# Gaussian measures of entanglement versus negativities: Ordering of two-mode Gaussian states

Gerardo Adesso and Fabrizio Illuminati
*Dipartimento di Fisica "E. R. Caianiello," Università di Salerno; CNR-Coherentia, Gruppo di Salerno; and INFN Sezione di Napoli-Gruppo Collegato di Salerno, Via S. Allende, 84081 Baronissi (SA), Italy*
(Received 20 June 2005; published 28 September 2005)

We study the entanglement of general (pure or mixed) two-mode Gaussian states of continuous-variable systems by comparing the two available classes of *computable* measures of entanglement: entropy-inspired Gaussian convex-roof measures and positive partial transposition–inspired measures (negativity and logarithmic negativity). We first review the formalism of Gaussian measures of entanglement, adopting the framework introduced in M. M. Wolf *et al.*, Phys. Rev. A **69**, 052320 (2004), where the Gaussian entanglement of formation was defined. We compute explicitly Gaussian measures of entanglement for two important families of nonsymmetric two-mode Gaussian state: namely, the states of extremal (maximal and minimal) negativities at fixed global and local purities, introduced in G. Adesso *et al.*, Phys. Rev. Lett. **92**, 087901 (2004). This analysis allows us to compare the different *orderings* induced on the set of entangled two-mode Gaussian states by the negativities and by the Gaussian measures of entanglement. We find that in a certain range of values of the global and local purities (characterizing the covariance matrix of the corresponding extremal states), states of minimum negativity can have more Gaussian entanglement of formation than states of maximum negativity. Consequently, Gaussian measures and negativities are definitely inequivalent measures of entanglement on nonsymmetric two-mode Gaussian states, even when restricted to a class of extremal states. On the other hand, the two families of entanglement measures are completely equivalent on symmetric states, for which the Gaussian entanglement of formation coincides with the true entanglement of formation. Finally, we show that the inequivalence between the two families of continuous-variable entanglement measures is somehow limited. Namely, we rigorously prove that, at fixed negativities, the Gaussian measures of entanglement are bounded from below. Moreover, we provide some strong evidence suggesting that they are as well bounded from above.

DOI: 10.1103/PhysRevA.72.032334 PACS number(s): 03.67.Mn, 03.65.Ud

## I. INTRODUCTION

Quantum information with continuous variables (CV's) [1,2] is a flourishing field dedicated to the manipulation of the information using quantum states governed by the laws of quantum mechanics. This approach contrasts with the usual methods involving discrete-spectrum observables (such as, e.g., polarization, spin, energy level) of single photons, atoms, or ions. The ability of quantum states with continuous spectra to implement quantum cryptography [3], quantum teleportation [4–8], entanglement swapping [6,9], dense coding [10], quantum state storage [11], and, to some extent, quantum computation [12] processes brings up new and exciting perspectives.

The crucial resource enabling a better-than-classical manipulation and processing of information is CV entanglement, introduced for the first time in the landmarck paper by Einstein, Podolski, and Rosen [13] in 1935. There, it was shown that the simultaneous eigenstate of relative position and total momentum of two particles (or of a two modes of the radiation field) contains perfect quantum correlations—i.e., infinite CV entanglement. While this state is clearly an unphysical, unnormalizable state, it can be approximated arbitrarily well by two-mode squeezed Gaussian states with large enough squeezing parameter. The special class of Gaussian states (which includes thermal, coherent, and squeezed states) thus emerges quite naturally in the CV scenario. These entangled states can be easily produced and manipulated experimentally, and moreover their mathematical description is greatly simplified due to the fact that, while still residing in a infinite-dimensional Hilbert space, their relevant properties (such as entanglement and mixedness) are completely determined by the finite-dimensional covariance matrix of two-point correlations between the canonically conjugated quadrature operators. Therefore, clarifying the characterization and quantification of CV entanglement in two-mode and, eventually, multimode Gaussian states stands as a major issue in the field of CV quantum information, as the amount of entanglement contained in a certain state directly quantifies its usefulness for information and communication tasks like teleportation [8].

