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Distance-based degrees of polarization for a quantum field

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It is well established that unpolarized light is invariant with respect to any SU(2) polarization transformation. This requirement fully characterizes the set of density matrices representing unpolarized states. We introduce the degree of polarization of a quantum state as its distance to the set of unpolarized states. We use two different candidates of distance, namely the Hilbert-Schmidt and the Bures metric, showing that they induce fundamentally different degrees of polarization. We apply these notions to relevant field states and we demonstrate that they avoid some of the problems arising with the classical definition.

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I. INTRODUCTION

The polarization properties of quantum fields have received much attention over the past few years, especially in the single-photon regime. The polarization state is a robust characteristic, which is relatively simple to manipulate without inducing more than marginal losses. It is thus hardly surprising that many outstanding experiments in quantum optics, such as Bell tests [1,2], quantum tomography [3], or quantum cryptography [4,5], have been performed using polarization states.

The polarization of a light beam in classical optics can be elegantly visualized by resorting to the Poincaré sphere, and is determined by the four Stokes parameters [6–9]. These parameters present unique advantages: they are easily measured, they can be extended to the quantum domain, where the Stokes parameters become the mean values of the Stokes operators, and, finally, they allow us to classify the states of light according to a degree of polarization [10–13].

This classical degree of polarization is simply the modulus of the Stokes vector. While this affords a very intuitive image, it also has serious drawbacks that can be traced back to the fact that the Stokes parameters are proportional to the second-order correlations of the field amplitudes. This may be sufficient for most classical problems, but for quantum fields higher-order correlations are crucial. For this reason, the Stokes parameters do not distinguish between quantum states having remarkably different polarization properties [14–16]. For example, the classical polarization degree of the state with exactly one photon in each of the horizontally and vertically polarization modes is zero, but it cannot be regarded as unpolarized. These unwanted consequences call for alternative measures.

Recently, Luis [17] has brought in a challenging idea to circumvent these difficulties. For the full characterization of polarization, a probability distribution [obtained via the SU(2) Q function] is defined on the Poincaré sphere. To some extent, the existence of such a probabilistic description is unavoidable in quantum optics from the very beginning, since the Stokes operators do not commute and thus no state can have a definite value of all of them simultaneously (except the vacuum). In this framework, the degree of polarization of a field state can be defined as the distance from its associated distribution to the uniform distribution corresponding to unpolarized light. We find this proposal interesting, but semiclassical in nature. In addition, we stress that the SU(2) Q function does not connect manifolds with different photon excitations, so the information it embodies cannot be, in general, complete.

The question of what unpolarized light is has a relatively long history [18–20]. Today, there is a wide consensus [21,22] in considering unpolarized light to be described by quantum states that is invariant with respect to any SU(2) polarization transformation. It turns out that this requirement fixes the set of density operators admissible to represent unpolarized fields. It is then suggestive to look at the degree of polarization as a distance from a given state to this set of unpolarized states.

The notion of distance measure has been successfully used in assessing nonclassicality [23–25], entanglement [26], quantum information [27–35], and localization [36–39], to cite only some relevant examples. These measures are useful both when comparing experiments with the corresponding theory and in comparing different experiments. We hope that they will soon be agreed upon by experimentalists and theorists alike.

In this paper, we connect the idea of distance with the problem of assessing the polarization characteristics of a quantum field, exploring a suitable definition that avoids at least some of the aforementioned difficulties that previous approaches based on Stokes parameters encounter.

II. SU(2) POLARIZATION STRUCTURE AND INVARIANCE PROPERTIES OF QUANTUM FIELDS

We assume a monochromatic plane wave propagating in the z direction, whose electric field lies in the xy plane. Under these conditions, we are dealing with a two-mode field that can be fully described by two complex amplitude operators. They are denoted by $\hat{a}_H$ and $\hat{a}_V$, where the subscripts H and V indicate horizontally and vertically polarized modes, respectively. The commutation relations of these operators are standard:

$$[\hat{a}_j,\hat{a}_k^\dagger]=\delta_{jk}, \quad j,k \in \{H,V\}. \tag{2.1}$$

The Stokes operators are then defined as the quantum counterparts of the classical variables, namely [10–13]

$$\hat{S}_0=\hat{a}_H^\dagger\hat{a}_H+\hat{a}_V^\dagger\hat{a}_V, \quad \hat{S}_1=\hat{a}_H^\dagger\hat{a}_V+\hat{a}_V^\dagger\hat{a}_H,$$

