Symbiotic solitons in heteronuclear multicomponent Bose-Einstein condensates

Víctor M. Pérez-García and Juan Belmonte Beitia
Departamento de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales, Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain

(Received 15 March 2005; published 30 September 2005)

We show that bright solitons exist in quasi-one-dimensional heteronuclear multicomponent Bose-Einstein condensates with repulsive self-interaction and attractive interspecies interaction. They are remarkably robust to perturbations of initial data and collisions and can be generated by the mechanism of modulational instability. Some possibilities for control and the behavior of the system in fully three-dimensional scenarios are also discussed.

DOI: 10.1103/PhysRevA.72.033620 PACS number(s): 03.75.Lm, 03.75.Kk

I. INTRODUCTION

Symbiosis is an assemblage of distinct organisms living together. Although the original definition of symbiosis by de Bary [1] did not include a judgment on whether the partners benefit or harm each other, currently most people use the term symbiosis to describe interactions from which both partners benefit.

In physics, waves in dispersive linear media tend to expand due to the different velocities at which the wave components propagate. This is not the case in many nonlinear media, in which certain wave packets, called *solitons*, are able to propagate undistorted due to the balance between dispersion and nonlinearity [2].

Stable solitons of different subsystems are sometimes able to "live together" and form stable complexes called vector solitons as happens with Manakov optical solitons [3,4] or stabilized vector solitons [5]. In some cases, a (large) robust soliton can be used to stabilize a (small) weakly unstable wave [6].

Multicomponent solitary waves also appear in Bose-Einstein condensates (BEC's). In fact, multicomponent BEC's support nonlinear waves which do not exist in single-component BEC's such as domain-wall solitons [7,8], dark-bright solitons [9], etc. Most of the previous analyses correspond to homonuclear multicomponent condensates for which the atom-atom interactions are repulsive. However, heteronuclear condensates offer a wider range of possibilities, the main one being the possibility of having a negative interspecies scattering length. This possibility has been theoretically explored in the context of Feschbach resonance management [10] and realized experimentally for boson-fermion mixtures [11,12].

In this paper we study the existence and properties of bright solitons in heteronuclear two-component BEC's with scattering lengths $a_{11},a_{22}>0$ and $a_{12}<0$. We would like to stress the fact that these coefficient combinations do not arise in other systems where similar model equations are used. For instance, in nonlinear optics, where the nonlinear Schrödinger equations used to describe the propagation of laser beams in nonlinear media are similar to the mean-field equations used to describe Bose-Einstein condensates, the nonlinear coefficients are always of the same sign. The closest analogy could happen in the so-called QPM (quasi-phase-matched) quadratically nonlinear media, where an *effective* cubic nonlinearity could be "engineered" which could have similar properties, but we do not know of any systematic studies of those systems.

Our analysis will show novel features with respect to those already found in single species BEC's [13]. For instance, even when solitons do not exist for each of the species, the coupling leads to robust vector solitons. Since the mutual cooperation between these structures is essential for their existence, we will refer to these solitons hereafter as *symbiotic solitons*. We also show how they appear by modulational instability and study some features of their collisions. We also comment on the possibility of obtaining these structures in multidimensional configurations.

II. MODEL AND ITS BASIC PROPERTIES

In this paper we will study two-component BEC's in the limit of strong transverse confinement ruled by [14]

$$i\frac{\partial u_1}{\partial t} = -\frac{1}{2}\frac{\partial^2 u_1}{\partial x^2} + (g_{11}|u_1|^2 + g_{12}|u_2|^2)u_1, \tag{1a}$$

$$i\frac{\partial u_2}{\partial t} = -\frac{\alpha}{2}\frac{\partial^2 u_2}{\partial x^2} + (g_{21}|u_1|^2 + g_{22}|u_2|^2)u_2, \tag{1b}$$

where x is the adimensional longitudinal spatial variable measured in units of $a_0=\sqrt{\hbar/m_1\omega_\perp}$, t is the time measured in terms of $1/\omega_\perp$, and $u_j(\mathbf{x},t)\equiv u_j(\mathbf{r},\tau)\sqrt{a_0^3}$. The dimensional reduction leads to [14] $g_{ij}=2a_{ij}\alpha^{i+j-2}/a_0$, with $\alpha=m_1/m_2$ and a_{ij} being the s-wave scattering lengths. The normalization for u_j is $\int|u_j|^2d^3x=N_j$ where N_j is the number of particles of each species.

