

Quantum and thermal fluctuations of trapped Bose-Einstein condensates

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We quantize a semiclassical system defined by the Hamiltonian obtained from the asymptotic self-similar solution of the Gross-Pitaevskii equation for a trapped Bose-Einstein condensate with a linear gain term. On the basis of a Schrödinger equation derived in a space of ellipsoidal parameters, we analytically calculate the quantum mechanical and thermal variance in the ellipsoidal parameters for Bose-Einstein condensates in various shapes of trap. We show that, except for temperatures close to zero, dimensionless dispersions do not depend on the frequencies of the trap and they have the same dependence on dimensionless temperatures.

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I. INTRODUCTION

A large number of theoretical works have investigated the elementary excitations of confined interacting Bose-Einstein condensates (BEC). These analyses have generally been based on zero-temperature mean-field theory. To calculate the low energy excitations of a dilute atomic Bose gas confined in a harmonic trap, Stringari [1] developed an approach based on solutions to the linearized Gross-Pitaevskii equation (GPE) [2] in the Thomas-Fermi approximation. Alternative approaches to the analytical study of excitation frequencies have also been provided by Fetter [3], who used variational wave functions in the generalized Bogoliubov approach, and by Pérez-García *et al.* [4], who also employed a variational technique, using a Gaussian trial wave function, to analyze the time-dependent GPE. Kagan *et al.* [5] and Castin and Dum [6] have used scaling transformations to study the collective excitations in the GPE approach. This approach has also been used by Dalfovo *et al.* [7] to consider the nonlinear regime of excitation and to examine the effects of mode coupling and harmonic generation, which have also been considered by Graham *et al.* [8].

A finite temperature field theory has also been applied to trapped Bose condensates to study density profiles, collective excitations, and damping rates. These works use finite-temperature Hartree-Fock-Bogoliubov (HFB) approximations, as well as other approximations originally introduced by Popov [9], Beliaev [10], and other authors (see for example the Hohenberg and Martin paper [11]). We mention here as examples only a few papers [12–16] based on the treatment of finite temperature BEC.

The results we present in this article are particularly topical due to the fact that the direct numerical calculation of the quantum statistics of a trapped BEC is an extremely difficult problem. A quantum mechanical one-dimensional calculation has been performed using phase-space methods for the coherence of a trapped BEC [17], but this approach was only successful over a short time. Quantum calculations of the evaporative cooling process have also been performed [18], but these were not viable once the condensate had formed. Fully quantum calculations have been performed for the processes of photoassociation and dissociation of atomic and molecular condensates [19–22], but the interesting dynamics

here occur over short times. It should also be noted that in all these quantum treatments apart from that of evaporative cooling, the initial quantum state was always decreed rather than calculated from first principles. As a quantity which will be of interest is the coherence or, alternatively, the linewidth of a continuous wave atom laser in the steady state, it is of some importance to develop methods of dealing with the multimode problem in the long time limit. In this article, working in a space of the ellipsoidal parameters of a trapped BEC, we derive a Schrödinger equation from a semiclassical Hamiltonian which itself comes from the GPE with an added gain term. This then enables us to calculate the excitation frequencies and quantum expectation values for the radial coordinate of axially symmetric trapped condensates.

An important requirement for achieving a continuous wave atom laser using trapped Bose-Einstein condensates is a pumped lasing mode of the condensate. In an early work in the field, Kneer *et al.* [23] modeled the trapped lasing mode semiclassically, using a generalized Gross-Pitaevskii equation with added loss and gain terms. They found that, by analogy to optical lasers, there was a threshold value of pumping above which the lasing mode became macroscopically occupied. They also found that, depending on the spatial dependence of the pumping, the lasing mode could undergo number, and hence spatial, oscillations. Drummond and Kheruntsyan [24] have also solved the GPE with gain to model the growth of a BEC. They found significant differences from the Thomas-Fermi solution, and also found that, as the condensate grows, it can develop a center of mass oscillation in the trap. What we will calculate here are the quantum uncertainties present in the radii of a trapped condensate, which will always be present along with the center of mass oscillations found by Drummond and Kheruntsyan.

We will show that the quantum fluctuations of a trapped BEC with a harmonic potential in the general case, when the number of condensed atoms N_c is a function of time, can be found using an asymptotic self-similar solution to the GPE with a linear gain term [25,26], thus obtaining a semiclassical Hamiltonian which lends itself to quantization. On the basis of a Schrödinger equation derived in a space of ellipsoidal parameters, we analytically calculate the quantum mechanical and thermal variance in the ellipsoidal parameters for Bose-Einstein condensates in various shapes of trap. In

in this paper we do not take into account collisional interactions of the BEC with the thermal cloud. We show that, except for temperatures close to zero, dimensionless transverse and longitudinal dispersions of the ellipsoidal parameters of a trapped BEC have the same dependence on dimensionless temperatures and do not depend on the frequencies of the trap. This scaling is an approximate but nevertheless very accurate law and it was found by numerical calculations.

II. SCHRÖDINGER EQUATION IN A SPACE OF ELLIPSOIDAL PARAMETERS

We begin with the Gross-Pitaevskii gain equation [23,24] for the semiclassical “wave function” of the condensate, normalized to the number, $N_c(t)$, of condensed atoms,

$$i \frac{\partial \psi}{\partial t} = \left(\frac{-\hbar \nabla^2}{2m} + \sum_{k=1}^3 \frac{m \omega_k^2 x_k^2}{2\hbar} + \frac{4\pi \hbar a_0}{m} |\psi|^2 + \frac{ig(t)}{2} \right) \psi. \quad (1)$$

In the above, m is the atomic mass, the ω_k are the trap frequencies along each axis, and a_0 is the atomic s -wave scattering length. The gain function is determined by the logarithmic derivative, $g(t) = d \ln N_c(t) / dt$. We now rewrite the wave function in Eq. (1) with a real amplitude and phase $\psi(\mathbf{x}, t) = A(\mathbf{x}, t) \exp[i\phi(\mathbf{x}, t)]$. In the amplitude equation we can then neglect the term $\hbar \nabla^2 A / 2m$ because $|\hbar \nabla^2 A / 2m| \ll 4\pi \hbar a_0 A^3 / m$ for the condition $E_c \gg 1$, where the large positive dimensionless parameter is $E_c = 8\pi n_c a_0 l_m^2$, with l_m being the minimum of the typical lengths along the three axes and $n_c = N_c / V_c$ the average condensate density. The condition $E_c \gg 1$ means that we are considering the strongly nonlinear hydrodynamic time-dependent regime. The GPE given by Eq. (1) has asymptotically self-similar solutions [25,26] for large positive values of E_c . When $g(t) = 0$ this approximation becomes equivalent to a time-dependent Thomas-Fermi solution, coinciding with the normal Thomas-Fermi approximation in the stationary case.

Under the above condition, the asymptotic self-similar solution for the amplitude variable can be written as

$$A(\mathbf{x}, t) = f_1(t) f_2(t) f_3(t) F(\xi_1, \xi_2, \xi_3), \quad (2)$$

and that for the phase variables as

$$\phi(\mathbf{x}, t) = \phi_0(t) + \sum_{k=1}^3 c_k(t) x_k^2, \quad (3)$$

where the ξ_k are self-similar dimensionless variables

$$\xi_k = x_k f_k^2(t) \exp\left(-\int_0^t g_k(t') dt'\right), \quad (4)$$

and the g_k are functions satisfying the condition

$$\sum_{k=1}^3 g_k(t) = g(t). \quad (5)$$

Applying a generalization of the method used to find solutions of the one-dimensional nonlinear Schrödinger equation with gain [26], we find (Appendix A) a system of ordinary

differential equations for the functions $f_k(t)$, $c_k(t)$, and the explicit expression $F(\boldsymbol{\xi}) = \sqrt{1 - \sum_k \theta_k \xi_k^2}$ for $\sum_k \theta_k \xi_k^2 \leq 1$ and zero otherwise. Here the θ_k are positive constants. The normalization condition gives us the equation $1 / \sqrt{\theta_1 \theta_2 \theta_3} = 15N_c(0) / 8\pi$ for the constants θ_k , which allows us to exclude these indeterminate parameters in the solutions for amplitude and phase. Introducing the ellipsoidal parameters of the trapped Bose gas by

$$a_k(t) = \theta_k^{-1/2} f_k^{-2}(t) \exp\left(\int_0^t g_k(t') dt'\right), \quad (6)$$

after some transformations we derive (more detail is given in Appendix A) an explicit self-similar asymptotic solution for the amplitude when the parameter E_c tends to infinity,

$$A(\mathbf{x}, t) = \left(\frac{15N_c(t)}{8\pi a_1(t) a_2(t) a_3(t)} \right)^{1/2} \left(1 - \sum_{k=1}^3 \frac{x_k^2}{a_k^2(t)} \right)^{1/2} S[q(\mathbf{x}, t)], \quad (7)$$

where

$$q(\mathbf{x}, t) = \left(1 - \sum_{k=1}^3 \frac{x_k^2}{a_k^2(t)} \right). \quad (8)$$