$$\hat{S}_2=i(\hat{a}_H\hat{a}_V^\dagger-\hat{a}_H^\dagger\hat{a}_V), \quad \hat{S}_3=\hat{a}_H^\dagger\hat{a}_H-\hat{a}_V^\dagger\hat{a}_V, \tag{2.2}$$

and their mean values are precisely the Stokes parameters $(\langle\hat{S}_0\rangle, \langle\hat{\mathbf{S}}\rangle)$, where $\hat{\mathbf{S}}=(\hat{S}_1,\hat{S}_2,\hat{S}_3)$. Using the relation (2.1), one immediately gets that the Stokes operators satisfy the commutation relations of angular momentum:

$$[\hat{\mathbf{S}},\hat{S}_0]=0, \quad [\hat{S}_1,\hat{S}_2]=2i\hat{S}_3, \tag{2.3}$$

and cyclic permutations. The noncommutability of these operators precludes the simultaneous exact measurement of their physical quantities. Among other consequences, this implies that no field state (leaving aside the two-mode vacuum) can have definite nonfluctuating values of all the Stokes operators simultaneously. This is expressed by the uncertainty relation

$$(\Delta\hat{\mathbf{S}})^2=(\Delta\hat{S}_1)^2+(\Delta\hat{S}_2)^2+(\Delta\hat{S}_3)^2 \geqslant 2\langle\hat{S}_0\rangle. \tag{2.4}$$

Contrary to what happens in classical optics, the electric vector of a monochromatic quantum field never describes a definite ellipse [17].

In mathematical terms, a SU(2) (or linear) polarization transformation is any transformation generated by the operators $\hat{\mathbf{S}}$. It is well known [40] that the operator $\hat{S}_2$ is the infinitesimal generator of geometrical rotations around the direction of propagation, whereas $\hat{S}_3$ is the infinitesimal generator of differential phase shifts between the modes. As indicated by Eq. (2.3), these two operators suffice to generate all SU(2) polarization transformations, which in experimental terms means that they can be accomplished with a combination of phase plates and rotators (which produce rotations of the electric field components around the propagation axis) [21].

The standard definition of the degree of polarization is [7–9]

$$\mathcal{P}_{\mathrm{sc}}=\frac{\sqrt{\langle\hat{\mathbf{S}}\rangle^2}}{\langle\hat{S}_0\rangle}=\frac{\sqrt{\langle\hat{S}_1\rangle^2+\langle\hat{S}_2\rangle^2+\langle\hat{S}_3\rangle^2}}{\langle\hat{S}_0\rangle}, \tag{2.5}$$

where the subscript sc indicates that this is a semiclassical definition, mimicking the form of the classical one. In the semiclassical description it is implicitly assumed that unpolarized light (i.e., the origin of the Poincaré sphere) is defined by the specific values [41]

$$\langle\hat{S}_1\rangle=\langle\hat{S}_2\rangle=\langle\hat{S}_3\rangle=0. \tag{2.6}$$

Sometimes the extra requirement that the Stokes parameters are temporally invariant is added to make the definition even more stringent [42]. In any case, this conception has several flaws that have been put forward before and give rise to strange concepts such as that of quantum states with "hidden" polarization [43]. Actually, this notion leads to the paradoxical conclusion that unpolarized light has a polarization structure, which is latent when the mean intensities are measured and detectable when the noise intensities are measured [44].

If, as anticipated in the Introduction, one looks at unpolarized light as field states that remain invariant under any SU(2) polarization transformation, then there is no more any "hidden" polarization. Any state satisfying this invariance condition will also fulfill the classical definition of an unpolarized state, but the converse is not true. It has been shown [18–20] that the density operator of such "quantum" unpolarized states can always be written as

$$\hat{\sigma}=\bigoplus_{N=0}^{\infty}\lambda_N\hat{\mathbb{1}}_N, \tag{2.7}$$

where N denotes the excitation manifold in which there are exactly N photons in the field. All the coefficients λ_N are real and non-negative, and to meet the unit-trace condition of the density operator they must satisfy

$$\sum_{N=0}^{\infty}(N+1)\lambda_N=1. \tag{2.8}$$

In the following, the basis states of the excitation manifold N will be denoted as $|N,k\rangle=|k\rangle_H\otimes|N-k\rangle_V$, $k=0,1,\ldots,N$. These states span a SU(2) invariant subspace of dimension $N+1$, and the generators $\hat{\mathbf{S}}$ act therein according to

$$\hat{S}_+|N,k\rangle=2\sqrt{(k+1)(N-k)}|N,k+1\rangle, \tag{2.9}$$

$$\hat{S}_-|N,k\rangle=2\sqrt{k(N-k+1)}|N,k-1\rangle,$$

$$\hat{S}_3|N,k\rangle=(2k-N)|N,k\rangle,$$

where $\hat{S}_\pm=\hat{S}_1\pm i\hat{S}_2$. These invariant subspaces will play a key role in the following.

