Free-electron quantum signatures in intense laser fields

Mohammad Mahmoudi,[1,2,3] Yousef I. Salamin,[1,2,4] and Christoph H. Keitel[1,2]
[1]*Max-Planck-Instiut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany*
[2]*Theoretische Quantendynamik, Physikalisches Institit der Universität Freiburg, Hermann-Herder-Strasse 3, 79104 Freiburg, Germany*
[3]*Physics Department, Zanjan University, P.O. Box 45195-313, Zanjan, Iran*
[4]*Physics Department, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates*
(Received 31 March 2005; published 2 September 2005)

Quantum signatures of a free electron in interaction with a continuous-wave radiation field are investigated by looking for negativities in the Wigner function of the system. The free-electron wave function in the radiation field is calculated fully analytically by solving the appropriate Schrödinger equation in the Krammers-Henneberger frame. It is found that pronounced quantum signatures show up already for a laser peak field of magnitude $E_0=1$–2 a.u. and a frequency $\omega=1$ a.u. However, the nonclassical behavior gets lost if the interaction with the radiation field is taken in the dipole approximation.

DOI: 10.1103/PhysRevA.72.033402 PACS number(s): 42.50.Hz, 42.50.Vk, 42.50.Ct, 52.75.Di

I. INTRODUCTION

For studying the interaction of a single bound electron with a laser field, at least a semiclassical approach whereby the electron is treated quantum mechanically and the laser field classically is essential in a wide range of applications. Among other things, the semiclassical treatments have dominated, for a long time, the investigation of phenomena like laser-assisted multiphoton and above-threshold ionization and harmonic generation.

On the other hand, it is generally believed that a free electron may be treated classically as long as the radiation energy is small compared with the (center-of-mass) electron-positron pair-production energy threshold ~ 1.022 MeV. Only laser photons of intensity as high as 2×10^{29} W/cm^2 may achieve this [1–3]. However, it was shown experimentally several years ago [4] that electron-positron pairs could be produced by making the threshold center-of-mass energy available in γ-γ collisions at intensities of the order of only 10^{19} W/cm^2. This has been achieved in two steps. First, low-energy (2.35 eV) laser photons were backscattered off a high-energy (46.6 GeV) electron beam (acquiring GeV energies themselves in the process), and then these high-energy photons were made to collide against a few of the low-energy laser photons.

For moderately intense laser pulses quantum electrodynamical effects due to the vacuum should not play a role anymore. Then the question arises as to whether the quantum nature of a free electron in an intense classical laser field may ever play a significant role [5]. Apart from rather small spin-induced dynamics [6,7], quantum features have always been considered negligible in such situations. In addition to powerful low-frequency laser fields [8], intense high-frequency laser fields are now available with a wavelength approaching the size of a laser-driven quantum wave packet. Currently, laser systems exist [8] at a number of laboratories around the world which promise to make investigating some of these effects possible [9].

This paper aims at showing that a free electron exhibits nonclassical behavior in the presence of an intense high-frequency laser field [10]. This will be demonstrated by the negativity exhibited by the Wigner function of the free electron in the given environment. It is well known that negativity in the Wigner function is an indicator of nonclassicality [11]. It will be shown that, starting with an initial free electronic state described by a Gaussian distribution, the Wigner function corresponding to the time-evolved wave function of the free electron in the laser field exhibits negativities, provided the dipole approximation is not made. Recall that the dipole approximation amounts to dropping the term $\boldsymbol{k}\cdot\boldsymbol{r}$, where $\boldsymbol{k}$ is the wave vector of the laser and $\boldsymbol{r}$ is the position vector of the free electron, from the description of the laser fields (or, equivalently, from the vector potential). With that, coupling of the free electron to the magnetic component of the laser field gets lost. However, at sufficiently high field intensities the $\boldsymbol{v}\times\boldsymbol{B}$ force can be quite strong ($\boldsymbol{v}$ is the velocity of the free electron and $\boldsymbol{B}$ is the magnetic field strength of the laser.) It causes shearing and distortion in the free electron when the latter is modeled by a quantum mechanical wave packet. So, without resorting to the dipole approximation, coupling to the laser magnetic field is retained and that results in the distortion of the free-electron quasi probability distribution, making it negative at times. An intuitive explanation of the effect is also presented in terms of quantum pathway interferences.

In the next section, our program for calculating the Wigner function of a free electron interacting with an intense laser field is outlined. Some crucial background material on the Krammer-Henneberger frame is given there. Next, the time-evolved wave function is worked out analytically, and the Wigner function is calculated numerically, with interaction with the field taken in the dipole approximation. The same thing is also done in Sec. II B, but without making the dipole approximation. We present and discuss our main results in Sec. III. Our conclusions based on those results are finally given in Sec. IV.