In the above, $S(q)$ is the Heaviside step function which equals one at positive q , and zero otherwise. In this case it serves to select when $q(\mathbf{x}, t)$ is positive. The phase variable $\phi(\mathbf{x}, t)$ in Eq. (3) is defined by the functions $\phi_0(t)$ and $c_k(t)$ via the relations

$$c_k(t) = \frac{m}{2\hbar} \frac{d}{dt} \ln a_k(t), \quad (9)$$

$$\phi_0(t) = \phi_0(0) - \frac{15\hbar a_0}{2m} \int_0^t \frac{N_c(t') dt'}{a_1(t') a_2(t') a_3(t')}, \quad (10)$$

where the ellipsoidal parameters are solutions of the equations ($k=1,2,3$):

$$\frac{d^2}{dt^2} a_k + \omega_k^2 a_k = \left(\frac{15a_0 \hbar^2}{m^2} \right) \frac{N_c(t)}{a_1 a_2 a_3 a_k}, \quad (11)$$

with initial conditions $a_k|_{t=0} = a_k(0)$ and

$$\left. \frac{da_k}{dt} \right|_{t=0} = \frac{2\hbar}{m} c_k(0) a_k(0). \quad (12)$$

In the case when the gain term $g(t)$ in the GPE is equal to zero ($N_c = \text{const}$), and using some approximations, the solutions of Eq. (1) were obtained in [4–6] by different methods. It should however be noted that Refs. [5,6,8] were interested in fluctuations of the condensate due to external trapping perturbations, while we will be calculating the quantum fluctuations which exist even for a completely stable trap. We will now consider the case where N_c is constant. The stationary solution of the system of Eqs. (11) for the ellipsoidal parameters are $a_k = \bar{a}_k$, where \bar{a}_k given for ($k=1,2,3$) by

$$\bar{a}_k = \omega_k^{-1} (G_0 \omega_1 \omega_2 \omega_3 / m)^{1/5}. \quad (13)$$

We defined here a new constant $G_0 = 15\hbar^2 a_0 N_c / m$. One can set $y_k = a_k - \bar{a}_k$ and linearize the system of equations (11) for the y_k ,

$$\frac{d^2}{dt^2} y_k + \sum_{s=1}^3 \Lambda_{ks} y_s = 0, \quad (14)$$

where

$$\Lambda_{ks} = \omega_k \omega_s + 2\omega_k^2 \delta_{ks}, \quad (15)$$

and δ_{ks} is the Kronecker delta function. We note here that we are not using the standard linearization procedure, which deals with fluctuations about a stationary solution, but are treating fluctuations about a deterministic bulk motion, in a manner similar to that previously used to treat the self-pulsing regime of optical second harmonic generation [28]. The characteristic frequencies Ω_k of the small oscillations given by Eq. (14) are the roots of the dispersion relation

$$\det(\Lambda - \Omega_k^2 \mathbf{I}) = 0, \quad (16)$$

where the matrix Λ is given by Eq. (15) and \mathbf{I} is the identity matrix. In the case of an axially symmetric trapping potential, with $\omega_1 = \omega_2 = \omega$, and setting $\omega_3 = \omega_0$, Eq. (16) leads directly to the characteristic frequencies

$$\Omega_k^6 - 3(\omega_0^2 + 2\omega^2)\Omega_k^4 + 8\omega^2(\omega^2 + 2\omega_0^2)\Omega_k^2 - 20\omega_0^2\omega^4 = 0. \quad (17)$$

We note that Eq. (1) with $g(t) = N_c^{-1}(dN_c/dt)$ is completely equivalent to the ordinary, gainless GPE with a time-dependent self-interaction parameter,

$$i\hbar \frac{\partial}{\partial t} \psi_N(\mathbf{x}, t) = \frac{\delta H}{\delta \psi_N^*(\mathbf{x}, t)}. \quad (18)$$

Here $H = H(t)$ is the real Hamiltonian function,

$$H(t) = \int \psi_N^*(\mathbf{x}) \left(-\frac{\hbar^2 \nabla^2}{2m} + \sum_{k=1}^3 \frac{m\omega_k^2 x_k^2}{2} + \frac{2\pi a_0 \hbar^2 N_c(t)}{m} |\psi_N(\mathbf{x})|^2 \right) \psi_N(\mathbf{x}) d\mathbf{x}, \quad (19)$$

where we have used unit normalization of the mean field density. The above expression follows directly from Eq. (1) and the time-dependent normalization condition $\psi_N = \psi / \sqrt{N_c(t)}$.

By use of the above solutions for the variables $A(\mathbf{x}, t)$ and $\phi(\mathbf{x}, t)$ we can calculate the integrals in Eq. (19). In the case where $E_c \gg 1$, this allows for the explicit formulation of the Hamiltonian as

$$H(t) = \sum_{k=1}^3 \left(\frac{p_k^2}{2m_a} + \frac{m_a}{2} \omega_k^2 a_k^2 \right) + \frac{G(t)}{a_1 a_2 a_3}, \quad (20)$$

where $p_k = m_a \dot{a}_k$, $m_a = m/7$ is the renormalized boson mass, and the self-interaction parameter is

$$G(t) = 15a_0 \hbar^2 N_c(t) / 7m. \quad (21)$$

The Hamiltonian equations,

$$\frac{da_k}{dt} = \frac{\partial H(t)}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H(t)}{\partial a_k}, \quad (22)$$

lead directly to Eq. (11), hence a_k and $p_k = m_a \dot{a}_k$ are the generalized coordinate and the canonical momentum of the classical system defined by the Hamiltonian of Eq. (20). Applying the standard quantization method to the Hamiltonian of Eq. (20) results in the Schrödinger type equation [27,29]:

$$i\hbar \frac{\partial \chi}{\partial t} = \left(-\frac{\hbar^2}{2m_a} \sum_{k=1}^3 \frac{\partial^2}{\partial a_k^2} + \frac{m_a}{2} \sum_{k=1}^3 \omega_k^2 a_k^2 + \frac{G(t)}{a_1 a_2 a_3} \right) \chi, \quad (23)$$

where the wave function $\chi(\mathbf{a}, t)$ has unit normalization.

In the case when N_c is constant we can use also alternative variables: a_k and $\pi_k = N_c p_k = \mu \dot{a}_k$, where the effective mass μ of the condensate is

$$\mu = N_c m_a = N_c m / 7. \quad (24)$$

In this case the Hamiltonian of the whole system $\mathcal{H} = N_c H$ is:

$$\mathcal{H} = \sum_{k=1}^3 \left(\frac{\pi_k^2}{2\mu} + \frac{\mu}{2} \omega_k^2 a_k^2 \right) + \frac{\mathcal{G}}{a_1 a_2 a_3}, \quad (25)$$

where \mathcal{G} is a new interaction constant given by

$$\mathcal{G} = N_c G = \frac{15a_0 \hbar^2 N_c^2}{7m}. \quad (26)$$

It is evident that the Hamiltonian equations (22) for the new variables have the form

$$\frac{da_k}{dt} = \frac{\partial \mathcal{H}}{\partial \pi_k}, \quad \frac{d\pi_k}{dt} = -\frac{\partial \mathcal{H}}{\partial a_k}. \quad (27)$$

Hence this quantization yields the Schrödinger-type equation for the new variables in the form

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2\mu} \sum_{k=1}^3 \frac{\partial^2}{\partial a_k^2} + \frac{\mu}{2} \sum_{k=1}^3 \omega_k^2 a_k^2 + \frac{\mathcal{G}}{a_1 a_2 a_3} \right) \Psi. \quad (28)$$

We note here that Eq. (28) is not the usual Schrödinger equation as it describes a wave function $\Psi(\mathbf{a}, t)$ of the trapped BEC in the space of ellipsoidal parameters a_k ($k=1,2,3$), taking into account the self-interaction of this many-body system. This is evident from the fact that the strength parameter of the self-interaction \mathcal{G} is proportional to the square number of condensed atoms, N_c . Thus, using the nonlinear GPE with added gain [Eq. (1)] we have derived (under the condition $\epsilon_c \equiv 1/E_c \ll 1$) a linear Schrödinger-type equation for the wave function $\Psi(\mathbf{a}, t)$. We should emphasize here the linear character of the wave equation derived by this method. We also note that Eqs. (23) and (28) are different because the first one describes the contribution of one particle to the distribution of the ellipsoidal parameters a_k but the second wave equation takes into account all particles. In next section we discuss this distinction in detail.

III. ASYMPTOTIC SOLUTION OF THE SCHRÖDINGER EQUATION

In this section we consider the solution of equations (23) and (28) in the quadratic approximation with respect to small deviations, $y_k = a_k - \bar{a}_k$, of the ellipsoidal parameters. The Schrödinger equation in the space of ellipsoidal parameters, Eq. (23), can be solved by an expansion of the internal and external potential energies,

$$V = \frac{m_a}{2} \sum_{k=1}^3 \omega_k^2 a_k^2 + \frac{G}{a_1 a_2 a_3}, \quad (29)$$

in a series with the variable y_k , subject to the condition $|y_k|/\bar{a}_k \ll 1$. In the quadratic approximation, and taking into account the stability condition

$$\left(\frac{\partial V}{\partial y_k} \right)_{y_k=0} = m_a \omega_k^2 \bar{a}_k - \frac{G}{\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_k} = 0, \quad (30)$$

for N_c constant, we find

$$V = V_0 + \frac{m_a}{2} \sum_{k=1}^3 \sum_{s=1}^3 \Lambda_{ks} y_k y_s. \quad (31)$$