III. QUANTUM DEGREE OF POLARIZATION AS A DISTANCE

As we have discussed above, measures of nonclassicality have been defined as the (minimum) distance to an appropriate set representing classical states [23–25]. Similarly, the minimum distance to the (convex) set of separable states has been used to introduce measures of entanglement [26]. In the same vein, we propose to quantify the degree of polarization as

$$\mathbb{P}(\hat{\rho}) \propto \inf_{\hat{\sigma}\in\mathcal{U}} D(\hat{\rho},\hat{\sigma}), \tag{3.1}$$

where $\mathcal{U}$ denotes the set of unpolarized states of the form (2.7) and $D(\hat{\rho},\hat{\sigma})$ is any measure of distance (not necessarily a metric) between the density matrices $\hat{\rho}$ and $\hat{\sigma}$, such that

$\mathbb{P}(\hat{\rho})$ satisfies some requirements motivated by both physical and mathematical concerns. The constant of proportionality in Eq. (3.1) must be chosen in such a way that $\mathbb{P}$ is normalized to unity, i.e., $\sup_{\hat{\rho}}\mathbb{P}(\hat{\rho})=1$.

In Ref. [34] a check list of six simple, physically motivated criteria that should be satisfied by any good measure of distance between quantum processes can be found. For our problem, we impose the following two conditions:

(C1) $\mathbb{P}(\hat{\rho})=0$ iff $\hat{\rho}$ is unpolarized.

(C2) Energy-preserving unitary transformations $\hat{U}_E$ leave $\mathbb{P}(\hat{\rho})$ invariant; that is, $\mathbb{P}(\hat{\rho})=\mathbb{P}(\hat{U}_E\hat{\rho}\hat{U}_E^{\dagger})$.

The first condition is to some extent trivial: it ensures that unpolarized and only unpolarized states have a zero degree of polarization. The second takes into account that the requirement that an unpolarized state is invariant under any SU(2) polarization transformation makes it also invariant under any energy-preserving unitary transformation [45].

It is clear that there are numerous nontrivial choices for $D(\hat{\rho},\hat{\sigma})$ (by nontrivial we mean that the choice is not a simple scale transformation of any other distance). None of them could be said to be more important than any other *a priori*, but the significance of each candidate would have to be seen through physical assumptions. To illustrate this point further, let us take an extreme example fulfilling the previous conditions [26]. Define the discrete distance

$$D_{\text{dis}}(\hat{\rho},\hat{\sigma}) = \begin{cases} 1, & \hat{\rho} \neq \hat{\sigma}, \\ 0, & \hat{\rho} = \hat{\sigma}. \end{cases} \tag{3.2}$$

If the degree of polarization is computed using this distance, we have

$$\mathbb{P}_{\text{dis}}(\hat{\rho}) = \begin{cases} 1, & \hat{\rho} \notin \mathcal{U}, \\ 0, & \hat{\rho} \in \mathcal{U}. \end{cases} \tag{3.3}$$

This therefore tells us only whether or not a given state $\hat{\rho}$ is unpolarized.

There are authors insisting that $D(\hat{\rho},\hat{\sigma})$ is a metric [34]. This requires three additional properties: (i) Positiveness: $D(\hat{\rho},\hat{\sigma})\geqslant 0$ and $D(\hat{\rho},\hat{\sigma})=0$ iff $\hat{\rho}=\hat{\sigma}$. (ii) Symmetry: $D(\hat{\rho},\hat{\sigma})=D(\hat{\sigma},\hat{\rho})$. (iii) Triangle inequality: $D(\hat{\rho},\hat{\tau})\leqslant D(\hat{\rho},\hat{\sigma})+D(\hat{\sigma},\hat{\tau})$. Most distances used in quantum mechanics have these properties, since they are based on an inner product. However, there exist pertinent examples in which D is not a metric. For example, the quantum relative entropy [46–49]

$$S(\hat{\rho}\,\|\,\hat{\sigma}) = \text{Tr}[\hat{\rho}(\ln\hat{\rho} - \ln\hat{\sigma})] \tag{3.4}$$

is not symmetric and does not satisfy the triangle inequality. Nevertheless, it generates a valuable measure of entanglement, and the corresponding degree of polarization satisfies both C1 and C2.