For the prototypical entangled states of a CV system, the two-mode Gaussian states, much is known about entanglement qualification, as the separability is completely characterized by the necessary and sufficient PPT (positivity of the partially transposed state) criterion [14] and also with regard to its quantifcation. Concerning the latter aspect, the negativity (quantifying the violation of the necessary and sufficient PPT condition for separability) is *computable* for all two-mode Gaussian states [15]. Moreover, for symmetric two-mode Gaussian states also the entanglement of formation is computable [16], and it turns out to be completely equivalent to the negativity for these states.

Another measure of CV entanglement, adapted for the class of Gaussian states, has been introduced in Ref. [17], where the Gaussian entanglement of formation (an upper bound to the true entanglement of formation) was defined as the cost of producing an entangled mixed state out of an ensemble of pure, Gaussian states. While the Gaussian entanglement of formation coincides with the true one for sym-

 

metric states, at present it is not known whether this equality holds for nonsymmetric states as well [18].

In this work, aimed at shedding light on the quantification of entanglement in two-mode Gaussian states, we compute the Gaussian entanglement of formation and, in general, the family of Gaussian entanglement measures, for two different classes of two-mode Gaussian states: namely, the states of extremal, maximal and minimal, negativities at fixed global and local purities [19,20]. We find that the two families of entanglement measures (negativities and Gaussian measures) are not equivalent for nonsymmetric states. Remarkably, they may induce a completely different *ordering* on the set of entangled two-mode Gaussian state: a nonsymmetric state $\varrho_A$ can be more entangled than another state $\varrho_B$, with respect to negativities and less entangled than the same state $\varrho_B$, with respect to Gaussian measures of entanglement. However, the inequivalence between the two families of measures is somehow bounded: we show that, at fixed negativities, the Gaussian entanglement measures are rigorously bounded from below. Moreover, we provide strong evidence hinting that they should be bounded from above as well.

The paper is organized as follows. In Sec. II we set up the notation and review the basic properties of Gaussian states of CV systems. In Sec. III we review the main results on the characterization of separability in Gaussian states, introducing also two families of measures of entanglement, respectively the negativities and the Gaussian entanglement measures. In Sec. IV we compute the latter for two-mode Gaussian states, solving the problem explicitly for the states of extremal negativities at fixed purities, described in Sec. IV A. In Sec. V we compare the orderings induced by negativities and Gaussian measures on the set of extremal two-mode Gaussian states. In Sec. VI we compare the two families of measures for generic two-mode Gaussian states, finding lower and upper bounds on one of them, when keeping the other fixed. Finally, in Sec. VII we summarize our results and discuss future perspectives.

## II. GAUSSIAN STATES: DEFINITIONS AND NOTATION

A continuous variable system is described by a Hilbert space $\mathcal{H}=\otimes_{i=1}^{n}\mathcal{H}_i$ resulting from the tensor product structure of infinite dimensional Fock spaces $\mathcal{H}_i$'s. Let $a_i$ be the annihilation operator acting on $\mathcal{H}_i$, and $\hat{q}_i=(a_i+a_i^\dagger)$ and $\hat{p}_i=(a_i-a_i^\dagger)/i$ be the related quadrature phase operators. The corresponding phase-space variables will be denoted by $q_i$ and $p_i$. Let $\hat{X}=(\hat{q}_1,\hat{p}_1,\ldots,\hat{q}_n,\hat{p}_n)$ denote the vector of the operators $\hat{q}_i$ and $\hat{p}_i$. The canonical commutation relations for the $\hat{X}_i$ can be expressed in terms of the symplectic form $\Omega$:

$$[\hat{X}_i,\hat{X}_j]=2i\Omega_{ij},$$

with

$$\Omega\equiv\bigoplus_{i=1}^{n}\omega,\quad \omega\equiv\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