Let us first consider constant amplitude solutions of Eq. (1), which are of the form

$$\phi_j(z,t) = A_j e^{i\beta_j t}, \tag{2a}$$

$$\beta_j = g_{jj}|A_j|^2 + g_{j,3-j}|A_{3-j}|^2, \tag{2b}$$

for $j=1,2$. We will study the evolution of small perturbations of ϕ_j of the form

$$u_j(z,t) = (A_j + \delta A_j(z,t))e^{i[\beta_j t+\delta\beta_j(z,t)]}. \tag{3}$$

Using Eq. (1) and retaining the first-order terms we get partial differential equations for $\delta A_1,\delta\beta_1,\delta A_2,\delta\beta_2$ which can be transformed to Fourier space to obtain

$$\delta A_j(z,t) = \int_{\mathbb{R}} a_0(k) e^{ikx} e^{\Omega(k)t} dk, \tag{4}$$

$a_0(k)$ being the Fourier transform of the initial perturbation. Perturbations remain bounded if $\mathrm{Re}[\Omega(k)] \leqslant 0$. Some algebra leads to

$$\Omega^2 = \frac{1}{2}[f_1 + f_2 \pm \sqrt{(f_1 - f_2)^2 + 4C^2}], \tag{5}$$

where $f_j = -(g_{jj}A_j^2 + k^2/4)k^2$ and $C^2 = A_1^2 A_2^2 g_{12}^2 k^4$. The so-called modulational instability (MI) occurs when $\Omega(k)^2 > 0$ for any k. For small wave numbers (worst situation) we get

$$g_{12}^2 > g_{11} g_{22}, \tag{6}$$

which is analogous to the miscibility criterion for two-component condensates [8]. However, the physical meaning of Eq. (6) is very different since now this instability is a signature of the tendency to form coupled objects between both atomic species. The role of MI in the formation of soliton trains and domains in BEC has been recognized in previous papers [8,13,15,16].

III. VECTOR SOLITONS

Equations (1) have sech-type solutions

$$u_j(x,t) = \left(\frac{N_j}{2\omega}\right)^{1/2} \mathrm{sech}\left(\frac{x}{\omega}\right) e^{i\lambda_j t}, \tag{7}$$

with $\lambda_1 = 1/(2\omega^2)$, $\lambda_2 = \alpha/(2\omega^2)$, and $\omega = 2/(-g_{11}N_1 - g_{12}N_2)$, provided the restriction

$$g_{12}(m_1 N_2 - m_2 N_1) = m_2 g_{22} N_2 - m_1 g_{11} N_1 \tag{8}$$

and the MI condition (6) are satisfied. Equation (8) implies that, given the number of particles in one component, the other is fixed.

Since the self-interaction coefficients are positive, these solitons are supported only by the mutual attractive interaction between both components. This type of vector soliton thus differs from others described for nonlinear Schrödinger equations of the form of Eqs. (1), such as the Manakov solitons [3], where all the nonlinear coefficients cooperate to form the solitonic solution.

The MI condition (6) implies that the formation of these solitons has a threshold in g_{12} and means that the cross interaction must be strong enough to be able to overcome the self-repulsion of each atomic cloud. There are no analogs to this condition in single-component systems since solitons exist for any value of the self-interaction coefficient $g < 0$. To fix ideas, taking a ^{87}Rb—^{41}K mixture with $a_{11} = 69a_0$ and $a_{22} = 99a_0$ the MI condition implies that $a_{12} < -83a_0$ in order to obtain solitons. In Fig. 1(a) it can be seen how the ratio N_2/N_1 is close to 0.4 in the range of values of $-83a_0 > a_{12} > -150a_0$. A hypothetical ^{7}Li—^{23}Na mixture with $a_{11} = 5a_0$ and $a_{22} = 52a_0$ (in appropriate quantum states) leads to the curve in Fig. 1(b), which shows a much larger range of variation.