However, for a detailed analysis we will consider the Hilbert-Schmidt metric

$$D_{\text{HS}}(\hat{\rho},\hat{\sigma}) = \|\hat{\rho}-\hat{\sigma}\|_{\text{HS}}^2 = \text{Tr}[(\hat{\rho}-\hat{\sigma})^2], \tag{3.5}$$

which has been previously studied in the contexts of entanglement [50–52]. Since $D_{\text{HS}}(\hat{\rho},\hat{\sigma})$ is a metric, condition C1 is satisfied. It follows from the unitary invariance of the Hilbert-Schmidt metric that also C2 is satisfied.

According to the general strategy stated in the definition (3.1), for a given state $\hat{\rho}$ we should find the unpolarized state $\hat{\sigma}$ that minimizes the distance

$$D_{\text{HS}}(\hat{\rho},\hat{\sigma}) = \text{Tr}(\hat{\rho}^2) + \text{Tr}(\hat{\sigma}^2) - 2\,\text{Tr}(\hat{\rho}\hat{\sigma}). \tag{3.6}$$

If we take into account that the purity of an unpolarized state is

$$\text{Tr}(\hat{\sigma}^2) = \sum_{N=0}^{\infty}(N+1)\lambda_N^2, \tag{3.7}$$

we easily get

$$D_{\text{HS}}(\hat{\rho},\hat{\sigma}) = \text{Tr}(\hat{\rho}^2) + \sum_{N=0}^{\infty}[(N+1)\lambda_N^2 - 2p_N\lambda_N], \tag{3.8}$$

where p_N is the probability distribution of the total number of photons

$$p_N = \sum_{k=0}^{N}\rho_{Nk,Nk}, \tag{3.9}$$

and $\rho_{Nk,N'k'}=\langle N,k|\hat{\rho}|N',k'\rangle$. Now, it is easy to obtain the coefficients λ_N that minimize this distance. The calculation is direct and the result is

$$\lambda_N = \frac{p_N}{N+1}. \tag{3.10}$$

The density operator $\hat{\sigma}_{\text{opt}}\in\mathcal{U}$ with these optimum coefficients λ_N satisfies the constraint (2.8), and hence minimizes the distance (3.6). Note that $\hat{\sigma}_{\text{opt}}$ can be written as

$$\hat{\sigma}_{\text{opt}} = \sum_{N=0}^{\infty}p_N\hat{\sigma}_{\text{opt}}^{(N)}, \tag{3.11}$$

with

$$\hat{\sigma}_{\text{opt}}^{(N)} = \frac{1}{N+1}\sum_{k=0}^{N}|N,k\rangle\langle N,k|. \tag{3.12}$$

With all this in mind, we can define the Hilbert-Schmidt degree of polarization by

$$\mathbb{P}_{\text{HS}}(\hat{\rho}) = \text{Tr}(\hat{\rho}^2) - \sum_{N=0}^{\infty}\frac{p_N^2}{N+1}, \tag{3.13}$$

which is determined not only by the purity $0<\text{Tr}(\hat{\rho}^2)\leqslant 1$ (as it happens for other measures [53]), but also by the distribution of the number of photons p_N. Although the maximum Hilbert-Schmidt distance between two density operators is 2, the minimum distance to an unpolarized state is normalized to unity.

Using Eqs. (3.7) and (3.10), the Hilbert-Schmidt degree of polarization can be recast as

$$\mathbb{P}_{\text{HS}}(\hat{\rho}) = \text{Tr}(\hat{\rho}^2) - \text{Tr}(\hat{\sigma}_{\text{opt}}^2), \tag{3.14}$$

which makes it easy to verify that it vanishes only for unpolarized states, in agreement with condition C1.

It has been shown [50–52] that the Hilbert-Schmidt distance is not monotonically decreasing under every completely positive trace-preserving map (what is called the CP nonexpansive property). This has motivated the quantum information community to identify the fidelity as a particularly important alternative approach to the definition of a distance measure for states [54].

In consequence, as our second candidate of distance we will employ the fidelity (or Uhlmann transition probability) [55]

$$F(\hat{\rho},\hat{\sigma}) = [\mathrm{Tr}(\hat{\sigma}^{1/2}\hat{\rho}\hat{\sigma}^{1/2})^{1/2}]^2. \tag{3.15}$$

A word of caution is necessary here. There is an ambiguity in the literature: both the quantity (3.15) and its square root have been referred to as the fidelity. The reader should take this into account when comparing different sources.