II. WIGNER FUNCTION

To investigate the quantum signatures of the free electron in a laser field, we employ the quantum-mechanical phase-space distribution, known as the Wigner function [12]. For a three-dimensional system, phase space is six dimensional and the said function may be written as

$$W(\boldsymbol{r},\boldsymbol{p},t) = \frac{1}{(\pi\hbar)^3}\int_{-\infty}^{\infty} d^3q\Psi^*(\boldsymbol{r}+\boldsymbol{q},t)\Psi(\boldsymbol{r}-\boldsymbol{q},t) \times \exp\left[\frac{2i\boldsymbol{q}\cdot\boldsymbol{p}}{\hbar}\right], \quad (1)$$

where $\boldsymbol{p}$ represents the generalized momenta conjugate to the spatial coordinates $\boldsymbol{r}$ of the particle and Ψ is its wave function. The latter is, of course, obtained from solving the appropriate Schrödinger equation. The Wigner function is a quasiprobability distribution whose negativity indicates nonclassical behavior.

We let the free electron have a mass m and a charge $-e$, and we model the laser field, by a vector potential $\mathbf{A}$ in the space-time coordinates $(\boldsymbol{r},t)$. In general, the Hamiltonian of such a system reads

$$H = \frac{(\boldsymbol{p}+e\boldsymbol{A}/c)^2}{2m} = \frac{\boldsymbol{p}^2}{2m} + \frac{e}{mc}\boldsymbol{A}\cdot\boldsymbol{p} + \frac{e^2A^2}{2mc^2} = H_f + H_i, \quad (2)$$

where $H_f = p^2/2m$ represents kinetic energy of the free electron, H_i describes its interaction with the radiation field, and c is the speed of light in vacuum. Note, further, that the Coulomb gauge ($\nabla\cdot\boldsymbol{A}=0$) has been employed. Throughout this work, Gaussian units will be used in the analysis while atomic units will be employed in the numerical calculations.

The next step would be to solve Schrödinger's equation

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t} \quad (3)$$

for Ψ and to use it in the Wigner function, Eq. (1). However, it turns out to be easier, and intuitively more appealing, to solve the Schrödinger equation in the Krammers-Henneberger (KH) frame [13], instead. This is a noninertial reference frame in which the well-known quiver motion of the free electron in the laser field may be eliminated (exactly within the dipole approximation and approximately without it). The wave function in the KH frame is related to its laboratory counterpart by the following unitary transformation

$$\Psi_{\mathrm{KH}}(\boldsymbol{r},t) = U\Psi(\boldsymbol{r},t), \quad U \equiv \exp\left(\frac{i}{\hbar}\int H_i(\boldsymbol{r},t')dt'\right). \quad (4)$$

When the transformation (4) is followed by a second transformation Ω (see the Appendix), the Schrödinger equation becomes

$$i\hbar\frac{\partial\Phi_{\mathrm{KH}}}{\partial t} \approx -\frac{\hbar^2}{2m}\nabla^2\Phi_{\mathrm{KH}}, \quad (5)$$

i.e., a noninteracting particle equation that admits an easy solution. Note that because the two terms making up H_i commute, the transformation may be written as $U=U_1U_2$ and

$$\Psi = U_2^\dagger U_1^\dagger \Omega^\dagger \Phi_{\mathrm{KH}}. \quad (6)$$

The above developments allow for calculation of the Wigner function to proceed along the following lines. First, a choice is made for the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ that best models the laboratory laser field. This function is then used to calculate the transformations U_1, U_2, and Ω. Next, a wave function Φ_{KH} that describes the free electron most realistically is adopted. Finally, Eq. (6) is used to calculate the laboratory wave function Ψ, and this is subsequently employed to obtain the Wigner function on the basis of Eq. (1). Several nontrivial steps of this procedure may be carried out analytically, while calculation of the Wigner function is done numerically.

The electric component of a continuous-wave (cw) radiation field, polarized in the x direction and propagating along $+z$, may be modeled by

$$\boldsymbol{E} = \hat{\boldsymbol{x}}E_0\sin(\omega\eta), \quad (7)$$

where

$$\eta = t - \frac{z}{c} \quad (8)$$

E_0 is a constant amplitude, and ω is the frequency. Two cases of free-electron–field interactions will be investigated in what follows—namely, one with and another without the dipole approximation (DA).