In the above, V_0 is the potential energy [Eq. (29)] at $y_k=0$:

$$V_0 = \frac{5}{2} \frac{G}{\bar{a}_1 \bar{a}_2 \bar{a}_3}, \quad (32)$$

and the matrix Λ_{ks} is given by Eq. (15). The eigenvectors of Λ_{ks} obey the equation

$$\sum_{s=1}^3 \Lambda_{is} u_s^{(k)} = \Omega_k^2 u_i^{(k)}, \quad (33)$$

with the normalization conditions

$$\sum_{s=1}^3 u_s^{(i)} u_s^{(k)} = \delta_{ik}, \quad (34)$$

so that the coordinate transformation

$$y_s = a_s - \bar{a}_s = \sum_{k=1}^3 u_s^{(k)} z_k \quad (35)$$

is orthogonal. Using Eqs. (31)–(35), we can now write Eq. (23) as

$$i\hbar \frac{\partial}{\partial t} \chi = \left(-\frac{\hbar^2}{2m_a} \sum_{k=1}^3 \frac{\partial^2}{\partial z_k^2} + \frac{m_a}{2} \sum_{k=1}^3 \Omega_k^2 z_k^2 + V_0 \right) \chi, \quad (36)$$

where the eigenfrequencies are given by Eq. (16), or, in explicit form,

$$\begin{aligned} & (\Omega_k^2 - 3\omega_1^2)(\Omega_k^2 - 3\omega_2^2)(\Omega_k^2 - 3\omega_3^2) - \omega_2^2 \omega_3^2 (\Omega_k^2 - 3\omega_1^2) \\ & - \omega_1^2 \omega_3^2 (\Omega_k^2 - 3\omega_2^2) - \omega_1^2 \omega_2^2 (\Omega_k^2 - 3\omega_3^2) - 2\omega_1^2 \omega_2^2 \omega_3^2 = 0. \end{aligned} \quad (37)$$

This allows us to write the general solution of Eq. (36) as

$$\chi(z_1, z_2, z_3, t) = \sum_{n_1 n_2 n_3} A_{n_1 n_2 n_3} \chi_{n_1 n_2 n_3}(z_1, z_2, z_3) \exp\left(-\frac{i}{\hbar} E_{n_1 n_2 n_3} t\right), \quad (38)$$

where $n_k=0,1,2,\dots$ at $k=1,2,3$ are the quantum numbers of a 3D harmonic oscillator, and $A_{n_1 n_2 n_3}$ are the arbitrary constants. The $\chi_{n_1 n_2 n_3}$ are the eigenfunctions of a 3D harmonic oscillator,

$$\begin{aligned} \chi_{n_1 n_2 n_3}(z_1, z_2, z_3) &= \left(\frac{m_a^3 \Omega_1 \Omega_2 \Omega_3}{\hbar^3 \pi^3} \right)^{1/4} \left(\frac{2^{-n_1 - n_2 - n_3}}{n_1! n_2! n_3!} \right)^{1/2} \\ &\times \prod_{k=1}^3 H_{n_k}(\xi_k) \exp\left(-\frac{1}{2} \xi_k^2\right), \end{aligned} \quad (39)$$

where $H_n(\xi)$ is a Hermite polynomial and the variable ξ_k is given by

$$\xi_k = \sqrt{\frac{m_a \Omega_k}{\hbar}} z_k. \quad (40)$$

The eigenenergies $E_{n_1 n_2 n_3}$ in Eq. (38) are

$$E_{n_1 n_2 n_3} = V_0 + \hbar \sum_{k=1}^3 \Omega_k \left(n_k + \frac{1}{2} \right). \quad (41)$$

Equations (37)–(41) now represent the general solution of the Schrödinger type equation (36) in the space of ellipsoidal parameters.

In the axially symmetric case with $(\omega_1 = \omega_2 = \omega, \omega_3 = \omega_0)$, the solution of Eq. (17) is

$$\Omega_{1,2}^2 = 2\omega^2 + \frac{3}{2} \omega_0^2 \pm \frac{1}{2} \sqrt{4(2\omega^2 - \omega_0^2)^2 + 5\omega_0^4}, \quad (42)$$

$$\Omega_3^2 = 2\omega^2, \quad (43)$$

where the positive (negative) sign in front of the square root belongs to Ω_1^2 (Ω_2^2). Note that these are the same frequencies as given previously in Ref. [1]. We will now consider some important limiting cases of the above solutions for different trapping geometries.

Case 1 (Disk form) $\omega_0 \gg \omega$

$$\Omega_1^2 \approx 3\omega_0^2 + \frac{2}{3}\omega^2, \quad \Omega_2^2 \approx \frac{10}{3}\omega^2, \quad \Omega_3^2 = 2\omega^2. \quad (44)$$

Case 2 (Spherical form) $\omega_0 = \omega$

$$\Omega_1^2 = 5\omega^2, \quad \Omega_2^2 = \Omega_3^2 = 2\omega^2. \quad (45)$$

Case 3 (Cigar form) $\omega \gg \omega_0$

$$\Omega_1^2 \approx 4\omega^2 + \frac{1}{2}\omega_0^2, \quad \Omega_2^2 \approx \frac{5}{2}\omega_0^2, \quad \Omega_3^2 = 2\omega^2. \quad (46)$$

We note here that the frequencies given above are not those along the orthogonal Cartesian axes, but refer to the orthogonal axes which come from diagonalization of the matrix of Eq. (15). As an example we will present here the explicit form of the quantized eigenfrequencies, $\Omega_{n_1 n_2 n_3} = E_{n_1 n_2 n_3} / \hbar$, of a cigar shaped condensate in the case where $\omega_1 = \omega_2 = \omega$.

Using Eq. (30) and setting $\bar{a}_1 = \bar{a}_2 = \bar{a}$, we find

$$\bar{a}^2 = \frac{\hbar}{2\omega m_a} \sqrt{2g_a}, \quad (47)$$

where g_a is the dimensionless constant

$$g_a = \frac{30a_0 N_c}{49\bar{a}_3}. \quad (48)$$

Combining Eqs. (32), (47), and (48), we find that

$$V_0 = \frac{5}{4} \sqrt{2g_a} \hbar \omega. \quad (49)$$

Note that this expression has been derived assuming only axial symmetry of the trap (i.e., $\omega_1 = \omega_2$), hence Eq. (49) is valid for all three condensate shapes considered above. In the case of a cigar shaped trap, and using the quadratic approximation for the potential energy, Eqs. (41), (46), and (49) result in

$$\Omega_{n_1 n_2 n_3} = \Omega_0 + \omega(2n_1 + \sqrt{2}n_3) + \sqrt{\frac{5}{2}} \omega_0 n_2, \quad (50)$$

where the constant Ω_0 is given by the expression

$$\Omega_0 = \left(1 + \frac{1}{\sqrt{2}} + \frac{5\sqrt{2g_a}}{4} \right) \omega + \frac{1}{2} \sqrt{\frac{5}{2}} \omega_0. \quad (51)$$

It is evident that the appropriate solution of Eq. (28) follows from the above equations by the substitution $m_a \rightarrow \mu$, $G \rightarrow \mathcal{G}$, and $g_a \rightarrow g_c$, where $g_c = g_a N_c^2$. Hence Eqs. (23) and (28) yield the same average ellipsoidal parameters \bar{a}_k by Eq. (30) or, in explicit form, by Eq. (13) and the same eigenfrequencies Ω_k given by Eq. (37). Moreover, Eq. (50) for the quantized eigenfrequencies in these two cases differs only by the first term Ω_0 , hence physically the differences of the quantized eigenfrequencies $\Omega_{n_1 n_2 n_3}$ are also the same. However, as seen in Eq. (40), the wave functions in these cases differ by the scale factor $N_c^{1/2}$, which should be taken into account if we use Eq. (23). Below we will consider the case of constant N_c and use Eq. (28) which automatically takes into account the total mass of the trapped BEC.

IV. QUANTUM FLUCTUATIONS AT LOW TEMPERATURES

The quantum fluctuations of the ellipsoidal parameters a_s of trapped Bose-Einstein condensates in the case of thermal equilibrium can be calculated directly from the Hamiltonian (25), where a_k and π_k in the quantum case are the generalized coordinates and corresponding canonical momenta. We note that in this paper we do not take into account collisional interactions of the BEC with the thermal cloud. As long as the deviations, $y_k = a_k - \bar{a}_k$, of the ellipsoidal parameters are small ($|y_k|/\bar{a}_k \ll 1$), which holds at low temperatures, we may use the results of the previous section to write the Hamiltonian operator in quadratic form:

$$\hat{\mathcal{H}} = \sum_{k=1}^3 \left(\frac{\tilde{\pi}_k^2}{2\mu} + \frac{\mu}{2} \Omega_k^2 z_k^2 \right) + \frac{5\mathcal{G}}{2\bar{a}_1 \bar{a}_2 \bar{a}_3}. \quad (52)$$

We do not assume here axial symmetry of the trap; the new canonical operators z_k , $\tilde{\pi}_k = -i\hbar \partial / \partial z_k$ are connected with the canonical operators a_k , $\pi_k = -i\hbar \partial / \partial a_k$ by a unitary transform given by Eqs. (33)–(35), and the eigenfrequencies are given by Eq. (37). According to quantum statistical mechanics the density matrix ρ at thermal equilibrium is

$$\rho = Z(\beta)^{-1} \exp(-\beta \hat{\mathcal{H}}), \quad \text{where } Z(\beta) = \text{Tr}[\exp(-\beta \hat{\mathcal{H}})], \quad (53)$$

where $\beta = (k_B T)^{-1}$. The probability density $w(z_1, z_2, z_3)$ then has the form

$$\begin{aligned} w(z_1, z_2, z_3) &= \langle z_1, z_2, z_3 | \rho | z_1, z_2, z_3 \rangle \\ &= \sum_{n_1 n_2 n_3} P_{n_1 n_2 n_3}(\beta) |\Psi_{n_1 n_2 n_3}(z_1, z_2, z_3)|^2. \end{aligned} \quad (54)$$