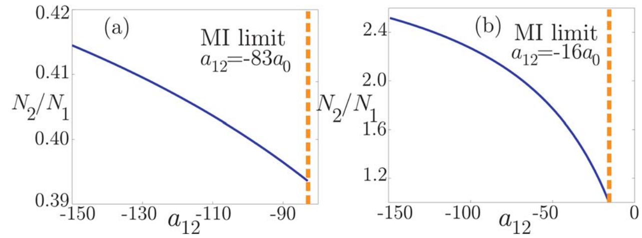

FIG. 1. (Color online) Dependence of the ratio N_2/N_1 of sech-type vector solitons on the interspecies scattering length a_{12}. (a) A ^{87}Rb—^{41}K mixture with $a_{11} = 69a_0$, $a_{22} = 99a_0$. (b) An hypotetical ^{7}Li—^{23}Na mixture with $a_{11} = 5a_0$ and $a_{22} = 52a_0$.

IV. SOLITON STABILITY

We can use the Vakhitov-Kolokov (VK) criterion to study the stability of solitons given by Eq. (7). To do this, we must study the sign of $\partial\lambda_j/\partial N_j$. For soliton solutions this can be done from the explicit form of λ_j. After some algebra we find $\lambda_1(N_1)$ and $\lambda_2(N_2)$ and obtain that $\partial\lambda_1/\partial N_1 > 0$, and $\partial\lambda_2/\partial N_2 > 0$ in all their range of existence, which *proves the linear stability* of the solitons for small perturbations and contradicts the naive intuition that the self-repulsion would lead to intrinsically unstable wavepackets.

We have studied numerically the robustness of symbiotic solitons to finite-amplitude perturbations. First we have perturbed both solutions with small-amplitude noise and found that, in agreement with the predictions of the VK criterion, they survive after the emission of the noise in the form of radiation. Next we have applied a stronger perturbation consisting of displacing mutually their centers and observe that a soliton is formed even for relative displacements of the order of the soliton size (Fig. 2). Finally we have started with sech-type initial data which are not solitons and observe that after the emission of some radiation solitons are formed.

V. GENERATION OF SYMBIOTIC SOLITONS BY MI

To study the generation of these solitons by MI in realistic systems we have considered a multicomponent Bose-Einstein condensate of ^{87}Rb and ^{41}K atoms for which the

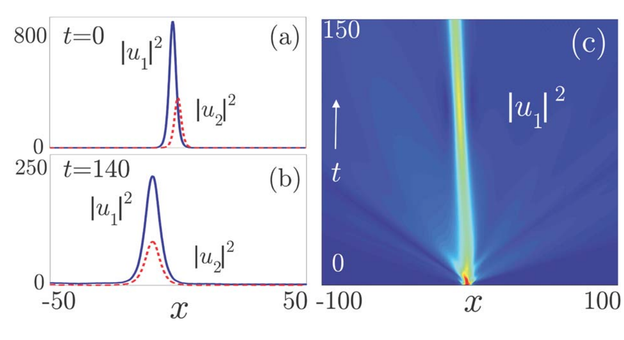

FIG. 2. (Color online) Evolution of displaced soliton initial data of the form $u_1 = [N_1/(2w)]^{1/2}\,\mathrm{sech}[(x+x_0)/w]$, $u_2 = [N_2/(2w)]^{1/2}\,\mathrm{sech}(x/w)$ for a ^{87}Rb—^{41}K mixture with $N_1 = 3000$, $N_2 = 1189$, $a_{11} = 69a_0$, $a_{12} = -90a_0$, and $a_{22} = 99a_0$ in a trap with $\omega_\perp = 215$ Hz.

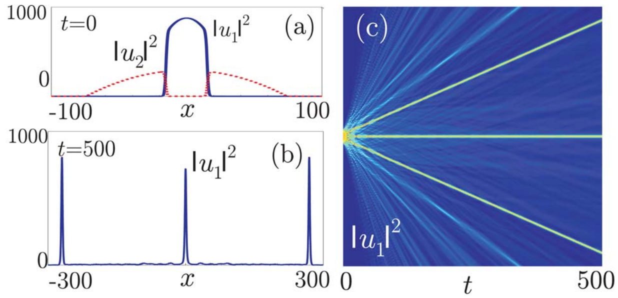

FIG. 3. (Color online) Evolution of the ground state of a ^{87}Rb—^{41}K mixture with $N_1=25\,000$ and $N_2=20\,000$ after switching the interspecies scattering length from $a_{12}=95a_0$ to $a_{12}=-90a_0$. (a) Initial state: $|u_1|^2$ (blue dotted line) and $|u_2|^2$ (red solid line). (b) Profile of $|u_1|^2$ for $t=500$ showing three remaining solitons. (c) Pseudocolor plot of $|u_1|^2$ for $x\in[-500,500]$ and $t\in[0,500]$.

interspecies scattering length a_{12} is controlled by the use of Feschbach resonances as proposed in [10]. To simplify the problem here we do not consider the effect of gravity.