A. Dipole approximation

In the DA, valid whenever $kz \ll 1$, the corresponding electric field follows, by differentiation, from the vector potential

$$\boldsymbol{A}(t) = \hat{\boldsymbol{x}}\frac{cE_0}{\omega}[\cos(\omega t) - 1]. \quad (9)$$

The presence of the constant term in Eq. (9) does not affect the observable electromagnetic fields and has the advantage of making $\mathbf{A}$ well behaved in the zero-frequency limit [14]. It is also simple enough to allow for considerable progress in the analytic investigation. Simple integrations and straightforward algebra lead to (see the Appendix)

$$\boldsymbol{\alpha} = \hat{\boldsymbol{x}}\frac{eE_0}{m\omega^2}[\sin(\omega t) - \omega t], \quad (10)$$

$$D = \frac{i}{\hbar}\frac{(eE_0)^2}{8m\omega^3}[\sin(2\omega t) - 8\sin(\omega t) + 6\omega t], \quad (11)$$

$$\boldsymbol{\beta} = 0, \quad (12)$$

$$U_2 = e^D. \quad (13)$$

Hence,

$$\Psi = e^{-D}\Phi_{\mathrm{KH}}(x-\alpha,y,z). \quad (14)$$

Evaluation of the full wave function follows immediately from Eq. (14) once a choice for Ψ_{KH} that satisfies Eq. (5) has been made. In order to help us gain some confidence in the procedure outlined above for calculation of the laboratory Ψ, let us consider a one-dimensional (1D) free-electron model. Assuming that the free electron is constrained to move in one dimension—say, along an x axis—a choice in the KH frame would be the momentum wave function

$$\Phi_{\mathrm{KH}}(x,t)=\frac{1}{\sqrt{2\pi}}e^{i(k_e x-\omega_e t)},\quad \hbar\omega_e=\frac{\hbar^2k_e^2}{2m},\tag{15}$$

where $\hbar k_e=p$ represents the momentum of the free electron. Recall here that the wave function (15) does not suitably represent a localized particle of well-defined momentum. For one thing, it leads to a wave velocity that is one-half the classical velocity of the particle it is supposed to represent [15]. In any case, it does lead to the following time-dependent solution:

$$\Psi=\frac{e^{-D}}{\sqrt{2\pi}}e^{i[k_e(x-\alpha)-\omega_e t]}.\tag{16}$$

This is precisely the solution developed by Rau and Unnikrishnan [16] using evolution operator methods. Note that it reduces to the free-electron plane wave function in the $\omega\rightarrow 0$ limit. Unlike the often-used Volkov wave function, this one does not retain any additional phase factors [17].

For a realistic representation of the localized free electron, a Gaussian wave packet moving along the z axis, and having nonzero extension along x, seems more appropriate. Our choice, taken straight from a textbook [15], is the 2D wave packet

$$\Phi_{\mathrm{KH}}=\frac{1}{a\sqrt{2\pi}(1+it/\tau)}\exp\left[\frac{i\tau}{t}\left(\frac{x^2+z^2}{4a^2}\right)\right]\times\exp\left\{-\frac{i\tau/4a^2t}{1+it/\tau}\left[x^2+\left(z-\frac{p_0}{m}t\right)^2\right]\right\},\tag{17}$$

where $\tau=2ma^2/\hbar$, a represents a reasonable *size* for the wave packet, and p_0 stands for the initial momentum of the free electron. Assuming it is born from an atom, possibly by ionization, a *size* for the free electron may be taken as $a\sim\Delta x\sim\Delta z$. This wave packet leads to a probability distribution which describes a point particle in the $a\rightarrow 0$ limit. In other words, it turns out to be zero everywhere except on the classical particle trajectory [15].

Using Eqs. (10), (13), and (17) in Eq. (14), we arrive at

$$\Psi^{(\mathrm{DA})}=\frac{e^{-D}}{a\sqrt{2\pi}(1+it/\tau)}\exp\left[\frac{i\tau}{t}\left(\frac{\bar{x}^2+z^2}{4a^2}\right)\right]\times\exp\left\{-\frac{i\tau/4a^2t}{1+it/\tau}\left[\bar{x}^2+\left(z-\frac{p_0}{m}t\right)^2\right]\right\},\tag{18}$$

where $\bar{x}=x-\alpha$. Finally, Eq. (18) is used in Eq. (6) and the Wigner function is calculated numerically. In Fig. 1, the quasiprobability distribution is shown, at $t=0.5$ T, where T is the laser period, for field amplitudes $E_0=0.5$, 1, and 2 a.u. Recall that $E_0=1$ a.u. is equivalent to an electric field strength of about 5.4×10^9 V/cm. This is much less than the QED *critical* field strength $E_{crit}=1.3\times 10^{16}$ V/cm at which a static electric field would spontaneously break down into electron-positron pairs [2,3]. Note that the initially symmetric distribution shows little signs of distortion due to interaction with the radiation field. No apparent signs of negativity are exhibited, however, in Fig. 1. It is also hard to detect any appreciable difference between the three distributions shown.

