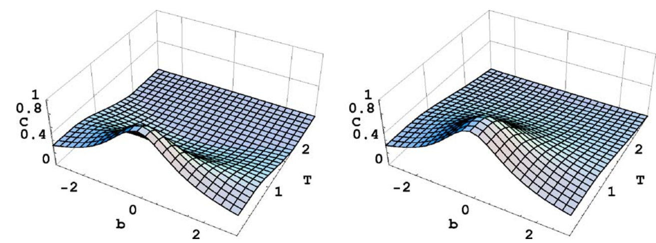

FIG. 3. (Color online) The concurrence in the *XXZ* spin model is plotted vs b and T. Coupling constant $J=1$, and the magnetic field $B=0$. The left panel corresponds to the $J_z=0$ case and the right panel corresponds to the $J_z=0.9$ case. T is plotted in units of the Boltzmann's constant k. We work in units where b is dimensionless.

$$e_4 = -J + 2|J|\sqrt{1+\delta^2}, \tag{11}$$

where $\delta=b/J$, $x=2b-2\sqrt{1+\delta^2}$, and $y=2b+\sqrt{1+\delta^2}$. For the ferromagnetic case $J=-1$, the ground-state concurrence is $C(|\varphi_3\rangle)=1/\sqrt{1+\delta^2}$. For the ferromagnetic case $J=1$, the ground-state concurrence is $C(|\varphi_4\rangle)=1/\sqrt{1+\delta^2}$. These results are same with those obtained in Ref. [20]. In Fig. 2, we give the plot of the thermal concurrence for $J_z=J=-1$ case.

In order to compare our result with that in Ref. [20], we let the inhomogeneous magnetic field $b=0.458$ (this accords to the value of ξ in Ref. [20]). We can see that the thermal entanglement develops and is maximized for zero magnetic field B and reaches the maximum value at $T=0$.

Case 3: *For any* J_z. With $B=0$, the concurrence as a function of b and T for two values of J_z are given in Fig. 3. They show that the concurrences are 1 for different J_z when $b=0$ and $T=0$. At the point, the ground state is $|\phi_3\rangle$ with energy $-J_z/2-1$, which is the maximally entangled state and the corresponding concurrences are 1. As the temperature increases, the concurrences decrease due to the mixing of other states with the maximally entangled state. We also know that the concurrence decreases with the increasing of $|b|$. From the two figures in Fig. 3, we can find that upon increasing J_z, the critical temperature T_c is improved (for $J_z=0$, T_c is about 2, but for $J_z=0.9$, T_c has a higher value). Thus, we can obtain a higher entanglement at a fixed temperature when J_z is increased.

Figure 4 shows the concurrence at a fixed temperature and

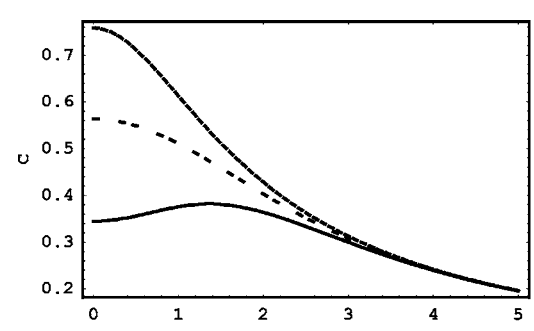

FIG. 4. The concurrence in the *XXZ* model is plotted vs b for various value of J_z, where $J=1$, $B=0.8$, and $T=0.6$. From top to bottom, J_z equals 0.9, 0.4, 0. T is plotted in units of the Boltzmann's constant k. We work in units where B and b are dimensionless.

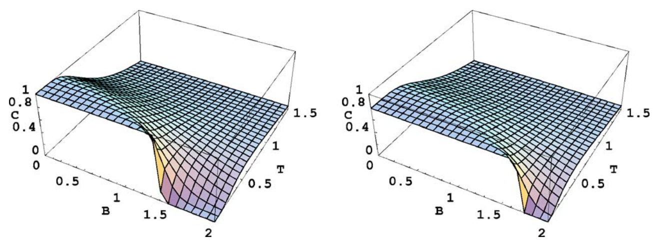

FIG. 5. (Color online) The concurrence in the *XXZ* spin model is plotted vs T and B, where $J=1$ and $J_z=0.4$. The left panel corresponds to $b=0$ case and the right panel corresponds to $b=0.8$ case. T is plotted in units of Boltzmann's constant k. We work in units where B and b are dimensionless.

magnetic field for three values of positive J_z. It is shown that the concurrences drop with the increasing value of b and arrive at zero at the same b value, which is called the critical inhomogeneous magnetic field, for various values of J_z. This is to say the critical inhomogeneous magnetic field is independent of J_z. Moreover, we can see that for a higher value of J_z, the system has a stronger entanglement, which is consistent with Fig. 3.

In Fig. 5 we give the plot of concurrence as a function of T and B for $b=0$ and $b=0.8$ when $J_z=0.4$. For $B=0$ and $b=0$, the maximally entangled state $|\Phi\rangle=(1/\sqrt{2})(|0,1\rangle-|1,0\rangle)$ is the ground state with eigenvalue $-J_z/2-|J|$. The maximum entanglement is at $T=0$, i.e., $C=1$. As T increases, the concurrence decreases, as seen from Fig. 5, due to the mixing of other states with the maximally entangled state. For a high value of B (in left Fig. 5, $B=1.40$ and in right Fig. 5, $B=1.70$) the state $|\phi_1\rangle$ becomes the ground state, which means there is no entanglement at $T=0$. However, by increasing T, the entangled state $|\phi_3\rangle$ and $|\phi_4\rangle$ will mix with the state, which makes the entanglement increase. From the two figures in Fig. 5, we can see that when b is raised, the critical magnetic field B_c increases, but the maximum entanglement value at which the system can arrive at becomes smaller.

IV. CONCLUSIONS

In conclusion, we have investigated the thermal entanglement in a two-qubit Heisenberg *XXZ* spin chain under an inhomogeneous magnetic field. The ground-state entanglement and thermal entanglement at a finite temperature are given. We find the entanglement exists for both antiferromagnetic and ferromagnetic cases. In addition, the entanglement is enhanced by increasing the interaction of z-component J_z. The critical inhomogeneous magnetic field is independent of J_z. The critical magnetic field B_c increases with the increasing $|b|$ but the maximum entanglement value that the system can arrive becomes smaller.

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