We start by constructing the ground state of the system for an elongated trap typical of the LENS setup [19,22] with $\omega_\perp=215$ Hz and $\omega=16.3$ Hz. For these atomic species $a_{11}=69a_0$ and $a_{22}=99a_0$. We adjust the interspecies scattering length to $a_{12}=95a_0$ during the condensation process. The ground state of this system for $N_1=25\,000$ and $N_2=20\,000$, shown in Fig. 3(a), agrees well with the theoretical predictions for these systems [20].

After the condensate is formed we change instantaneously this quantity to a negative value and at the same time switch off the longitudinal trapping potential and observe numerically the evolution of the ground state.

First we choose $a_{12}=-90a_0$ and observe the evolution starting from the ground state with $a_{12}=95$. Since the intercomponent repulsive force is not present now, the sharp domain wall separating both species [see Fig. 3(a)] decays through a highly oscillatory process related to the formation of a shock wave [21]. The final outcome is the formation of a soliton train [see Figs. 3(b) and 3(c)] of which three solitons of about 20 μm size and each with about 3000 rubidium and 1200 potassium atoms remain in our simulation domain after 500 adimensional time units [Fig. 3(c)]. Other smaller and wider solitons exit our integration region traveling at a faster speed.

The final number of solitons depends on the value of a_{12} chosen during the condensation process (which controls the overlapping of the species) and the number of particles, N_1 and N_2, and the negative scattering length a_{12} chosen to destabilize the system. For instance, choosing $a_{12}=-70a_0$, which is below the theoretical limit for MI the evolution of the wave packet is purely dispersive [see Fig. 4(a)]. Choosing $a_{12}=-87a_0$, above the MI limit but below the choice of Fig. 3 leads to the formation of a single soliton [Fig. 4(b)]. It seems that the larger the scattering length, the larger the number of solitons which arise after the decay of the initial configuration. The many degrees of freedom present in these systems open many posibilities for controlling the number and sizes of solitons by appropriately choosing the values of a_{12} before and after the condensate is released and the initial number of particles, N_1 and N_2.

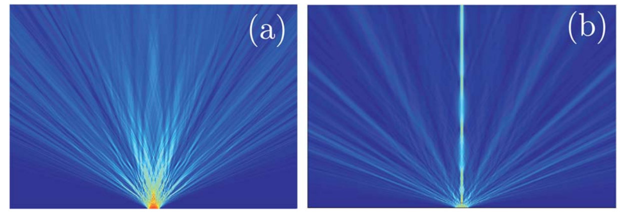

FIG. 4. (Color online) Pseudocolor plots of the evolution of the ground state of a ^{87}Rb—^{41}K mixture with $N_1=25\,000$ and $N_2=20\,000$ after switching the interspecies scattering length from $a_{12}=95a_0$ to (a) $a_{12}=-70a_0$ (below the MI limit) and (b) $a_{12}=-87a_0$ (slightly above the MI limit $a_*=-83a_0$).

VI. COLLISIONS OF SYMBIOTIC SOLITONS

The robustness of symbiotic solitons manifests also in their collisional behavior; however, now the fact that they have an internal structure makes the interaction of these vector solitons very rich. Since each soliton is a compound object, the collisions are partially a coherent phenomenon because of the direct overlapping of the same type of atoms and an incoherent one because of the incoherent nature of interaction between different types of atoms. A related subject of recent interest in optics is that of partially coherent solitons [23].

It is not our intention in this paper to make a detailed and systematic study of the collisions of symbiotic solitons but just to present a few examples and to show that in many situations the symbiotic solitons behave robustly during collisions.