Note that here and in what follows z_k and a_k are c -number variables and that in Eq. (54) we have used the definitions

$$P_{n_1 n_2 n_3}(\beta) = Z(\beta)^{-1} \exp(-\beta E_{n_1 n_2 n_3}), \quad (55)$$

$$\Psi_{n_1 n_2 n_3}(z_1, z_2, z_3) = \langle z_1, z_2, z_3 | n_1, n_2, n_3 \rangle, \quad (56)$$

$$\hat{\mathcal{H}} |n_1, n_2, n_3\rangle = E_{n_1 n_2 n_3} |n_1, n_2, n_3\rangle. \quad (57)$$

Calculating the sum in Eq. (54) with the results found in the previous section allows us to write the probability density in the following form:

$$w(z_1, z_2, z_3) = C \exp \left\{ - \sum_{k=1}^3 \left[\frac{\mu \Omega_k}{\hbar} \tanh \left(\frac{1}{2} \beta \hbar \Omega_k \right) \right] z_k^2 \right\}, \quad (58)$$

where the normalization constant is

$$C = \prod_{k=1}^3 \left[\frac{\mu \Omega_k}{\pi \hbar} \tanh \left(\frac{1}{2} \beta \hbar \Omega_k \right) \right]^{1/2}, \quad (59)$$

and the characteristic frequencies Ω_k are given by Eq. (37). Taking into account that $a_s = \bar{a}_s + y_s$, we find that $\langle a_s \rangle = \bar{a}_s$ and that the dispersions of the ellipsoidal parameters are

$$D(a_s) = \langle a_s^2 \rangle - \langle a_s \rangle^2 = \langle y_s^2 \rangle = \sum_{k=1}^3 (u_s^{(k)})^2 \langle z_k^2 \rangle. \quad (60)$$

Note that the notation $\langle \dots \rangle$ signifies averages over the probability density w defined in Eq. (58): $\langle F \rangle = \int \int \int F w dz_1 dz_2 dz_3$. Using Eqs. (58) and (59) one may find the standard deviation $\Delta z_k = \sqrt{\langle z_k^2 \rangle}$ as

$$\Delta z_k = \sqrt{\frac{\hbar}{2\mu\Omega_k} \coth \left(\frac{1}{2} \beta \hbar \Omega_k \right)}. \quad (61)$$

Hence the dispersions of the ellipsoidal parameters can be written:

$$D(a_s) = \sum_{k=1}^3 \frac{\hbar(u_s^{(k)})^2}{2\mu\Omega_k} \coth\left(\frac{1}{2}\beta\hbar\Omega_k\right). \quad (62)$$

The solution of the linear homogeneous system Eqs. (33) for eigenvectors $u_j^{(k)}$ in the axially symmetric trap is

$$u_1^{(k)} = u_2^{(k)} = \frac{(\Omega_k^2 - 2\omega^2)\omega\omega_0}{(\Omega_k^2 - 3\omega^2)^2 - \omega^4} u_3^{(k)}. \quad (63)$$

Using

$$\frac{1}{(\Omega_k^2 - 3\omega^2)^2 - \omega^4} = \frac{3\omega_0^2 - \Omega_k^2}{2\omega_0^2\omega^2(2\omega^2 - \Omega_k^2)}, \quad (64)$$

which follows from Eq. (17) and taking into account the normalization condition $\sum_{s=1}^3 (u_s^{(k)})^2 = 1$ we find

$$u_1^{(k)} = u_2^{(k)} = \frac{\pm(\Omega_k^2 - 3\omega_0^2)}{\sqrt{2(\Omega_k^2 - 3\omega_0^2)^2 + 4\omega_0^2\omega^2}}, \quad (65)$$

$$u_3^{(k)} = \frac{\pm 2\omega_0\omega}{\sqrt{2(\Omega_k^2 - 3\omega_0^2)^2 + 4\omega_0^2\omega^2}}. \quad (66)$$

In the axially symmetric case all three eigenfrequencies Ω_k are different. However, as follows from Eqs. (65) and (66), the determinant of the matrix u_{sk} ($=u_s^{(k)}$) is zero. The reason for this degeneracy is connected with the fact that the denominator of Eq. (63) is zero at $\Omega_3 = \sqrt{2}\omega$. To avoid this degeneracy we will use Eqs. (65) and (66) when $k=1,2$ and for $k=3$ we find the eigenvector $u^{(3)}$ as the vector product $u^{(3)} = u^{(1)} \times u^{(2)}$. This leads directly to the equations

$$u_1^{(3)} = \frac{1}{\sqrt{2}}, \quad u_2^{(3)} = -\frac{1}{\sqrt{2}}, \quad u_3^{(3)} = 0. \quad (67)$$

Using these equations it is easy to verify that $\Lambda u^{(3)} = \Omega_3^2 u^{(3)}$. Taking the equations $\Omega_3^2 = 2\omega^2$ and $\Omega_1^2 \Omega_2^2 = 10\omega^2 \omega_0^2$ into account [which follow from Eq. (17)] and using an explicit solution (42) and (43) for three eigenfrequencies Ω_k and explicit solutions (63)–(67) for eigenvectors $u_j^{(k)}$ one can write, after some algebra, the weight matrix w_{sk} as

$$w_{sk} \equiv (u_s^{(k)})^2 = \begin{pmatrix} \Gamma_1/2 & \Gamma_2/2 & 1/2 \\ \Gamma_1/2 & \Gamma_2/2 & 1/2 \\ \Gamma_2 & \Gamma_1 & 0 \end{pmatrix}, \quad (68)$$

where $\Gamma_1 = (\Omega_1^2 - 3\omega_0^2)/(\Omega_1^2 - \Omega_2^2)$ and $\Gamma_2 = (3\omega_0^2 - \Omega_2^2)/(\Omega_1^2 - \Omega_2^2)$. We note that the weight matrix w_{sk} is degenerate in the axially symmetric case since $D(a_1) = D(a_2)$, but the matrix $u_{sk} = u_s^{(k)}$ is not degenerate [see Eqs. (63)–(67)]. One can

check that all normalization conditions are fulfilled for the weight factors $w_{sk} \equiv (u_s^{(k)})^2$:

$$\sum_{s=1}^3 (u_s^{(k)})^2 = 1 \quad \text{at } k = 1, 2, 3, \quad (69)$$

$$\sum_{k=1}^3 (u_s^{(k)})^2 = 1 \quad \text{at } s = 1, 2, 3.$$

These normalization conditions directly follow from Eq. (68) and the identity $\Gamma_1 + \Gamma_2 = 1$. Using Eqs. (62) and (68) we can write the variances $(\Delta a_s)^2 = D(a_s)$ in the axially symmetric case in the form:

$$(\Delta a_{\perp})^2 = \frac{\hbar}{2\mu} \left[\left(\frac{\Gamma_1}{2} \right) \frac{\coth\left(\frac{1}{2}\beta\hbar\Omega_1\right)}{\Omega_1} + \left(\frac{\Gamma_2}{2} \right) \frac{\coth\left(\frac{1}{2}\beta\hbar\Omega_2\right)}{\Omega_2} + \left(\frac{1}{2} \right) \frac{\coth\left(\frac{1}{2}\beta\hbar\Omega_3\right)}{\Omega_3} \right], \quad (70)$$

$$(\Delta a_3)^2 = \frac{\hbar}{2\mu} \left[\Gamma_2 \frac{\coth\left(\frac{1}{2}\beta\hbar\Omega_1\right)}{\Omega_1} + \Gamma_1 \frac{\coth\left(\frac{1}{2}\beta\hbar\Omega_2\right)}{\Omega_2} \right], \quad (71)$$

where

$$\Gamma_1 = \frac{\Omega_1^2 - 3\omega_0^2}{\Omega_1^2 - \Omega_2^2}, \quad \Gamma_2 = \frac{3\omega_0^2 - \Omega_2^2}{\Omega_1^2 - \Omega_2^2} = 1 - \Gamma_1. \quad (72)$$

Here $\Delta a_{\perp} = \Delta a_1 = \Delta a_2$ due to the axial symmetry. We will present expressions for the general case where all three trapping frequencies are different in Appendix B. We note that the effective mass $\mu = N_c m / 7$ is a function of the temperature T in Eqs. (70) and (71) because $N_c = N_c(T)$. One can fit the temperature variation of the function $N_c(T)$ to the functional form $N_c(T) = N[1 - (T/T_c)^\alpha]$ (for $T < T_c$), where T_c and α are fitting parameters. Obviously, this fitting formula is a simple generalization of the ideal Bose gas result where T_c is the critical temperature for Bose-Einstein condensation. For example, in the case of the ideal Bose gas we have $\alpha = 3/2$, for liquid helium $\alpha \approx 11/2$, and $\alpha \approx 2.3$ for weakly interacting Bose gas in the isotropic harmonic potential trap [12]. These results for the variances simplify for high temperatures (the classical limit), when the conditions

$$\frac{k_B T}{\hbar\Omega_k} \gg 1, \quad k = 1, 2, 3, \quad (73)$$

are fulfilled. In this case, using the approximation $\coth(\beta\hbar\Omega_k/2) \approx 2/(\beta\hbar\Omega_k)$, one can find from Eqs. (70)–(72) asymptotically exact equations:

$$\Delta a_{\perp} \approx \frac{1}{\omega} \sqrt{\frac{2k_B T}{5\mu}}, \quad \Delta a_3 \approx \frac{1}{\omega_0} \sqrt{\frac{2k_B T}{5\mu}}. \quad (74)$$