The fidelity has many attractive properties. First, it is symmetric in its arguments $F(\hat{\rho},\hat{\sigma})=F(\hat{\sigma},\hat{\rho})$, a fact that is not obvious from Eq. (3.15), but which follows from other equivalent expressions. It can also be shown that $0 \leqslant F(\hat{\rho},\hat{\sigma}) \leqslant 1$, with equality in the second inequality iff $\hat{\rho} = \hat{\sigma}$. This means that the fidelity is not a metric as such, but serves rather as a generalized measure of the overlap between two quantum states. A common way of turning it into a metric is through the Bures metric,

$$D_{\mathrm{B}}(\hat{\rho},\hat{\sigma}) = 2[1-\sqrt{F(\hat{\rho},\hat{\sigma})}]. \tag{3.16}$$

The origin of this distance can be seen intuitively by considering the case where $\hat{\rho}$ and $\hat{\sigma}$ are both pure states. The Bures metric is just the Euclidean distance between the two pure states, with respect to the usual norm on the state space.

Since the larger the fidelity $F(\hat{\rho},\hat{\sigma})$, the smaller the Bures distance $D_{\mathrm{B}}(\hat{\rho},\hat{\sigma})$, we can define the Bures degree of polarization as

$$\mathbb{P}_{\mathrm{B}}(\hat{\rho}) = 1 - \sup_{\hat{\sigma}\in\mathcal{U}} \sqrt{F(\hat{\rho},\hat{\sigma})}. \tag{3.17}$$

An alternative definition would be $1-\sup_{\hat{\sigma}\in\mathcal{U}} F(\hat{\rho},\hat{\sigma})$, which arises naturally in the context of quantum computation [34]. It is clear that these definitions order the states $\hat{\rho}$ in the same way. Unfortunately, we have not found a general expression of the unpolarized state $\hat{\sigma}$ that gives the maximum fidelity. Such a task must be performed case by case and will be illustrated with some selected examples in the next section. Let us conclude this section by noting that the fidelity can be used to define a measure of entanglement [26], whereas this is not the case with the Hilbert-Schmidt metric [51]. Since the unpolarized states are separable, $\mathbb{P}_{\mathrm{B}}(\hat{\rho})$ thus gives an upper bound on the entanglement of $\hat{\rho}$.

IV. EXAMPLES

It is clear from Eq. (3.13) that all pure N-photon states have the same Hilbert-Schmidt degree of polarization. For such states, denoted $|\Psi^{(N)}\rangle$, we have

$$\mathbb{P}_{\mathrm{HS}}(|\Psi^{(N)}\rangle) = \frac{N}{N+1}. \tag{4.1}$$

The Bures degree of polarization for these states can also be readily found:

$$\mathbb{P}_{\mathrm{B}}(|\Psi^{(N)}\rangle) = 1 - \frac{1}{\sqrt{N+1}}. \tag{4.2}$$

The vacuum is the only unpolarized state, in agreement with condition C1. Note also that the expressions (4.1) and (4.2) apply, e.g., to the states $|n\rangle_H \otimes |n\rangle_V$. Since for them $\langle\hat{\mathbf{S}}\rangle=0$, classically they would be unpolarized for every n (that is, $\mathcal{P}_{\mathrm{sc}}=0$, even in the limit $n \gg 1$). In our distance-based approach, the degree of polarization is a function of all moments of the Stokes operators and not only of the first one, as it happens for $\mathcal{P}_{\mathrm{sc}}$, which causes this quite different behavior. We also observe that all these states lying in the $(N+1)$-dimensional invariant subspace satisfy $\mathbb{P} \to 1$ when their intensity is increased.

Next, we define the diagonal states as those that can be expressed as

$$\hat{\rho}_{\mathrm{diag}} = \sum_{N=0}^{\infty}\sum_{k=0}^{N} p_{Nk} |\Psi_k^{(N)}\rangle\langle\Psi_k^{(N)}|, \tag{4.3}$$

where we let $p_{Nk} \geqslant p_{Nk+1}$, for all $k<N$, and $\{|\Psi_k^{(N)}\rangle\}_{k=0}^{N}$ is an arbitrary orthonormal basis in the excitation manifold N. It then follows from C2 that any two diagonal states whose probability distribution $\{p_{Nk}\}_{k=0}^{N}$ coincide, must have the same degree of polarization. For any diagonal state, we have

$$\mathbb{P}_{\mathrm{HS}}(\hat{\rho}_{\mathrm{diag}}) = \sum_{N=0}^{\infty}\sum_{k=0}^{N} p_{Nk}^2 - \sum_{N=0}^{\infty} \frac{p_N^2}{N+1} \leqslant \sum_{N=0}^{\infty} \frac{Np_N^2}{N+1}. \tag{4.4}$$