We have simulated head-on collisions of equal symbiotic solitons of opposite velocities given by

$$u_j=\sqrt{\frac{N_j}{2w}}\,\mathrm{sech}\left(\frac{x+x_0}{w}\right)e^{iv\sqrt{m_j}x+i\alpha_{j,+}} + \sqrt{\frac{N_j}{2w}}\,\mathrm{sech}\left(\frac{x-x_0}{w}\right)e^{-iv\sqrt{m_j}x+i\alpha_{j,-}} \quad (9)$$

for $j=1,2$. $\alpha_{j,\pm}$ are the relative phases, and N_2 is given by Eq. (8). In all the simulations shown here we have chosen $N_1=3000$, $N_2=1189$, $w=1.723$, $a_{11}=69a_0$, $a_{12}=-90a_0$, and $a_{22}=99a_0$ which means that the symbiotic solitons which will collide are composed of a "large" soliton in component 1 and a smaller one in the second component. The larger component is also wider, and thus when these objects collide the interaction is first dominated by the collisions of the outer parts, there being a smaller influence of the smaller (inner) soliton in component 2.

In Fig. 5 we show some examples of these collisions in which the relative phases of the solitons are shown to play a crucial role as happens in collisions of solitons of the scalar cubic nonlinear Schrödinger equation. In Fig. 5(a) we show the collision of two symbiotic solitons with equal phases and slow speeds which leads to the formation of an oscillating bound state. This state is a very interesting object since it

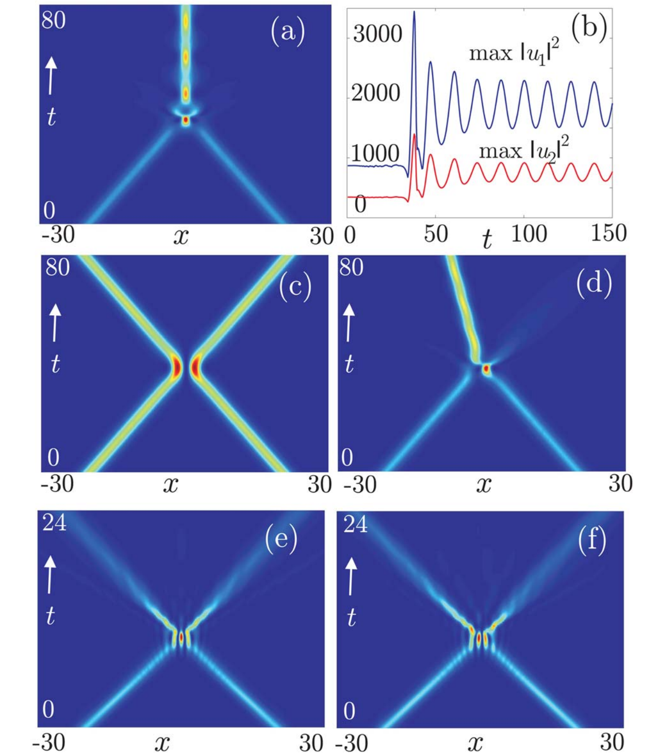

FIG. 5. (Color online) Head-on collisions of symbiotic solitons with $N_1=3000$, $N_2=1189$, $w=1.723$, $a_{11}=69a_0$, $a_{12}=-90a_0$, and $a_{22}=99a_0$. (a)–(d) Slow collisions for $v=0.05$ and (a), (b) $\boldsymbol{\theta}\equiv(\alpha_{1,+},\alpha_{1,-},\alpha_{2,+},\alpha_{2,-})=(0,0,0,0)$, (c) $\boldsymbol{\theta}=(0,\pi,0,0)$, and (d) $\boldsymbol{\theta}=(\pi/2,0,0,0)$. Moderate speed collisions (e) $v=0.2$, $\boldsymbol{\theta}=(0,0,0,0)$ and (f) $v=0.2$, $\boldsymbol{\theta}=(\pi/2,0,0,0)$.

seems to perform undamped oscillations for very long times [Fig. 5(b)], and it could correspond to a stable breatherlike solution of our model.

The picture changes radically in Fig. 5(c) in which the phase difference between the components is set to π, specifically $(\alpha_{1,+},\alpha_{1,-},\alpha_{2,+},\alpha_{2,-})=(0,\pi,\pi,0)$. This choice leads to mutual repulsion between the solitons also following the pattern of interactions of scalar solitons. Setting $\boldsymbol{\theta}=(0,0,\pi,0)$ (not shown in the figure) leads also to mutual repulsion but the outgoing speeds are slightly reduced due to the attractive interaction between the solitons in the second component.