These equations are connected with the classical limit when the temperature is sufficiently high [Eq. (73)] because in this case the standard deviations do not depend on the constant \hbar . Equations (62) also yield a simple formula for the average standard deviation Δa :

$$\Delta a \equiv \sqrt{(\Delta a_1)^2 + (\Delta a_2)^2 + (\Delta a_3)^2} = \left(\sum_{k=1}^3 \frac{\hbar \coth\left(\frac{1}{2}\beta\hbar\Omega_k\right)}{2\mu\Omega_k} \right)^{1/2}. \quad (75)$$

In the limiting case $T \rightarrow 0$ we find that $\coth\left(\frac{1}{2}\beta\hbar\Omega_k\right) \rightarrow 1$, which simplifies the above expressions. For example, for zero temperature the above formula has the form

$$\begin{aligned} (\Delta a)^2 &= \frac{\hbar}{2\mu} \sum_{k=1}^3 \Omega_k^{-1} \\ &= \frac{\hbar}{2\mu} \left(\frac{1}{\sqrt{2}\omega} + \sqrt{\frac{2}{5\omega_0^2} + \frac{3}{10\omega^2} + \frac{\sqrt{2}}{\sqrt{5}\omega\omega_0}} \right). \end{aligned} \quad (76)$$

The formulas for Δa given by Eqs. (75) and (76) call to mind the equations for average standard deviations of a three-dimensional quantum oscillator at temperatures T and $T=0$, respectively, with characteristic frequencies Ω_k and mass μ .

To conclude this section we consider the limiting cases for the formulas (70) and (71) which connected with different trapping geometries Eqs. (44)–(46).

Case 1 (Disc form) $\omega_0 \gg \omega$. In the limiting case when $\omega/\omega_0 \rightarrow 0$ we find that $\Gamma_1 \rightarrow 0$ and $\Gamma_2 \rightarrow 1$, which leads to significant simplification of the formulas (70) and (71). For quantum fluctuations ($T=0$), one can find

$$(\Delta a_{\perp})^2 = \frac{\hbar}{4\mu} \left(\frac{1}{\Omega_2} + \frac{1}{\Omega_3} \right) = \frac{1}{4} \left(\sqrt{\frac{3}{10}} + \frac{1}{\sqrt{2}} \right) \frac{\hbar}{\mu\omega}, \quad (77)$$

$$(\Delta a_3)^2 = \frac{\hbar}{2\mu\Omega_1} = \frac{1}{2\sqrt{3}} \frac{\hbar}{\mu\omega_0}. \quad (78)$$

Case 2 (Spherical form) $\omega_0 = \omega$. In the spherically symmetric case $\Omega_1 = \sqrt{5}\omega$, $\Omega_2 = \Omega_3 = \sqrt{2}\omega$, the expressions for the basis vectors take the form

$$u^{(1)} = \begin{pmatrix} \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \\ \pm \frac{1}{\sqrt{3}} \end{pmatrix}, \quad u^{(2)} = \begin{pmatrix} \mp \frac{1}{\sqrt{6}} \\ \mp \frac{1}{\sqrt{6}} \\ \pm \frac{2}{\sqrt{6}} \end{pmatrix}, \quad u^{(3)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \quad (79)$$

It is readily verified that, despite the degeneracy, $\Omega_2 = \Omega_3 = \sqrt{2}\omega$, the three vectors are orthogonal. Hence the degeneracy does not cause any difficulty and, using Eq. (62), we find

$$\begin{aligned} (\Delta a_1)^2 &= (\Delta a_2)^2 = (\Delta a_3)^2 \\ &= \frac{\hbar}{6\mu\omega} \left[\frac{1}{\sqrt{5}} \coth\left(\frac{\sqrt{5}}{2}\beta\hbar\omega\right) + \sqrt{2} \coth\left(\frac{1}{\sqrt{2}}\beta\hbar\omega\right) \right], \end{aligned} \quad (80)$$

where the three variances are equal because of the spherical symmetry. In particular for quantum fluctuations ($T=0$) this equation takes the form

$$(\Delta a_1)^2 = (\Delta a_2)^2 = (\Delta a_3)^2 = \frac{1}{3} \left(\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{5}} \right) \frac{\hbar}{\mu\omega}. \quad (81)$$

We note here that the above expressions for the spherically symmetric case also follow from Eqs. (70) and (71) with $\Gamma_1 = 2/3$ and $\Gamma_2 = 1/3$.

Case 3 (Cigar form) $\omega \gg \omega_0$. In the limiting case when $\omega_0/\omega \rightarrow 0$ we find that $\Gamma_1 \rightarrow 1$ and $\Gamma_2 \rightarrow 0$ which again leads to significant simplification of the formulas (70) and (71). For quantum fluctuations ($T=0$) we find

$$(\Delta a_{\perp})^2 = \frac{\hbar}{4\mu} \left(\frac{1}{\Omega_1} + \frac{1}{\Omega_3} \right) = \frac{1}{4} \left(\frac{1}{2} + \frac{1}{\sqrt{2}} \right) \frac{\hbar}{\mu\omega}, \quad (82)$$

$$(\Delta a_3)^2 = \frac{\hbar}{2\mu\Omega_2} = \frac{1}{\sqrt{10}} \frac{\hbar}{\mu\omega_0}. \quad (83)$$

Finally we note that Eqs. (52) and (53) yield the expression for the standard deviations of the energies E_k for the modes $k=1,2,3$ as

$$\Delta E_k = \frac{\hbar\Omega_k}{2} \sqrt{\coth^2\left(\frac{1}{2}\beta\hbar\Omega_k\right) - 1}, \quad (84)$$

which coincides with the classical formula for a quantum harmonic oscillator with the frequency Ω_k . Hence in the classical limit given by Eq. (73), in contrast to Eqs. (74) we have a linear dependence of the deviation ΔE_k on the temperature: $\Delta E_k \approx k_B T$.

V. SCALING OF THERMAL FLUCTUATIONS OF TRAPPED BEC

We have considered that the condensate is at thermal equilibrium at a temperature T which may be zero. Beginning with a mean-field, Gross-Pitaevskii theory, and developing equations which lend themselves to quantization, we have been able to predict dynamical effects which are not calculable in the GPE approach. Our approach gives values for the excitation frequencies equal to those found using other methods, but also predicts a quantum mechanical uncertainty in the physical dimensions of the condensate. We note that the collective modes of the condensate are weakly temperature dependent in a range of temperatures below T_c , even when the condensate is strongly depleted. All our calculations were based on the effective Hamiltonian

$$\hat{\mathcal{H}} = -\frac{\hbar^2}{2\mu} \sum_{k=1}^3 \frac{\partial^2}{\partial a_k^2} + \frac{\mu}{2} \sum_{k=1}^3 \omega_k^2 a_k^2 + \frac{\mathcal{G}}{a_1 a_2 a_3}, \quad (85)$$

which, for small deviations of the ellipsoidal parameters, has the form (52). We can also introduce the bosonic annihilation

and creation operators for the diagonalized fluctuations of the ellipsoidal parameters

$$\hat{b}_k = \sqrt{\frac{\mu\Omega_k}{2\hbar}} z_k + i \sqrt{\frac{1}{2\mu\Omega_k\hbar}} \tilde{\pi}_k, \quad (86)$$

$$\hat{b}_k^\dagger = \sqrt{\frac{\mu\Omega_k}{2\hbar}} z_k - i \sqrt{\frac{1}{2\mu\Omega_k\hbar}} \tilde{\pi}_k,$$

which allows us to write a linearized Hamiltonian as

$$\hat{\mathcal{H}} = \sum_{k=1}^3 \hbar\Omega_k \left(\hat{b}_k^\dagger \hat{b}_k + \frac{1}{2} \right) + \frac{5\mathcal{G}}{2\bar{a}_1\bar{a}_2\bar{a}_3}. \quad (87)$$