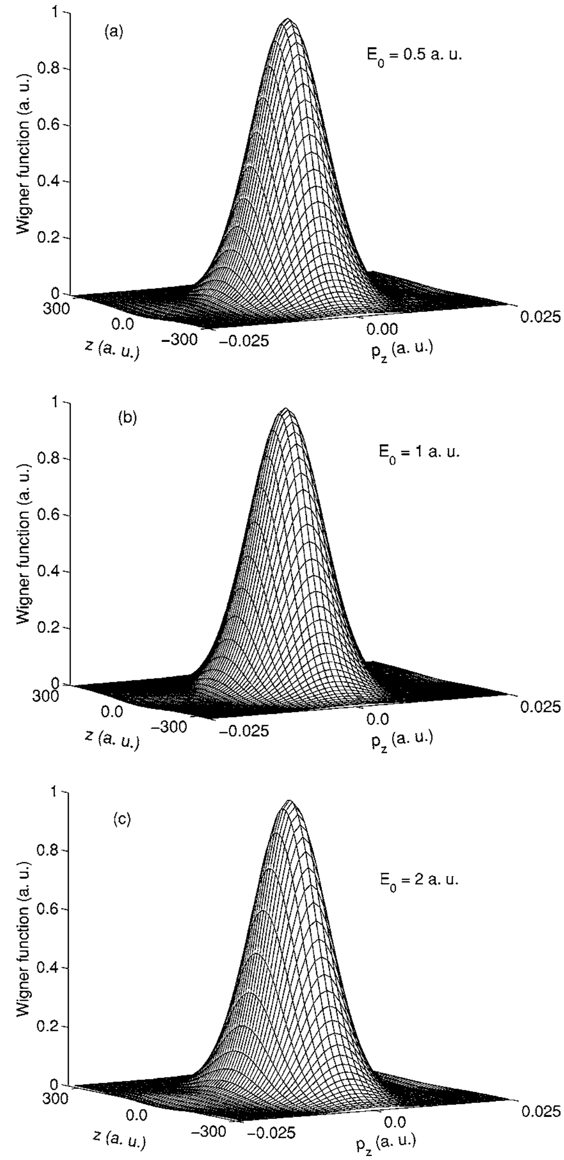

FIG. 1. The Wigner function of a free electron initially (at $t=0$) at rest at the origin of coordinates and subsequently subjected to a plane-wave, dipole-approximation, laser field. The plots are for a wave packet that has interacted with a field of frequency $\omega=1$ a.u. for a time equal to 0.5 T, where T is a laser- field cycle. The snapshots correspond to $x=0=p_x$ and an initial wave packet modeled by a two-dimensional Gaussian of width equal to $a=100$ a.u. in either spatial dimension.

B. Nondipole approximation

Recall that making the dipole approximation amounts to ignoring the effects on the free-electron dynamics due to the magnetic component of the laser field [18,19]. The magnetic component of the Lorentz force on the free electron becomes comparable to the electric component when the free electron's velocity approaches the speed of light, a situation we know happens under conditions of interaction with a high-intensity laser field.

Next, we abandon the dipole approximation altogether. Thus, in place of Eq. (9) we will now model the radiation field by the nondipole-approximation (NDA) vector potential

$$\boldsymbol{A} = \hat{\boldsymbol{x}}\frac{cE_0}{\omega}[\cos(\omega\eta) - 1], \tag{19}$$

where η is given in Eq. (8). This choice for the vector potential is also well behaved in the $\omega \to 0$ limit; it vanishes for a free electron initially at rest at the origin of coordinates, and it is admissible by the principle of gauge invariance. Using it we get the following NDA counterparts for Eqs. (10)–(13):

$$\boldsymbol{\alpha} = \hat{\boldsymbol{x}}\frac{eE_0}{m\omega^2}[\sin(\omega\eta) - \omega\eta], \tag{20}$$

$$D = \frac{i}{\hbar}\frac{(eE_0)^2}{8m\omega^3}[\sin(2\omega\eta) - 8\sin(\omega\eta) + 6\omega\eta], \tag{21}$$

$$\boldsymbol{\beta} = -\hat{z}\left(\frac{i\hbar}{mc}\right)D, \tag{22}$$

$$U_2 = e^D. \tag{23}$$

Note that Eqs. (19)–(21) follow from Eqs. (9)–(11), respectively, by replacing ωt with $\omega\eta$, as in Ref. [14]. Eventually, the following expression for the laboratory wave function $\Psi^{(\mathrm{NDA})}$ results:

$$\Psi^{(\mathrm{NDA})} = \frac{e^{-D}}{a\sqrt{2\pi}(1 + it/\tau)}\exp\left[\frac{i\tau}{t}\left(\frac{\bar{x}^2 + \bar{z}^2}{4a^2}\right)\right] \times \exp\left\{-\frac{i\tau/4a^2t}{1 + it/\tau}\left[\bar{x}^2 + \left(\bar{z} - \frac{p_0}{m}t\right)^2\right]\right\}, \tag{24}$$

with

$$\bar{x} = x - \alpha \quad \text{and} \quad \bar{z} = z - \beta. \tag{25}$$

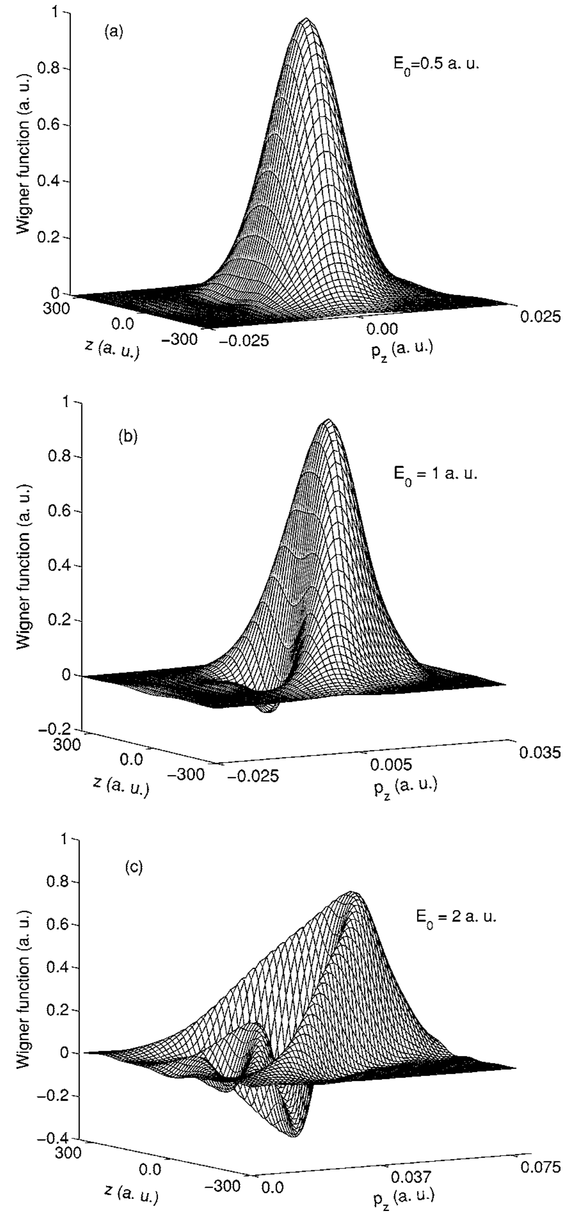

FIG. 2. Same as Fig. 1, but without the dipole approximation.

III. RESULTS AND DISCUSSION

To look for nonclassical behavior, the Wigner function of the system is considered next. Results from numerical calculations using Eq. (24) in Eq. (1) are shown in Fig. 2. The quasiprobability distribution is shown for three laser field intensities, corresponding to E_0=0.5, 1 and 2 a.u., respectively. In Figs. 2(b) and 2(c), the Wigner function exhibits negativities, clear manifestations of the quantum signatures. The nonclassical behavior is more pronounced in (c) than in (b), as expected. Note that our calculations have employed a wave packet of size $2a \sim 200$ a.u. and laser fields of wavelength $\lambda \sim 850$ a.u. The case for abandoning the dipole approximation is made even stronger by the observation that the quantum signatures become more pronounced for wave packet sizes smaller than the laser wavelength.

To further elucidate these conclusions we refer to Fig. 3. Within the dipole approximation, Fig. 3(a), the forces (indicated by the arrows, which also indicate free-electron quantum pathways) affecting all parts of the wave packet (solid circle) are roughly the same in both magnitude and direction. However, beyond the dipole approximation, Fig. 3(b), while, at this stage, still a negligible ponderomotive force acts, parts of the wave packet on the left-hand side may be subject to forces different from those to which the parts on the right-hand side are exposed, at the same time. However, only beyond the dipole approximation and with significant dynamics in the laser propagation direction 3(c), may fractions of the wave packet move in both polarization and propagation directions. Thus, there may be quantum pathway interferences leading to negativities in the Wigner function. Suppose, further, that measurement, at some time, reveals that the free electron is roughly at point B in Fig. 3(c). Then, one would not be certain whether it was at A or at A' at some earlier time.