The states of a CV system can be equivalently described by a positive trace-class operator (the density matrix $\varrho$) or by quasiprobability distributions such as the Wigner function [21]. States with Gaussian characteristic functions and quasiprobability distributions are referred to as Gaussian states. Such states are at the heart of information processing in CV systems [2] and are the subject of our analysis. By definition, a Gaussian state $\varrho$ is completely characterized by the first and second statistical moments of the quadrature field operators, which will be denoted, respectively, by the vector of first moments $\bar{X}\equiv(\langle\hat{X}_1\rangle,\langle\hat{X}_1\rangle,\ldots,\langle\hat{X}_n\rangle,\langle\hat{X}_n\rangle)$ and the covariance matrix (CM) $\boldsymbol{\sigma}$ of elements

$$\boldsymbol{\sigma}_{ij}\equiv\frac{1}{2}\langle\hat{X}_i\hat{X}_j+\hat{X}_j\hat{X}_i\rangle-\langle\hat{X}_i\rangle\langle\hat{X}_j\rangle, \tag{1}$$

where, for any observable $\hat{o}$, the expectation value $\langle\hat{o}\rangle\equiv\mathrm{Tr}(\varrho\hat{o})$. Notice that the entries of the CM can be expressed as energies by multiplying them by the quantity $\hbar\omega$, where $\omega$ is the frequency of the considered mode. In fact, for any $n$-mode state (even non Gaussian) the quantity $\hbar\omega\mathrm{Tr}\boldsymbol{\sigma}/4$ is simply the average of the noninteracting Hamiltonian $\Sigma_{i=1}^{n}(a_i^\dagger a_i+1/2)$. First moments can be arbitrarily adjusted by local unitary operations (displacements), which cannot affect any property related to entropy or entanglement. Therefore, they will be unimportant to the present scope and we will set them to 0 in the following, without any loss of generality.

The canonical commutation relations and the positivity of the density matrix $\varrho$ imply

$$\boldsymbol{\sigma}+i\Omega\geqslant 0. \tag{2}$$

Inequality (2) is the necessary and sufficient constraint the matrix $\boldsymbol{\sigma}$ has to fulfill to be a CM corresponding to a physical Gaussian state [22,23]. More in general, the previous condition is necessary for the CM of *any*, generally non-Gaussian, state. We note that such a constraint implies $\boldsymbol{\sigma}\geqslant 0$.

A major role in the theoretical and experimental manipulation of Gaussian states is played by unitary operations which preserve the Gaussian character of the states on which they act. Such operations are all those generated by Hamiltonian terms at most quadratic in the field operators. As a consequence of the Stone–Von Neumann theorem, any such unitary operation at the Hilbert-space level corresponds, in phase space, to a symplectic transformation—i.e., to a linear transformation $S$ which preserves the symplectic form $\Omega$, so that $\Omega=S^T\Omega S$. Symplectic transformations on a $2n$-dimensional phase space form the (real) symplectic group $\mathrm{Sp}_{(2n,\mathbb{R})}$. Such transformations act linearly on first moments and by congruences on covariance matrices: $\boldsymbol{\sigma}\mapsto S^{\mathbf{T}}\boldsymbol{\sigma}S$. One has $\mathrm{Det}S=1$, $\forall S\in\mathrm{Sp}_{(2n,\mathbb{R})}$. Ideal beam splitters, phase shifters, and squeezers are all described by some kind of symplectic transformation. A particularly important symplectic transformation is the one realizing the decomposition of a Gaussian state in normal modes. Through this decompostion, thanks to the Williamson theorem [24], the CM of an $n$-mode Gaussian state can always be written in the so-called Williamson normal or diagonal form

$$\boldsymbol{\sigma}=S^T\boldsymbol{\nu}S, \tag{3}$$

where $S\in\mathrm{Sp}_{(2n,\mathbb{R})}$ and $\boldsymbol{\nu}$ is the CM,

$$\boldsymbol{\nu} = \mathrm{diag}(\boldsymbol{\nu}_1, \nu_1, \ldots, \nu_n, \nu_n), \tag{4}$$

corresponding to a tensor product of thermal states with a diagonal density matrix $\varrho^{\otimes}$ given by

$$\varrho^{\otimes} = \bigotimes_i \frac{2}{\nu_i + 1} \sum_{k=0}^{\infty} \left( \frac{\nu_i - 1}{\nu_i + 1} \right) |k\rangle_{ii}\langle k|, \tag{5}$$

where $|k\rangle_i$ denotes the number state of order $k$ in the Fock space $\mathcal{H}_i$.