Finally, we give the explicit expressions for the fluctuations of the ellipsoidal parameters in dimensionless form. For transverse fluctuations we introduce the characteristic length l , the dimensionless temperature Θ and the dimensionless frequency ξ as

$$l = \sqrt{\frac{\hbar}{\mu\omega}}, \quad \Theta = \frac{k_B T}{\hbar\omega}, \quad \xi = \frac{\omega_0}{\omega}, \quad (88)$$

then the dimensionless transverse dispersion given by Eq. (70) is

$$\begin{aligned} \left(\frac{\Delta a_\perp}{l} \right)^2 &= \left(\frac{2 - \frac{3}{2}\xi^2 + f(\xi)}{8f(\xi)} \right) \frac{\coth\left(\frac{1}{2\Theta}\sqrt{2 + \frac{3}{2}\xi^2 + f(\xi)}\right)}{\sqrt{2 + \frac{3}{2}\xi^2 + f(\xi)}} \\ &+ \left(\frac{-2 + \frac{3}{2}\xi^2 + f(\xi)}{8f(\xi)} \right) \frac{\coth\left(\frac{1}{2\Theta}\sqrt{2 + \frac{3}{2}\xi^2 - f(\xi)}\right)}{\sqrt{2 + \frac{3}{2}\xi^2 - f(\xi)}} \\ &+ \frac{\coth\left(\frac{1}{2\Theta}\right)}{4\sqrt{2}}, \end{aligned} \quad (89)$$

where function $f(\xi)$ has the form

$$f(\xi) = \sqrt{(2 - \xi^2)^2 + \frac{5}{4}\xi^4}. \quad (90)$$

For longitudinal fluctuations one may introduce the characteristic length l_0 , the dimensionless temperature Θ_0 and the dimensionless frequency ξ_0 as

$$l_0 = \sqrt{\frac{\hbar}{\mu\omega_0}}, \quad \Theta_0 = \frac{k_B T}{\hbar\omega_0}, \quad \xi_0 = \frac{\omega}{\omega_0} = \frac{1}{\xi}, \quad (91)$$

then the dimensionless longitudinal dispersion given by Eq. (71) is

$$\begin{aligned} \left(\frac{\Delta a_3}{l_0} \right)^2 &= \left(\frac{\frac{3}{2} - 2\xi_0^2 + f_0(\xi_0)}{4f_0(\xi_0)} \right) \frac{\coth\left(\frac{1}{2\Theta_0}\sqrt{\frac{3}{2} + 2\xi_0^2 + f_0(\xi_0)}\right)}{\sqrt{\frac{3}{2} + 2\xi_0^2 + f_0(\xi_0)}} \\ &+ \left(\frac{-\frac{3}{2} + 2\xi_0^2 + f_0(\xi_0)}{4f_0(\xi_0)} \right) \frac{\coth\left(\frac{1}{2\Theta_0}\sqrt{\frac{3}{2} + 2\xi_0^2 - f_0(\xi_0)}\right)}{\sqrt{\frac{3}{2} + 2\xi_0^2 - f_0(\xi_0)}}, \end{aligned} \quad (92)$$

where the function $f_0(\xi_0)$ is given as

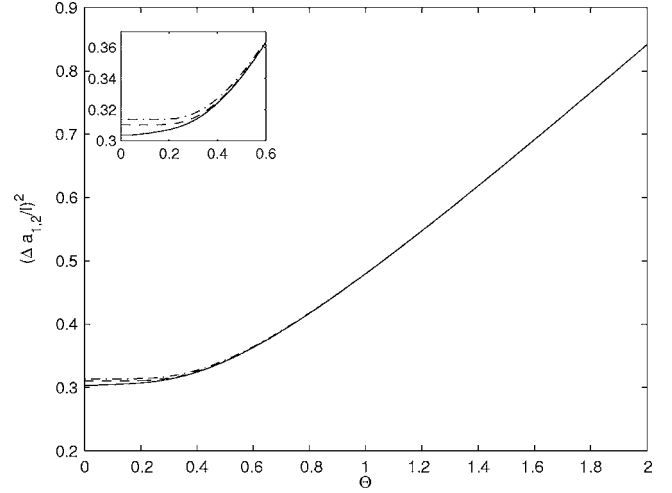


FIG. 1. The dimensionless transverse dispersion $(\Delta a_\perp/l)^2$ as a function of the dimensionless temperature Θ for different values of $\xi=0.1, 1$, and 10 . A greater parameter ξ gives the curve with a greater dispersion. There is very little dependence on ξ , with the upper (chain) line, the middle (broken) line, and the lower (full) line being for $\xi=10, 1$, and 0.1 , respectively. The subfigure shows the same curves for $0 \leq \Theta \leq 0.6$.

$$f_0(\xi_0) = \sqrt{(1 - 2\xi_0^2)^2 + \frac{5}{4}}. \quad (93)$$

We note that in all these equations the effective mass $\mu = Nm/7$ is a function of the temperature T (see previous section). Considering Eqs. (89) and (92) for fixed parameters ξ, ξ_0 and Θ, Θ_0 we can find the dimensionless transverse and longitudinal dispersions as functions of Θ, Θ_0 and ξ, ξ_0 , respectively. These curves are represented by Figs. 1–4 for different values of the parameters. We can see that the curve in Fig. 1 is similar to that in Fig. 2 (they practically coincide)

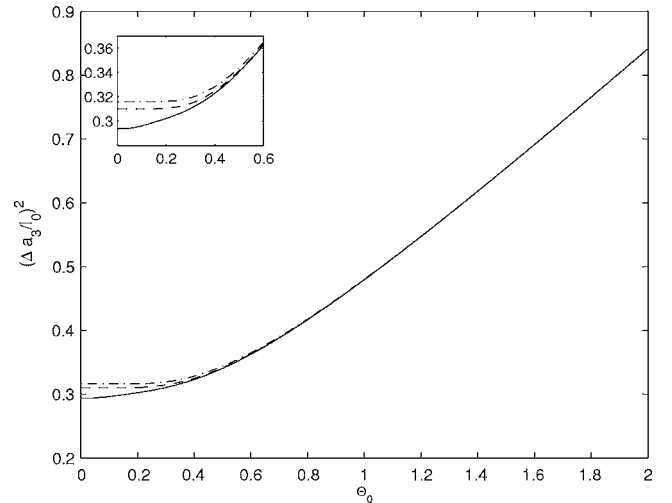


FIG. 2. The dimensionless longitudinal dispersion $(\Delta a_3/l_0)^2$ as a function of the dimensionless temperature Θ_0 . Here the upper (chain) line, the middle (broken) line, and the lower (full) line are for $\xi_0=10, 1$, and 0.1 , respectively. The subfigure shows the same curves for $0 \leq \Theta_0 \leq 0.6$.

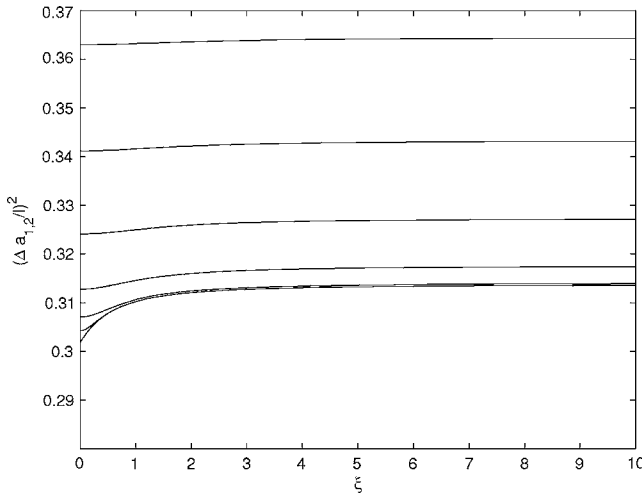


FIG. 3. The dimensionless transverse dispersion $(\Delta a_{\perp}/l)^2$ as a function of the dimensionless parameter $\xi = \omega_0/\omega$ for different values of dimensionless temperatures $\Theta = 0, 0.1, \dots, 0.6$. A greater temperature gives a curve with a greater value of dispersion $(\Delta a_{\perp}/l)^2$.

when Θ and Θ_0 are greater than 0.6. Figures 3 and 4 also demonstrate a very weak dependence of the dimensionless transverse and longitudinal dispersions on the frequency parameters ξ and ξ_0 when Θ and Θ_0 are greater than 0.6. This is the result of our choice of the scaling parameters and the nature of the transverse and longitudinal fluctuations. Hence the dimensionless transverse and longitudinal dispersions have the same behavior considered as functions of dimensionless temperature, except in a small region near zero. For these dimensionless variables the asymptotically exact Eqs. (74) have the form

$$\left(\frac{\Delta a_{\perp}}{l}\right)^2 \approx \frac{2}{5}\Theta, \quad \left(\frac{\Delta a_3}{l_0}\right)^2 \approx \frac{2}{5}\Theta_0, \quad (94)$$

when $\Theta \gg 1$ and $\Theta_0 \gg 1$ respectively. However, Fig. 1 and Fig. 2 show that these formulas at $\Theta = 1$ and $\Theta_0 = 1$ both have

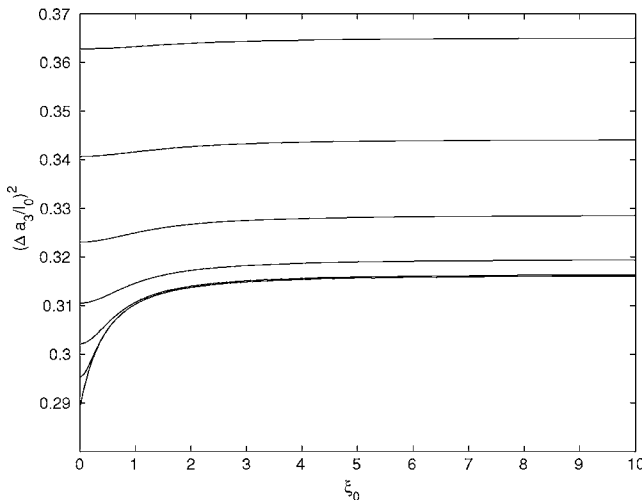


FIG. 4. The dimensionless longitudinal dispersion $(\Delta a_3/l_0)^2$ as a function of the dimensionless parameter $\xi_0 = \omega/\omega_0$ for different values of dimensionless temperatures $\Theta_0 = 0, 0.1, \dots, 0.6$. A greater temperature gives a curve with a greater value of dispersion $(\Delta a_3/l_0)^2$.

an accuracy of about 20%; at $\Theta = 2$ and $\Theta_0 = 2$ they both have an accuracy about 5% and at higher temperatures become practically exact. Hence one may consider the intervals of temperatures $\Theta \geq 2$ and $\Theta_0 \geq 2$ as the regions of classical fluctuations for transverse and longitudinal components, respectively.