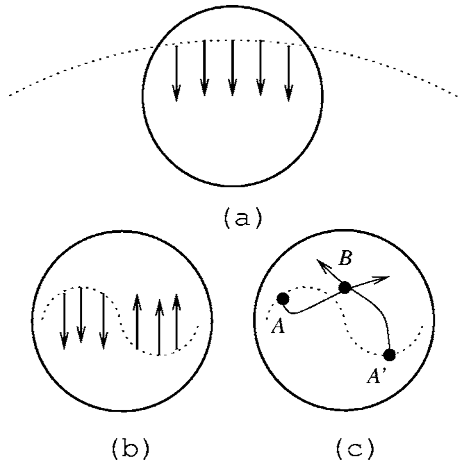

FIG. 3. A schematic of the interaction of a free-electron wave packet (solid lines) with a laser wave (dotted lines). (a) In the DA ($\lambda \gg a$), where a is the size of the wave packet, (b) in the NDA, but in the regime in which the magnetic force ($\boldsymbol{v}\times\boldsymbol{B}$) is neglected, and (c) in the NDA case, where $\lambda \leqslant a$ and with the $\boldsymbol{v}\times\boldsymbol{B}$ force playing a significant role. The arrows indicate possible free-electron quantum pathways.

IV. CONCLUSION

We have shown that the quantum effects associated with a free electron in laser fields of present-day intensities and frequencies may be retained if the dipole approximation is not used in modeling the laser-electron interaction. The quantum signatures (pointing to nonclassical behavior) get washed away when the dipole approximation is made. This could be attributed to the fact that the dipole approximation amounts to neglecting the magnetic effects associated with the laser field. Such effects are retained when the dipole approximation is abandoned in strong laser pulses, and they alter the free-electron dynamics quite appreciably. This is evidenced by distortions in the wave packet and in negativities exhibited by the quasiprobability distributions [20].

ACKNOWLEDGMENTS

M.M. acknowledges partial support for this work from the German Science Foundation and Zanjan University under Grant No. 8145. He also thanks A. R. Rau and A. H. Darooneh for useful discussions. Y.I.S. gratefully acknowledges support for this work from the German Alexander von Humboldt Stiftung, and C.H.K. is funded by the German Science Foundation.

APPENDIX

Under the KH transformation, the Schrödinger equation becomes

$$UH_fU^\dagger\Psi_{\mathrm{KH}} = i\hbar\frac{\partial\Psi_{\mathrm{KH}}}{\partial t}. \qquad \text{(A1)}$$

Note that, with a vector potential of the form

$$\boldsymbol{A} = \hat{\boldsymbol{x}}\frac{cE_0}{\omega}(\cos\xi - 1), \quad \xi = \omega t - kz, \qquad \text{(A2)}$$

the two terms making up H_i commute and the transformation may be written as $U=U_1U_2$, where

$$U_1 = e^{\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}, \quad U_2 = e^D \qquad \text{(A3)}$$

and where

$$\boldsymbol{\alpha} = \frac{e}{mc}\int \boldsymbol{A}(\boldsymbol{r},t')dt', \qquad \text{(A4)}$$

$$D = \frac{ie^2}{2mc^2\hbar}\int A^2(\boldsymbol{r},t')dt'. \qquad \text{(A5)}$$

In Eq. (A4) the operator substitution $\boldsymbol{p}=-i\hbar\boldsymbol{\nabla}$ has been made and the excursion parameter $\boldsymbol{\alpha}(\boldsymbol{r},t)$ has been introduced. Obviously, α and D can only involve the spatial z coordinate in addition to the time. The following derivatives will turn out to be useful for the subsequent analysis:

$$\frac{\partial\alpha}{\partial z} = \frac{eE_0}{mc}\int \sin\xi dt' \sim \mathcal{O}\left(\frac{1}{c}\right), \qquad \text{(A6)}$$

$$\frac{\partial^2\alpha}{\partial z^2} = -\frac{eE_0\omega}{mc^2}\int \cos\xi dt' \sim \mathcal{O}\left(\frac{1}{c^2}\right), \qquad \text{(A7)}$$

$$\frac{\partial D}{\partial z} = \frac{i}{\hbar}\frac{(eE_0)^2}{m\omega c}\int \sin\xi(\cos\xi - 1)dt' \sim \mathcal{O}\left(\frac{1}{c}\right), \qquad \text{(A8)}$$

$$\frac{\partial^2 D}{\partial z^2} = -\frac{i}{\hbar}\frac{(eE_0)^2}{mc^2}\int [\cos(2\xi) - \cos\xi]dt' \sim \mathcal{O}\left(\frac{1}{c^2}\right). \qquad \text{(A9)}$$