To deal with the Bures degree of polarization for this example, we first note that

$$\sqrt{F(\hat{\rho}_{\mathrm{diag}},\hat{\sigma})} = \sum_{N=0}^{\infty}\sum_{k=0}^{N} \sqrt{\lambda_N p_{Nk}} = \sum_{N=0}^{\infty} s_N\sqrt{\lambda_N}, \tag{4.5}$$

where

$$s_N = \sum_{k=0}^{N} \sqrt{p_{Nk}}. \tag{4.6}$$

The extremal points of Eq. (4.5) are then determined by

$$\frac{s_N}{2\sqrt{\lambda_N}} - \mu(N+1) = 0, \tag{4.7}$$

where μ is a Lagrange multiplier that takes into account the constraint (2.8). Solving for λ_N and imposing again Eq. (2.8) to fix the value of μ, the optimum parameters λ_N are found to be

$$\lambda_N = \frac{s_N^2}{(N+1)^2 \sum_{k=0}^{\infty} \frac{s_k^2}{k+1}}. \tag{4.8}$$

In this way, we finally arrive at

$$\mathbb{P}_{\mathrm{B}}(\hat{\rho}_{\mathrm{diag}}) = 1 - \sqrt{\sum_{N=0}^{\infty} \frac{s_N^2}{N+1}}. \tag{4.9}$$

One can easily prove that

$$\sqrt{p_N} \leqslant s_N \leqslant \sqrt{(N+1)p_N}, \tag{4.10}$$

so we have the bound

$$\mathbb{P}_{\mathrm{B}}(\hat{\rho}_{\mathrm{diag}}) \leqslant 1 - \sqrt{\sum_{N=0}^{\infty} \frac{p_N}{N+1}} < 1. \tag{4.11}$$

This and Eq. (3.13) show that $\Sigma_{N=0}^{\infty} p_N^2 (N+1)^{-1} \to 0$ is necessary for both $\mathbb{P}_{\mathrm{B}}(\hat{\rho}_{\mathrm{diag}})$ and $\mathbb{P}_{\mathrm{HS}}(\hat{\rho})$ to approach unity. The latter also requires the purity to approach unity, whereas it is clear from Eq. (4.11) that this is not necessary in order to have $\mathbb{P}_{\mathrm{B}}(\hat{\rho}) \to 1$.

As another relevant example, let us consider the case in which both modes are in (quadrature) coherent states. The product of two quadrature coherent states, which we shall denote by $|\alpha_H, \alpha_V\rangle$, can be expressed as a Poissonian superposition of SU(2) coherent states [56],

$$|\alpha_H, \alpha_V\rangle = \sum_{N=0}^{\infty} e^{-\bar{N}/2} \frac{\bar{N}^{N/2}}{\sqrt{N!}} |N, \theta, \phi\rangle, \tag{4.12}$$

where $\bar{N} = |\alpha_H|^2 + |\alpha_V|^2$ is the average number of excitations and the SU(2) coherent states are defined as [57]

$$|N, \theta, \phi\rangle = \sum_{k=0}^{N} \binom{N}{k}^{1/2} \left(\sin\frac{\theta}{2}\right)^{N-k} \left(\cos\frac{\theta}{2}\right)^{k} e^{-ik\phi} |N, k\rangle, \tag{4.13}$$

and the state parameters are connected by the relations

$$\alpha_H = e^{-i\phi/2} \sqrt{\bar{N}} \sin\frac{\theta}{2}, \quad \alpha_V = e^{i\phi/2} \sqrt{\bar{N}} \cos\frac{\theta}{2}. \tag{4.14}$$

Taking into account that

$$\sum_{N=0}^{\infty} \frac{p_N^2}{N+1} = \frac{I_1(2\bar{N})}{\bar{N}} e^{-2\bar{N}}, \tag{4.15}$$

where $I_1(z)$ is the modified Bessel function, Eq. (3.13) reduces to

$$\mathbb{P}_{\mathrm{HS}} = 1 - \frac{I_1(2\bar{N})}{\bar{N}} e^{-2\bar{N}}. \tag{4.16}$$

When $\bar{N} \gg 1$ we can retain the first term in the asymptotic expansion of $I_1(z)$ to obtain

$$\mathbb{P}_{\mathrm{HS}} \simeq 1 - \frac{1}{2\sqrt{\pi}\bar{N}^{3/2}}. \tag{4.17}$$

This tends again to unity. However, one may have expected a $\bar{N}^{-1}$ behavior for coherent states, while the scaling is $\bar{N}^{-3/2}$.