Thus, from Figs. 1–4 it follows that the dimensionless transverse and longitudinal dispersions with high accuracy do not depend on the frequency parameters ξ and ξ_0 of the trap and they have the same dependence on dimensionless temperatures. This scaling is an approximate but nevertheless very accurate law and it was found in this section by direct numerical calculations of Eqs. (70) and (71). We can express this scaling of thermal fluctuations of trapped BEC by the approximate, but very accurate formulas:

$$\left(\frac{\Delta a_{\perp}}{l}\right)^2 = \Xi(\Theta), \quad \left(\frac{\Delta a_3}{l_0}\right)^2 = \Xi(\Theta_0), \quad (95)$$

where Θ and Θ_0 are greater than 0.6. One can find the scaling function $\Xi(x)$ by making the choice that ξ and ξ_0 in Eqs. (89) and (92) are equal to 1 ($\omega = \omega_0$) that corresponds to the middle (broken) curves of Figs. 1 and 2:

$$\Xi(x) = \frac{1}{6\sqrt{5}} \coth\left(\frac{\sqrt{5}}{2x}\right) + \frac{1}{3\sqrt{2}} \coth\left(\frac{1}{\sqrt{2}x}\right). \quad (96)$$

At $x \geq 2$ this equation yields $\Xi(x) \approx (2/5)x$, in accordance with the asymptotic limit [see Eqs. (94)]. Using, for example, a characteristic frequency of the trap $\nu = \omega/2\pi = 100$ Hz one can find that the characteristic temperature connected with the dimensionless temperature $\Theta = 0.6$ to be $T \approx 2.9$ nK.

VI. CONCLUDING REMARKS

In this paper we have studied the dynamics of the collective modes of a trapped dilute Bose gas in an approach which goes beyond the Gross-Pitaevskii theory. Our treatment is essentially different from previous ones (see for example the papers [12–16] mentioned above). The approach developed in this paper was first used to find the roton structure of the excitation spectrum in liquid helium II [27].

We found in the previous section a scaling of the thermal fluctuations of trapped BEC which takes place when the dimensionless temperatures Θ and Θ_0 are greater than 0.6. This scaling means that the dimensionless dispersions do not depend on the frequencies ω_k of the trap and the transverse and longitudinal ones both have the same dependence on the dimensionless temperatures. In the general case when the three frequencies ω_k , ($k = 1, 2, 3$) of the trap are different one can define the scaling parameters:

$$l_k = \sqrt{\frac{\hbar}{\mu\omega_k}}, \quad \Theta_k = \frac{k_B T}{\hbar\omega_k}, \quad k = 1, 2, 3. \quad (97)$$

Equations (95) can then be generalized as

$$\left(\frac{\Delta a_k}{l_k}\right)^2 = \Xi(\Theta_k), \quad k = 1, 2, 3, \quad (98)$$

where the scaling function $\Xi(x)$ is given by Eq. (96) and $\Theta_k \geq 0.6$. This generalization is proved by the results ob-

tained in Appendix B. We note again that in all these equations the effective mass $\mu = N_c m / 7$ is a function of the temperature T .

Finally, we discuss a simple estimation of the damping effect for low-energy excitations at temperatures $k_B T \lesssim \mu_0$, where μ_0 is the chemical potential of the trap and we suppose that $N_{ex}/N_c \ll 1$, where N_{ex} is the number of excited atoms and N_c is the number of condensed atoms. We also assume here that $\tau_{col}\Omega_k \gg 1$, where τ_{col} is collision time and hence in this case collective excitations have hydrodynamic nature. The solution of the classical Eq. (14) has the form given by Eq. (35) where $z_k = A_k \cos(\Omega_k t + \Phi_k)$ and A_k are the arbitrary constants. However, when the damping is provided by the interaction of the low-energy excitations of a trapped BEC with the thermal excitations, the amplitudes A_k ($k=1, 2, 3$) depend slowly on time: $A_k = A_{ok} e^{-\lambda_k t}$, $\lambda_k / \Omega_k \ll 1$. Hence the damping parameters can be defined as $\lambda_k = -\delta A_k / (A_k \delta t)$. When the conditions mentioned above are fulfilled using the ergodic hypothesis one may assume that $\delta A_k / \delta t \sim \Delta z_k / \Delta t$, where $\Delta E_k \Delta t \sim \hbar$. Thus the damping parameters can be estimated as $\lambda_k \sim \Delta z_k \Delta E_k$, where the standard deviations Δz_k and ΔE_k are given by Eqs. (61) and (84). Hence for the collisionless regime $\tau_{col}\Omega_k \gg 1$ and $N_{ex}/N_c \ll 1$ in the region of classical fluctuation Θ , $\Theta_0 \geq 2$ and $k_B T \lesssim \mu_0$, the temperature dependence of the damping parameters has the form $\lambda_k \sim T^{3/2}$, which is in agreement with the result reported in [38]. We consider this discussion of the damping effect for low-energy excitations to emphasize that the approach developed in this paper is relevant to this problem. However, the full treatment of it would require consideration of the interaction of the BEC with the noncondensed thermal cloud.

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APPENDIX A: SOLUTION OF GROSS-PITAEVSKII GAIN EQUATION

The modeling of an atom laser which includes a semiclassical treatment of pumping and losses for a trapped BEC leads to a generalized GPE with a linear gain term [23]. Here we consider in detail the self-similar solution of this model because it can be generalized to describe a number of different experimental situations [30–37].

Introducing the real amplitude $A(\mathbf{x}, t)$ and phase $\phi(\mathbf{x}, t)$, so that $\psi = A \exp(i\phi)$ we may write Eq. (1) as the system of equations

$$\frac{\partial A}{\partial t} = -2\gamma(\nabla A)(\nabla \phi) - \gamma A \nabla^2 \phi + \frac{g(t)}{2} A, \quad (\text{A1})$$

$$-\left[\gamma(\nabla \phi)^2 + \frac{\partial}{\partial t} \phi \right] A = -\gamma \nabla^2 A + \sum_{k=1}^3 \beta_k x_k^2 A + k A^3, \quad (\text{A2})$$

where

$$\gamma = \frac{\hbar}{2m}, \quad \beta_k = \frac{m\omega_k^2}{2\hbar}, \quad k = \frac{4\pi\hbar a_0}{m}, \quad (\text{A3})$$

with the normalization condition

$$N_c(t) = \int A^2(\mathbf{x}, t) d\mathbf{x}. \quad (\text{A4})$$

Equations (A1) and (2)–(4) then lead to a set of equations (with $k=1, 2, 3$)

$$\frac{df_k}{dt} = -2\gamma c_k f_k + \frac{1}{2} g_k(t) f_k, \quad (\text{A5})$$

where

$$\sum_{k=1}^3 g_k(t) = g(t) \equiv \frac{\dot{N}_c(t)}{N_c(t)}. \quad (\text{A6})$$

We now introduce the characteristic dimensionless parameter

$$E_c = 8\pi n_c a_0 l_m^2, \quad (\text{A7})$$

where l_m is the minimum of the lengths along the three axes of the trapped BEC and n_c is the average condensate density. For the conditions $a_0 > 0$ (positive s -wave scattering length) and $E_c \gg 1$, we can neglect the term $-\gamma \nabla^2 A$ on the right-hand side of Eq. (A2). Taking into account Eqs. (2)–(4), we can now write Eq. (A2) in the form

$$\sum_{k=1}^3 \left(4\gamma c_k^2 + \frac{dc_k}{dt} + \beta_k \right) (f_1 f_2 f_3 f_k^2)^{-2} \exp\left(2 \int_0^t g_k(t') dt' \right) \xi_k^2 + (f_1 f_2 f_3)^{-2} \frac{d\phi_0}{dt} = -k F^2(\xi_1, \xi_2, \xi_3). \quad (\text{A8})$$

Due to the fact that the right-hand side of Eq. (A8) depends only on the ξ_k , we can write a system of ordinary differential equations (for $k=1, 2, 3$),

$$\frac{dc_k}{dt} + 4\gamma c_k^2 + \beta_k = k \theta_k^2 f_k^4 (f_1 f_2 f_3)^2 \exp\left(-2 \int_0^t g_k(t') dt' \right). \quad (\text{A9})$$

We also find that

$$\frac{d\phi_0}{dt} = -k (f_1 f_2 f_3)^2, \quad (\text{A10})$$

and

$$F(\xi_1, \xi_2, \xi_3) = \left[1 - \sum_{k=1}^3 \theta_k \xi_k^2 \right]^{1/2} S \left(1 - \sum_{k=1}^3 \theta_k \xi_k^2 \right), \quad (\text{A11})$$

where the θ_k are positive constants and $S(x)$ is the Heaviside step function. Note here that by writing the equation for the phase in the form of Eq. (A10), we are fixing a normalization condition for the function $F(\xi_1, \xi_2, \xi_3)$. This does not alter the generality of the solution because Eq. (2) allows for a scaling factor in the definitions of f_1, f_2, f_3 and F . In accord with Eq. (A11), we fix this scaling factor by the condition $F(0, 0, 0) = 1$.