The quantities $\nu_i$'s form the symplectic spectrum of the CM $\boldsymbol{\sigma}$, and they can be computed as the eigenvalues of the matrix $|i\Omega\boldsymbol{\sigma}|$. Such eigenvalues are in fact invariant under the action of symplectic transformations on the matrix $\boldsymbol{\sigma}$. The symplectic eigenvalues $\nu_i$ encode essential information on the Gaussian state $\boldsymbol{\sigma}$ and provide powerful, simple ways to express its fundamental properties. For instance, in terms of the symplectic eigenvalues $\nu_i$, the uncertainty relation (2) reads

$$\nu_i \geqslant 1. \tag{6}$$

Moreover, the entropic quantities of Gaussian states can be as well expressed in terms of their symplectic eigenvalues and invariants [20]. Notably, the purity $\mathrm{Tr}\varrho^2$ of a Gaussian state $\varrho$ is simply given by the symplectic invariant $\mathrm{Det}\boldsymbol{\sigma} = \Pi_{i=1}^{n} \nu_i^2$, being [25]

$$\mu \equiv \mathrm{Tr}\varrho^2 = \frac{1}{\sqrt{\mathrm{Det}\boldsymbol{\sigma}}}. \tag{7}$$

#### Two-mode states

This work is focused on two-mode Gaussian states: we thus briefly review here some of their basic properties. The expression of the two-mode CM $\boldsymbol{\sigma}$ in terms of the three $2 \times 2$ matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, which will be useful in the following, takes the form

$$\boldsymbol{\sigma} \equiv \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^{\mathsf{T}} & \boldsymbol{\beta} \end{pmatrix}. \tag{8}$$

For any two-mode CM $\boldsymbol{\sigma}$ there is a local symplectic operation $S_l = S_1 \oplus S_2$ which brings $\boldsymbol{\sigma}$ to the so-called standard form $\boldsymbol{\sigma}_{sf}$ [14,26]:

$$S_l^T \boldsymbol{\sigma} S_l = \boldsymbol{\sigma}_{sf} \equiv \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}. \tag{9}$$

States whose standard form fulfills $a = b$ are said to be symmetric. Let us recall that any pure state ($\mu = 1$) is symmetric and fulfills $c_+ = -c_- = \sqrt{a^2 - 1}$. The correlations $a$, $b$, $c_+$, and $c_-$ are determined by the four local symplectic invariants $\mathrm{Det}\boldsymbol{\sigma} = (ab - c_+^2)(ab - c_-^2)$, $\mathrm{Det}\boldsymbol{\alpha} = a^2$, $\mathrm{Det}\boldsymbol{\beta} = b^2$, and $\mathrm{Det}\boldsymbol{\gamma} = c_+ c_-$. Therefore, the standard form corresponding to any CM is unique (up to a common sign flip in $c_-$ and $c_+$).

For two-mode states, the uncertainty principle (2) can be recast as a constraint on the $\mathrm{Sp}_{(4,\mathbb{R})}$ invariants $\mathrm{Det}\boldsymbol{\sigma}$ and $\Delta(\boldsymbol{\sigma}) = \mathrm{Det}\boldsymbol{\alpha} + \mathrm{Det}\boldsymbol{\beta} + 2\mathrm{Det}\boldsymbol{\gamma}$ [27]:

$$\Delta(\boldsymbol{\sigma}) \leqslant 1 + \mathrm{Det}\boldsymbol{\sigma}. \tag{10}$$