One can ask if the Hilbert-Schmidt and Bures measures order some pairs of states differently. In the Appendix we show that this is indeed the case, and the induced degrees of polarization are therefore fundamentally different.

V. CONCLUSION

In summary, we have shown that quantum optics entails polarization states that cannot be suitably described by the (semi)classical formalism based on Stokes parameters. We have advocated the use of a degree of polarization based on an appropriate distance to the set of unpolarized states. Such a definition is closely related to other recent proposals in different areas of quantum optics and is well behaved even when the classical formalism fails.

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APPENDIX: PROOF OF THE DEFINITIONS' DIFFERENT ORDERING OF STATES

In this Appendix, we will show that the Hilbert-Schmidt and the Bures distances induce fundamentally different degrees of polarization. To this end, we consider the states

$$\hat{\rho}_{N_1 N_2} = \sum_{j,k=1}^{2} \rho_{jk} |\Psi^{(N_j)}\rangle\langle\Psi^{(N_k)}|, \tag{A1}$$

where $|\Psi^{(N_1)}\rangle$ and $|\Psi^{(N_2)}\rangle$ are orthogonal pure states with N_1 and N_2 photons, respectively. We here assume that $N_1 \neq N_2$, and note that $\hat{\rho}_{NN}$ is a diagonal state of the form (4.3). To simplify calculations, we shall use the notation

$$p = \rho_{11}, \quad 1 - p = \rho_{22}, \quad q = \rho_{12} = \rho_{21}^{*}. \tag{A2}$$

The states $|\Psi^{(N_1)}\rangle$ and $|\Psi^{(N_2)}\rangle$ then correspond to $p=0$ and $p=1$, respectively, and the purity becomes

$$\mathrm{Tr}(\hat{\rho}_{N_1 N_2}^2) = p^2 + (1-p)^2 + 2|q|^2. \tag{A3}$$

We note in passing that $1 - 2p(1-p) \leqslant \mathrm{Tr}(\hat{\rho}_{N_1 N_2}^2) \leqslant 1$ and $0 \leqslant |q|^2 \leqslant p(1-p)$.

In the basis $(|\Psi^{(N_1)}\rangle, |\Psi^{(N_2)}\rangle)$, we can write

$$\hat{\sigma}^{1/2} \hat{\rho} \hat{\sigma}^{1/2} = \begin{pmatrix} \lambda_{N_1} p & \sqrt{\lambda_{N_1} \lambda_{N_2}} q \\ \sqrt{\lambda_{N_1} \lambda_{N_2}} q^* & \lambda_{N_2}(1-p) \end{pmatrix}. \tag{A4}$$

Since the eigenvalues of this matrix are

$$\chi_{\pm} = \frac{1}{2}\{\lambda_{N_1} p + \lambda_{N_2}(1-p) \pm \sqrt{[\lambda_{N_1} p - \lambda_{N_2}(1-p)]^2 + 4\lambda_{N_1}\lambda_{N_2}|q|^2}\}, \tag{A5}$$

the fidelity can be expressed as

$$F(\hat{\rho}, \hat{\sigma}) = \chi_+ + 2\sqrt{\chi_+ \chi_-} + \chi_- = \lambda_{N_1} p + \lambda_{N_2}(1-p) + 2\sqrt{\lambda_{N_1}\lambda_{N_2}[p(1-p) - |q|^2]}. \tag{A6}$$

For any fixed λ_{N_1}, λ_{N_2}, and p, the fidelity decreases as $|q|^2$ increases. This could have been expected, since the unpolarized states do not have any off-diagonal elements.

The restriction (2.8) implies for this problem that

$$\lambda_{N_2} = \frac{1-(N_1+1)\lambda_{N_1}}{N_2+1}. \tag{A7}$$

In consequence, the coefficients that optimize the fidelity are determined by

$$\frac{\partial F}{\partial \lambda_{N_1}} = 0 = p - \frac{(1+N_1)(1-p)}{1+N_2} + [1-2\lambda_{N_1}(1+N_1)]\sqrt{\frac{p(1-p)-|q|^2}{\lambda_{N_1}[1-\lambda_{N_1}(1+N_1)](1+N_2)}}. \tag{A8}$$