Let us now consider the normalization condition, Eq. (A4), where the amplitude is given by Eqs. (A1) and (A11). Calculating the integral on the right-hand side of Eq. (A4), we can write the number of atoms in the trapped BEC as

$$N_c(t) = \left(\frac{8\pi}{15}\right) (\theta_1 \theta_2 \theta_3)^{-1/2} \exp\left(\int_0^t g(t') dt'\right). \quad (\text{A12})$$

Introducing the definition

$$N_k(t) = N_k(0) \exp\left(\int_0^t g_k(t') dt'\right), \quad (\text{A13})$$

where

$$N_k(0) = \left(\frac{8\pi}{15}\right)^{1/3} \theta_k^{-1/2}, \quad (\text{A14})$$

we can rewrite Eq. (A12) in the form

$$N_c(t) = \prod_{k=1}^3 N_k(t). \quad (\text{A15})$$

Hence the self-similar variables ξ_k are defined as

$$\xi_k = \frac{N_k(0) f_k^2(t)}{N_k(t)} x_k, \quad (\text{A16})$$

and the ellipsoidal parameters have the form

$$a_k(t) = \left(\frac{15}{8\pi}\right)^{1/3} \frac{N_k(t)}{f_k^2(t)}. \quad (\text{A17})$$

Combining Eqs. (13) and (A11) with the definitions of Eqs. (A13), (A16), and (A17), we can represent the amplitude $A(\mathbf{x}, t)$ in the form given by Eqs. (7) and (8). Equations (A5), (A9), and (A10) and the definition of Eq. (A17) yield Eqs. (9)–(11) and the initial conditions of Eq. (12). Hence, by using the variables defined in Eq. (A17) we are able to exclude the indeterminate parameters θ_k in our self-similar solution.

APPENDIX B: GENERAL ASYMMETRIC CASE

In the general case where all three trapping frequencies are different, the equations for the eigenvectors, $u^{(k)}$, have the form

$$\begin{aligned} (3\omega_1^2 - \Omega_k^2)u_1^{(k)} + \omega_1\omega_2u_2^{(k)} + \omega_1\omega_3u_3^{(k)} &= 0, \\ \omega_1\omega_2u_1^{(k)} + (3\omega_2^2 - \Omega_k^2)u_2^{(k)} + \omega_2\omega_3u_3^{(k)} &= 0, \\ \omega_1\omega_3u_1^{(k)} + \omega_3\omega_2u_2^{(k)} + (3\omega_3^2 - \Omega_k^2)u_3^{(k)} &= 0. \end{aligned} \quad (\text{B1})$$

The solution to these equations is

$$\begin{aligned} u_1^{(k)} &= \frac{(\Omega_k^2 - 2\omega_2^2)\omega_1\omega_3}{\Delta_k} u_3^{(k)}, \\ u_2^{(k)} &= \frac{(\Omega_k^2 - 2\omega_1^2)\omega_2\omega_3}{\Delta_k} u_3^{(k)}, \end{aligned} \quad (\text{B2})$$

where

$$\Delta_k = (\Omega_k^2 - 3\omega_1^2)(\Omega_k^2 - 3\omega_2^2) - \omega_1^2\omega_2^2, \quad (\text{B3})$$

and $u_3^{(k)}$ may be found from the normalization condition

$$\sum_j (u_j^{(k)})^2 = 1. \quad (\text{B4})$$

We note here that these expressions are not valid in the axially symmetric case where $\omega_1 = \omega_2 = \omega$ and $\omega_3 = \omega_0$, because Ω_3^2 then becomes equal to $2\omega^2$, so that $\Delta_3 = 0$ and the solutions are undefined. However, for $k=1,2$, we find the normalized solutions

$$\begin{aligned} u_1^{(k)} &= \pm \frac{(\Omega_k^2 - 2\omega_2^2)\omega_1\omega_3}{\sqrt{Q_k}}, \\ u_2^{(k)} &= \pm \frac{(\Omega_k^2 - 2\omega_1^2)\omega_2\omega_3}{\sqrt{Q_k}}, \\ u_3^{(k)} &= \pm \frac{\Delta_k}{\sqrt{Q_k}}, \end{aligned} \quad (\text{B5})$$

where

$$Q_k = (\Omega_k^2 - 2\omega_2^2)^2 \omega_1^2 \omega_3^2 + (\Omega_k^2 - 2\omega_1^2)^2 \omega_2^2 \omega_3^2 + \Delta_k^2. \quad (\text{B6})$$

Using the orthonormality of the eigenvectors, we can give a general definition for $u^{(3)}$ via the product $u^{(1)} \times u^{(2)}$:

$$\begin{aligned} u_1^{(3)} &= [(\Omega_1^2 - 2\omega_1^2)\Delta_2 - (\Omega_2^2 - 2\omega_1^2)\Delta_1] \frac{\omega_2\omega_3}{\sqrt{Q_1Q_2}}, \\ u_2^{(3)} &= [(\Omega_2^2 - 2\omega_2^2)\Delta_1 - (\Omega_1^2 - 2\omega_2^2)\Delta_2] \frac{\omega_1\omega_3}{\sqrt{Q_1Q_2}}, \\ u_3^{(3)} &= [(\Omega_1^2 - 2\omega_2^2)(\Omega_2^2 - 2\omega_1^2) \\ &\quad - (\Omega_2^2 - 2\omega_2^2)(\Omega_1^2 - 2\omega_1^2)] \frac{\omega_1\omega_2\omega_3^2}{\sqrt{Q_1Q_2}}. \end{aligned} \quad (\text{B7})$$

We now find that the matrix u_{sk} ($=u_s^{(k)}$), defined by Eq. (B5) for $k=1,2$ and by Eq. (B7) for $k=3$, is not degenerate (the determinant being equal to 1), when Q_1 and Q_2 are not equal to zero.

Using Eqs. (62) and (B5)–(B7), we may now find the standard deviations in the general case with all three frequencies different:

$$\begin{aligned} \Delta a_1 &= \left[\frac{\hbar(\Omega_1^2 - 2\omega_2^2)^2 \omega_1^2 \omega_3^2}{2\mu\Omega_1Q_1} \coth\left(\frac{1}{2}\beta\hbar\Omega_1\right) \right. \\ &\quad + \frac{\hbar(\Omega_2^2 - 2\omega_2^2)^2 \omega_1^2 \omega_3^2}{2\mu\Omega_2Q_2} \coth\left(\frac{1}{2}\beta\hbar\Omega_2\right) \\ &\quad \left. + \frac{\hbar\omega_2^2\omega_3^2 [(\Omega_1^2 - 2\omega_1^2)\Delta_2 - (\Omega_2^2 - 2\omega_1^2)\Delta_1]^2}{2\mu\Omega_3Q_1Q_2} \right]^{1/2}, \end{aligned} \quad (\text{B8})$$

$$\Delta a_2 = \left[\frac{\hbar(\Omega_1^2 - 2\omega_1^2)\omega_2^2\omega_3^2}{2\mu\Omega_1 Q_1} \coth\left(\frac{1}{2}\beta\hbar\Omega_1\right) + \frac{\hbar(\Omega_2^2 - 2\omega_1^2)\omega_2^2\omega_3^2}{2\mu\Omega_2 Q_2} \coth\left(\frac{1}{2}\beta\hbar\Omega_2\right) + \frac{\hbar\omega_1^2\omega_2^2[(\Omega_2^2 - 2\omega_2^2)\Delta_1 - (\Omega_1^2 - 2\omega_2^2)\Delta_2]^2}{2\mu\Omega_3 Q_1 Q_2} \right]^{1/2}, \quad (\text{B9})$$

$$\Delta a_3 = \left[\frac{\hbar\Delta_1^2}{2\mu\Omega_1 Q_1} \coth\left(\frac{1}{2}\beta\hbar\Omega_1\right) + \frac{\hbar\Delta_2^2}{2\mu\Omega_2 Q_2} \coth\left(\frac{1}{2}\beta\hbar\Omega_2\right) + \frac{\hbar\omega_1^2\omega_2^2\omega_3^4[(\Omega_1^2 - 2\omega_2^2)(\Omega_2^2 - 2\omega_1^2) - (\Omega_2^2 - 2\omega_2^2)(\Omega_1^2 - 2\omega_1^2)]^2}{2\mu\Omega_3 Q_1 Q_2} \coth\left(\frac{1}{2}\beta\hbar\Omega_3\right) \right]^{1/2}. \quad (\text{B10})$$

To conclude, we note that these expressions for the standard deviations are valid only when Q_1 and Q_2 are not equal to zero. For example in the spherically symmetric case $Q_2=0$ and hence in this degenerate case we should use Eq. (80). Thus, despite the generality of the results given here, in the axially and spherically symmetric cases we should use the equations derived in Sec. IV.

Actually the above general equations can be used even in the spherically symmetric case as long as we use an accurate limiting procedure. In the spherically symmetric case Eq. (B3) can be written as $\Delta_k = (\Omega_k^2 - 4\omega^2)(\Omega_k^2 - 2\omega^2)$; hence the normalized Eq. (B2) for $k=1,2$ yields

$$u_1^{(k)} = u_2^{(k)} = \pm \frac{\omega^2}{\sqrt{(\Omega_k^2 - 4\omega^2)^2 + 2\omega^4}}, \quad (\text{B11})$$

$$u_3^{(k)} = \pm \frac{(\Omega_k^2 - 4\omega^2)}{\sqrt{(\Omega_k^2 - 4\omega^2)^2 + 2\omega^4}}.$$

Defining $u^{(3)}$ via the product $u^{(1)} \times u^{(2)}$, one may verify that these eigenvectors lead to Eq. (80). We also note that the three expressions given above can be combined to give the same result as in Eq. (75) for the average standard deviation $\Delta a = [\sum_{i=1}^3 (\Delta a_i)^2]^{1/2}$.

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