Setting the phase difference between the larger components to be $\pi/2$ we obtain an energy transfer between both solitons and some emission of radiation, leading to an outgoing larger soliton with nonzero speed [Fig. 5(d)].

Moderate speed collisions [Figs. 5(e) and 5(f)] still lead to bound solitons while for larger speeds the picture is not so clear. In the former cases we see that after a transient in which a higher-order soliton is formed (shown as interference fringes in the plots) they decay into more stable symbiotic vector solitons. In the two moderate-speed collisions shown here the outgoing solitons are wider than the incoming ones. The collisional behavior in moderate speed collisions seems to be less affected by the phase differences as shown in Figs. 5(e) and 5(f)). In fact, the major difference between both pictures is the energy transfer between the solitons which can be seen in Fig. 5(f); anyway, the effect is not as dramatic as in Fig. 5(d) in which an almost complete energy transfer was observed.

VII. PROSPECTS FOR MULTIDIMENSIONAL SYMBIOTIC SOLITONS

A very interesting question arising naturally is, do these symbiotic solitons exist in multidimensional scenarios? In principle the answer is not evident since the only effect acting against stabilization of multidimensional soliton structures would be collapse, but one could think that in this case collapse could be inhibited because of the *repulsive* self-interaction; thus, a deeper analysis is in order.

The adimensional model equations in two and three dimensions take the form

$$i\frac{\partial u_j}{\partial t}=\left(-\frac{1}{2m_j}\Delta+V_j+g_{j,j}|u_j|^2+g_{j,k}|u_k|^2\right)u_j, \qquad (10)$$

with $j=1,2$ and $k=2,1$ correspondingly.

Let us first consider this problem in two spatial dimensions. To study collapse rigorously one usually tries to compute the exact evolution of the wave packet widths rigorously [24]. For the multicomponent case and $m_1=m_2=m$, this was studied by group-theoretical methods by [25]. In our case, from the general formulas obtained by Gosh we get a sufficient condition for collapse, which is

$$\mathcal{H}=\int_{\mathbb{R}^n}\left[\sum_{j=1,2}(|\nabla u_j|^2/(2m)+V_j|u_j|^2+g_{jj}|u_j|^4/2)+g_{12}|u_1|^2|u_2|^2\right]<0. \qquad (11)$$

In principle, this is a bad result for obtaining localized structures since it means that arbitrarily close to any stationary solution (for which $\mathcal{H}=0$), there would be collapsing solutions and thus stationary solutions, if they exist, would be unstable. As is usual in the framework of collapse problems the situation would be even worse in three spatial dimensions with solutions of an arbitrarily small number of particles undergoing collapse, provided they are initially sufficiently localized.

This means that in principle symbiotic solitons could only be obtained in quasi-one-dimensional geometries because of the transverse stabilization effect provided by the trap in a similar way as ordinary bright solitons do.

VIII. CONCLUSIONS AND EXTENSIONS

In this paper we have studied vector solitons in heteronuclear two-component BEC's which are supported by their attractive mutual interaction. These symbiotic solitons are linearly stable and remarkably robust and can be generated through a modulational instability phenomenon with many possibilities for control. Collisions of these vector solitons show their robustness and open different ways for their manipulation and the design of novel quantum states such as breatherlike states. We have also considered multidimensional configurations and shown that collapse may avoid the formation of fully multidimensional symbiotic solitons.

We think that the conceptual ideas behind our work can also be used to understand boson-fermion mixtures. For instance, a_{12} is known to be negative and large for quantum-degenerate mixtures of ^{87}Rb and ^{40}K [17]. In those systems numerical simulations have proven the formation of localized wave packets [18] which could share the same essential mechanisms for the formation of solitary waves.

ACKNOWLEDGMENTS

We acknowledge V. Vekslerchik, R. Hulet, and B. Malomed for discussions. This work has been partially supported by Grant Nos. BFM2003-02832 (Ministerio de Educación y Ciencia, Spain) and PAI05-001 (Consejería de Educación y Ciencia de la Junta de Comunidades de Castilla-La Mancha).

[1] H. A. de Bary, *Die Erscheinung der Symbiose* (Karl J. Tübner, Strassburg, 1879).