The symplectic eigenvalues of a two-mode Gaussian state will be denoted as $\nu_-$ and $\nu_+$, with $\nu_- \leqslant \nu_+$, with the uncertainty relation (6) reducing to

$$\nu_- \geqslant 1. \tag{11}$$

A simple expression for the $\nu_{\mp}$ can be found in terms of the two $\mathrm{Sp}_{(4,\mathbb{R})}$ invariants (invariants under global, two-mode symplectic operations) [15,27]

$$2\nu_{\mp}^2 = \Delta(\boldsymbol{\sigma}) \mp \sqrt{\Delta^2(\boldsymbol{\sigma}) - 4\mathrm{Det}\boldsymbol{\sigma}}. \tag{12}$$

## III. ENTANGLEMENT OF GAUSSIAN STATES

In this section we recall the main results on the qualification and quantification of entanglement for Gaussian states of CV systems.

### A. Qualification: PPT criterion

The positivity of the partially transposed state (Peres-Horodecki PPT criterion [28]) is necessary and sufficient for the separability of two-mode Gaussian states [14] and, more generally, of all $(1+n)$-mode Gaussian states under $(1 \times n)$-mode bipartitions [29] and of symmetric and bisymmetric $(m+n)$-mode Gaussian states under $(m \times n)$-mode bipartitions [30]. In general, the partial transposition $\tilde{\varrho}$ of a bipartite quantum state $\varrho$ is defined as the result of the transposition performed on only one of the two subsystems in some given basis. In phase space, the action of partial transposition amounts to a mirror reflection of one of the four canonical variables [14]. The CM $\boldsymbol{\sigma}$ is then transformed into a new matrix $\tilde{\boldsymbol{\sigma}}$ which differs from $\boldsymbol{\sigma}$ by a sign flip in $\mathrm{Det}\boldsymbol{\gamma}$. Therefore the invariant $\Delta(\boldsymbol{\sigma})$ is changed into $\tilde{\Delta}(\boldsymbol{\sigma}) \equiv \Delta(\tilde{\boldsymbol{\sigma}}) = \mathrm{Det}\boldsymbol{\alpha} + \mathrm{Det}\boldsymbol{\beta} - 2\,\mathrm{Det}\boldsymbol{\gamma}$. Now, the symplectic eigenvalues $\tilde{\nu}_{\mp}$ of $\tilde{\boldsymbol{\sigma}}$ read

$$\tilde{\nu}_{\mp} = \sqrt{\frac{\tilde{\Delta}(\boldsymbol{\sigma}) \mp \sqrt{\tilde{\Delta}^2(\boldsymbol{\sigma}) - 4\mathrm{Det}\boldsymbol{\sigma}}}{2}}. \tag{13}$$

The PPT criterion for separability thus reduces to a simple inequality that must be satisfied by the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed state,

$$\tilde{\nu}_- \geqslant 1, \tag{14}$$

which is equivalent to

$$\tilde{\Delta}(\boldsymbol{\sigma}) \leqslant \mathrm{Det}\boldsymbol{\sigma} + 1. \tag{15}$$

Moreover, the above inequalities imply $\mathrm{Det}\boldsymbol{\gamma} = c_+ c_- < 0$ as a necessary condition for a two-mode Gaussian state to be en-

tangled. Therefore, the quantity $\tilde{\nu}_-$ encodes all the qualitative characterization of the entanglement for arbitrary (pure or mixed) two-mode Gaussian states.

### B. Negativities

From a quantitative point of view, a measure of entanglement which can be computed for general Gaussian states is provided by the *negativity* $\mathcal{N}$, first introduced in Ref. [31], later thoroughly discussed and extended in Refs. [15,32] to CV systems. The negativity of a quantum state $\varrho$ is defined as