The operator $UH_fU^\dagger = U_1U_2H_fU_2^\dagger U_1^\dagger$ on the left-hand side of Eq. (A1) may be much simplified with the help of the following two commutation relations:

$$[U_1,H_f] = \frac{\hbar^2}{2m}\left[\nabla^2U_1 + \frac{2i}{\hbar}\boldsymbol{\nabla}\, U_1\cdot\boldsymbol{p}\right], \qquad \text{(A10)}$$

$$[U_2,H_f] = \frac{\hbar^2}{2m}\left[\nabla^2U_2 + \frac{2i}{\hbar}\boldsymbol{\nabla}\, U_2\cdot\boldsymbol{p}\right]. \qquad \text{(A11)}$$

Hence, the following derivatives are needed (with p_x the x component of the free-electron momentum operator):

$$\boldsymbol{\nabla}U_1 = \hat{z}\left(\frac{i}{\hbar}\right)\frac{\partial\alpha}{\partial z}U_1p_x, \qquad \text{(A12)}$$

$$\nabla^2U_1 = \left(\frac{i}{\hbar}\right)\left[\frac{\partial^2\alpha}{\partial z^2} + \left(\frac{i}{\hbar}\right)\left(\frac{\partial\alpha}{\partial z}\right)^2\right]U_1p_x, \qquad \text{(A13)}$$

$$\boldsymbol{\nabla}U_2 = \hat{z}\frac{\partial D}{\partial z}U_2, \qquad \text{(A14)}$$

$$\nabla^2 U_2 = \left[\frac{\partial^2 D}{\partial z^2} + \left(\frac{\partial D}{\partial z} \right)^2 \right] U_2. \tag{A15}$$

Furthermore, employing a Taylor series expansion for the operator U_1, namely,

$$U_1 = e^{\frac{i}{\hbar}\alpha p_x} = \sum_{n=0}^{\infty} \left(\frac{i}{\hbar} \right)^n \frac{\alpha^n}{n!} p_x^n, \tag{A16}$$

it is easy to see that the following commutation relations hold as well (p_z stands for the z component of the free electron momentum operator):

$$[U_1, p_z] = -\frac{\partial \alpha}{\partial z} p_x U_1, \quad [U_1, p_x] = 0. \tag{A17}$$

With the help of Eqs. (A10)–(A17) we may now write

$$\begin{aligned} UH_fU^\dagger = H_f + \frac{\hbar^2}{2m}\Bigg\{ \frac{i}{\hbar}\left[\frac{\partial^2 \alpha}{\partial z^2} + \frac{i}{\hbar}\left(\frac{\partial \alpha}{\partial z} \right)^2 \right] p_x \\ - \frac{2}{\hbar^2}\frac{\partial \alpha}{\partial z}\left(p_x p_z - \frac{\partial \alpha}{\partial z} p_x^2 \right) \Bigg\} + \frac{\hbar^2}{2m} U_1 \\ \times \left[\frac{\partial^2 D}{\partial z^2} + \left(\frac{\partial D}{\partial z} \right)^2 + \frac{2i}{\hbar}\left(\frac{\partial D}{\partial z} \right) p_z \right] U_1^\dagger, \end{aligned} \tag{A18}$$

exactly. Next we take the bold step of dropping all terms in Eq. (18) that are proportional to the inverse square of the speed of light, together with terms involving $p_x p_z$. That leaves us with

$$UH_fU^\dagger \approx H_f + \frac{\hbar^2}{2m} U_1 \left[\frac{2i}{\hbar}\left(\frac{\partial D}{\partial z} \right) p_z \right] U_1^\dagger, \tag{A19}$$

This simplifies even further using

$$\left[\frac{\partial D}{\partial z}, p_z \right] = i\hbar \frac{\partial^2 D}{\partial z^2} \approx 0, \quad \left[\frac{\partial D}{\partial z}, U_1 \right] = 0. \tag{A20}$$

Hence,

$$UH_fU^\dagger \approx H_f + H', \quad H' = \left(\frac{i\hbar}{m} \right) \frac{\partial D}{\partial z} p_z. \tag{A21}$$

Obviously, because it involves the operator p_z, H' leads to a translation in the z direction. But this is not a *simple space translation*, for it implicitly depends on the local space-time coordinates through the term $\partial D/\partial z$. Actually, the same thing may be said of the space translation resulting from the action of U_1.