[2] A. Scott, *Nonlinear Science* (Oxford University Press, Oxford, 1999).

[3] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys. JETP **38**, 248 (1974)].

[4] G. P. Agrawall and Y. Kivshar, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, San Diego, 2003).

[5] G. D. Montesinos, V. M. Pérez-García, and H. Michinel, Phys. Rev. Lett. **92**, 133901 (2004).

[6] Z. H. Musslimani, M. Segev, D. N. Christodoulides, and M. Soljacic, Phys. Rev. Lett. **84**, 1164 (2000); J. Malmberg, A. Carlsson, D. Anderson, M. Lisak, E. Ostrovskaya, and Y. Kivshar, Opt. Lett. **25**, 643 (2000); J. J. García-Ripoll, V. M. Pérez-García, E. A. Ostrovskaya, and Y. S. Kivshar, Phys. Rev. Lett. **85**, 82 (2000); J. Yang, and D. E. Pelinovsky, Phys. Rev. E **67**, 016608 (2003).

[7] S. Coen and M. Haelterman, Phys. Rev. Lett. **87**, 140401 (2001).

[8] K. Kasamatsu and M. Tsubota, Phys. Rev. Lett. **93**, 100402 (2004).

[9] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Phys. Rev. Lett. **86**, 2926 (2001); Th. Busch and J. R. Anglin, *ibid.* **87**, 010401 (2001).

[10] A. Simoni, F. Ferlaino, G. Roati, G. Modugno, and M. Inguscio, Phys. Rev. Lett. **90**, 163202 (2003).

[11] S. Inouye, J. Goldwin, M. L. Olsen, C. Ticknor, J. L. Bohn, and D. S. Jin, Phys. Rev. Lett. **93**, 183201 (2004).

[12] C. A. Stan, M. W. Zwierlein, C. H. Schunck, S. M. F. Raupach, and W. Ketterle, Phys. Rev. Lett. **93**, 143001 (2004).

[13] G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature (London) **417**, 150 (2002); L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).

[14] V. M. Pérez-García, H. Michinel, and H. Herrero, Phys. Rev. A **57**, 3837 (1998).

[15] C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation: Self-focusing and wave collapse* (Springer, Berlin, 2000).

[16] G. Theocharis, Z. Rapti, P. G. Kevrekidis, D. J. Frantzeskakis, and V. V. Konotop, Phys. Rev. A **67**, 063610 (2003); L. D. Carr and J. Brand, *ibid.* **70**, 033607 (2004); Phys. Rev. Lett. **92**, 040401 (2004); P. G. Kevrekidis, G. Theocharis, D. J. Frantzeskakis, and A. Trombettoni, Phys. Rev. A **70**, 023602 (2004).

[17] G. Roati, F. Riboli, G. Modugno, and M. Inguscio, Phys. Rev. Lett. **89**, 150403 (2002); J. Goldwin, S. Inouye, M. L. Olsen, B. Newman, B. D. DePaola, and D. S. Jin, Phys. Rev. A **70**, 021601(R) (2004).

[18] T. Karpiuk, K. Brewczyk, S. Ospelkaus-Schwarzer, K. Bongs, M. Gajda, and K. Rzazewski, Phys. Rev. Lett. **93**, 100401 (2004).

[19] G. Modugno, M. Modugno, F. Riboli, G. Roati, and M. Inguscio, Phys. Rev. Lett. **89**, 190404 (2002).

[20] M. Trippenbach, K. Goral, K. Rzazewski, B. Malomed, and Y. B. Band, J. Phys. B **33**, 4017 (2000).

[21] A. M. Kamchatnov, R. A. Kraenkel, and B. A. Umarov, Phys. Rev. E **66**, 036609 (2002).

[22] F. Riboli and M. Modugno, Phys. Rev. A **65**, 063614 (2002).

[23] T.-S. Ku, M.-F. Shih, A. A. Sukhorukov, and Y. S. Kivshar, Phys. Rev. Lett. **94**, 063904 (2005).

[24] J. J. García-Ripoll, V. M. Pérez-García, and P. Torres, Phys. Rev. Lett. **83**, 1715 (1999).

[25] P. K. Ghosh, Phys. Rev. A **65**, 053601 (2002).