We first consider pure states, for which $|q|^2=p(1-p)$. Choosing λ_{N_1} according to

$$\lambda_{N_1}=0, \quad p<\frac{1+N_1}{2+N_1+N_2},$$

$$0 \leqslant \lambda_{N_1} \leqslant \frac{1}{1+N_1}, \quad p=\frac{1+N_1}{2+N_1+N_2},$$

$$\lambda_{N_1}=\frac{1}{1+N_1}, \quad p>\frac{1+N_1}{2+N_1+N_2}, \tag{A9}$$

then maximizes the fidelity:

$$\sup_{\hat{\sigma}\in\mathcal{U}} F(\hat{\rho},\hat{\sigma}) = \begin{cases} \frac{1-p}{1+N_2}, & p \leqslant \frac{1+N_1}{2+N_1+N_2}, \\ \frac{p}{1+N_1}, & p \geqslant \frac{1+N_1}{2+N_1+N_2}. \end{cases} \tag{A10}$$

On the other hand, when $|q|^2 \neq p(1-p)$, i.e., when $0<p<1$, the solution of Eq. (A8) is

$$\lambda_{N_1} = \frac{1}{2(N_1+1)}\left(1 - \frac{(1+N_1)(1-p)-(1+N_2)p}{\sqrt{[1+N_1(1-p)+N_2 p]^2-4(1+N_1)(1+N_2)|q|^2}}\right). \tag{A11}$$

Depending on the parameters, this solution can take any value in the interval $0<\lambda_{N_1}<1/(1+N_1)$. In fact, one can check that the choice (A11) gives the closest unpolarized state. Combining Eqs. (A6) and (A7), and (A11) thus allows one to obtain the fidelity and hence $\mathbb{P}_{\mathrm{B}}$.

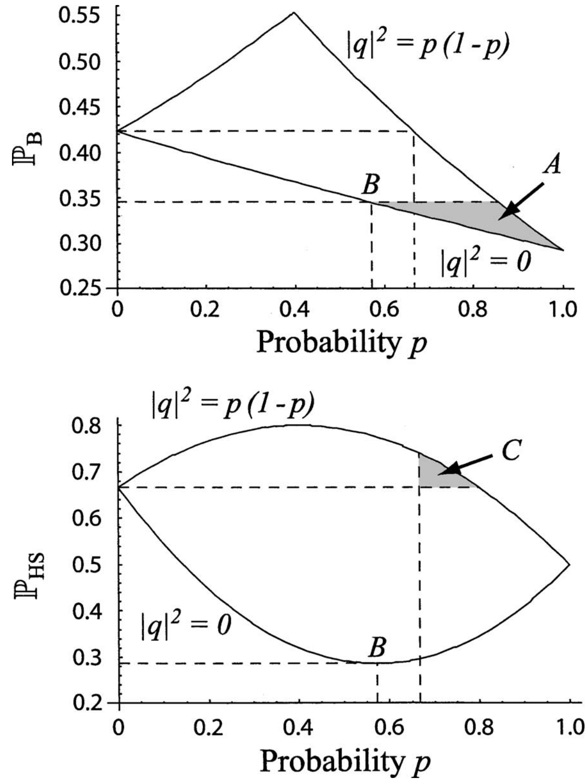

FIG. 1. Polarization degrees for two-dimensional states with $N_1=1$ and $N_2=2$. For any given p, the maximum and minimum fidelities are given by $|q|^2=p(1-p)$ and $|q|^2=0$, respectively. Region A corresponds to states satisfying $p>4/7$ and $\mathbb{P}_{\mathrm{B}}<1-\sqrt{3/7}$. For any state $\hat{\rho}_A$ in this region, we have $\mathbb{P}_{\mathrm{B}}(\hat{\rho}_A)<\mathbb{P}_{\mathrm{B}}(\hat{\rho}_B)$ while $\mathbb{P}_{\mathrm{HS}}(\hat{\rho}_A)>\mathbb{P}_{\mathrm{HS}}(\hat{\rho}_B)$, where the state $\hat{\rho}_B$ is characterized by $p=4/7$ and $|q|^2=0$. In region C, the states satisfy $p>2/3$ and $\mathbb{P}_{\mathrm{HS}}>2/3$. For any such state $\hat{\rho}_C$, we have $\mathbb{P}_{\mathrm{HS}}(\hat{\rho}_C)>\mathbb{P}_{\mathrm{HS}}(|\Psi^{(2)}\rangle)$ and $\mathbb{P}_{\mathrm{B}}(\hat{\rho}_C)<\mathbb{P}_{\mathrm{B}}(|\Psi^{(2)}\rangle)$, where $|\Psi^{(2)}\rangle$ is the (arbitrary) pure two-photon state corresponding to $p=0$.

In Fig. 1, we have plotted the Hilbert-Schmidt and Bures degree of polarization for some two-dimensional states. From the explanation in the caption, we see that the two measures order some pairs of states differently. The Hilbert-Schmidt and Bures distances thus induce two fundamentally different degrees of polarization.

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