After all these simplifications, Eq. (A1) takes on the following approximate form:

$$i\hbar \frac{\partial \Psi_{\mathrm{KH}}}{\partial t} \approx (H_f + H')\Psi_{\mathrm{KH}}. \tag{A22}$$

Guided by the U_1 algebra, we now let

$$\Phi_{\mathrm{KH}} = \Omega \Psi_{\mathrm{KH}}, \quad \Omega \equiv \exp\left(\tfrac{i}{\hbar} \int H' dt' \right) \tag{A23}$$

Under this transformation, Eq. (A22) becomes

$$i\hbar \frac{\partial \Phi_{\mathrm{KH}}}{\partial t} \approx \Omega H_f \Omega^\dagger \Phi_{\mathrm{KH}}. \tag{A24}$$

Note that if one writes $H' = f(z) p_z$, then

$$[f, p_z] = i\hbar \frac{\partial f}{\partial z} = -\frac{\hbar^2}{m}\frac{\partial^2 D}{\partial z^2} p_z \approx 0. \tag{A25}$$

From this follows that $[\Omega, H_f] \approx 0$. Finally,

$$i\hbar \frac{\partial \Phi_{\mathrm{KH}}}{\partial t} \approx -\frac{\hbar^2}{2m}\nabla^2 \Phi_{\mathrm{KH}}. \tag{A26}$$

Thus the laboratory wave function may be written as

$$\begin{aligned} \Psi &= U_2^\dagger U_1^\dagger \Omega^\dagger \Phi_{\mathrm{KH}}(x,y,z;t) = U_2^\dagger U_1^\dagger \Phi_{\mathrm{KH}}(x,y,z-\beta;t) \\ &= U_2^\dagger \Phi_{\mathrm{KH}}(x-\alpha, y, z-\beta; t), \end{aligned} \tag{A27}$$

where Ω has been rewritten as $\Omega = e^{\boldsymbol{\beta}\cdot\boldsymbol{\nabla}}$, with

$$\boldsymbol{\beta} \equiv \hat{z}\left(\frac{i\hbar}{m} \right) \int \frac{\partial D}{\partial z} dt' = -\hat{z}\left(\frac{i\hbar}{mc} \right) D \sim O\left(\frac{1}{c} \right). \tag{A28}$$

Note that, in the dipole approximation, $H'=0$ implies $\Omega=1$ and, hence, $\boldsymbol{\beta}=0$.

[1] G. Breit and J. A. Wheeler, Phys. Rev. **46**, 1087 (1934).
[2] F. Sauter, Z. Phys. **69**, 742 (1931).
[3] W. Heisenberg and H. Euler, Z. Phys. **98**, 718 (1936).
[4] D. L. Burke *et al.*, Phys. Rev. Lett. **79**, 1626 (1997).
[5] G. R. Mocken and C. H. Keitel, J. Comput. Phys. **199**, 558 (2004); J. S. Roman, L. Roso, and H. R. Reiss, J. Phys. B **33**, 1869 (2000); M. Verschl and C. H. Keitel, Laser Phys. **15**, 1 (2005).
[6] M. W. Walser and C. H. Keitel, J. Phys. B **33**, L221 (2000).
[7] J. San Roman, L. Roso, and L. Plaja, J. Phys. B **37**, 435 (2004).
[8] M. D. Perry *et al.*, Opt. Lett. **24**, 160 (1999); J. Andruszkow *et al.*, Phys. Rev. Lett. **85**, 3825 (2000); Th. Stöhlker *et al.*, *ibid.* **86**, 983 (2001); M. Drescher *et al.*, Science **291**, 1058561 (2001).
[9] B. Milosevic and F. Ehlotzky, Adv. At., Mol., Opt. Phys. **49**, 373 (2003).
[10] *The X-ray Free Electron Laser TESLA Technical Design Report*, edited by G. Materlik and T. Tschentscher, (DESY, Hamburg, 2001), Vol. 5.
[11] See A. Kenfack and K. Zyczkowski, J. Opt. B: Quantum Semiclassical Opt. **6**, 396 (2004) and references therein.

[12] E. P. Wigner, Phys. Rev. **40**, 749 (1932); M. Hillery, R. F. O'Connel, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).
[13] H. A. Kramers, *Quantum Mechanics* (North-Holland, Amsterdam, 1956); W. C. Henneberger, Phys. Rev. Lett. **21**, 838 (1968).
[14] A. R. P. Rau and C. Rangan, Phys. Rev. A **64**, 037402 (2001).
[15] R. L. Liboff, *Introductory Quantum Mechanics* (Adison-Wesley, New York, 1998).
[16] A. R. P. Rau and K. Unnikrishnan, Phys. Lett. A **222**, 304 (1996).
[17] P. W. Milonni and Bala Sundaram, in *Progress in Optics XXXI*, edited by E. Wolf (Elsevier, Amsterdam, 1993), p. 1.
[18] Y. I. Salamin, Phys. Rev. A **56**, 4910 (1997).
[19] H. R. Reiss, Phys. Rev. A **63**, 013409 (2001).
[20] J. B. Watson, C. H. Keitel, P. L. Knight, and K. Burnett, Phys. Rev. A **52**, 4023 (1995).