$$\mathcal{N}(\varrho) = \frac{\|\tilde{\varrho}\|_1 - 1}{2}, \tag{16}$$

where $\tilde{\varrho}$ is the partially transposed density matrix and $\|\hat{o}\|_1 = \mathrm{Tr}|\hat{o}|$ stands for the trace norm of the Hermitian operator $\hat{o}$. The quantity $\mathcal{N}(\varrho)$ is equal to $|\Sigma_i \lambda_i|$, the modulus of the sum of the negative eigenvalues of $\tilde{\varrho}$, quantifying the extent to which $\tilde{\varrho}$ fails to be positive. Strictly related to $\mathcal{N}$ is the *logarithmic negativity* $E_{\mathcal{N}}$, defined as $E_{\mathcal{N}} \equiv \ln\|\tilde{\varrho}\|_1$, which constitutes an upper bound to the *distillable entanglement* of the quantum state $\varrho$ and is related to the entanglement cost under PPT preserving operations [33]. Both the negativity and logarithmic negativity have been proven to be monotone under LOCC (local operations and classical communications) [15,32,34], a crucial property for a *bona fide* measure of entanglement. Moreover, the logarithmic negativity possesses the nice property of being additive.

For any two-mode Gaussian state $\varrho$ it is easy to show that both the negativity and logarithmic negativity are simple decreasing functions of $\tilde{\nu}_-$ [15,20]:

$$\|\tilde{\varrho}\|_1 = \frac{1}{\tilde{\nu}_-} \Rightarrow \mathcal{N}(\varrho) = \max\left[0, \frac{1-\tilde{\nu}_-}{2\tilde{\nu}_-}\right], \tag{17}$$

$$E_{\mathcal{N}}(\varrho) = \max[0, -\ln \tilde{\nu}_-]. \tag{18}$$

These expressions directly quantify the amount by which the necessary and sufficient PPT condition (14) for separability is violated. The symplectic eigenvalue $\tilde{\nu}_-$ thus completely qualifies and quantifies (in terms of negativities) the entanglement of a two-mode Gaussian state $\boldsymbol{\sigma}$: for $\tilde{\nu}_- \geqslant 1$ the state is separable; otherwise, it is entangled. Finally, in the limit of vanishing $\tilde{\nu}_-$, the negativities grow unboundedly.

### C. Entanglement of formation

In the special instance of symmetric two-mode Gaussian states, the *entanglement of formation* (EOF) [35] can be computed as well [16]. We recall that the EOF $E_F$ of a quantum state $\varrho$ is defined as

$$E_F(\varrho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{19}$$

where the minimum is taken over all the pure states realizations of $\varrho$:

$$\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

The asymptotic regularization of the entanglement of formation coincides with the *entanglement cost* $E_C(\varrho)$, defined as the minimum number of singlets (maximally entangled antisymmetric two-qubit states) which is needed to prepare the state $\varrho$ through LOCC [36].

The optimal convex decomposition of Eq. (19) has been found for symmetric two-mode Gaussian states and turns out to be Gaussian; that is, the absolute minimum is realized within the set of pure two-mode Gaussian states [16], yielding

$$E_F = \max[0, h(\tilde{\nu}_-)], \tag{20}$$

with

$$h(x) = \frac{(1+x)^2}{4x} \ln\left[\frac{(1+x)^2}{4x}\right] - \frac{(1-x)^2}{4x} \ln\left[\frac{(1-x)^2}{4x}\right]. \tag{21}$$

Such a quantity is, again, a monotonically decreasing function of $\tilde{\nu}_-$, thus providing a quantification of the entanglement of symmetric states *equivalent* to the one provided by the negativities.

As a consequence of this equivalence, it is tempting to conjecture that there exists a unique quantification of entanglement for two-mode Gaussian states, embodied by the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed CM, and that the different measures simply provide trivial rescalings of the same unique quantification. In particular, the *ordering* induced on the set of entangled Gaussian state is uniquely defined for the subset of symmetric two-mode states, and it is independent of the chosen measure of entanglement. However, regrettably, in Sec. V we will indeed show that different measures of entanglement induce, in general, different orderings on the set of nonsymmetric two-mode Gaussian states.

### D. Gaussian convex-roof extended measures

In this subsection we consider a family of entanglement measures exclusively defined for Gaussian states of CV systems. The formalism of *Gaussian entanglement measures* (Gaussian EM's) has been introduced in Ref. [17] where the Gaussian EOF has been defined and analyzed. Furthermore, the framework developed in Ref. [17] is general and enables to define generic Gaussian EM's of bipartite entanglement by applying the Gaussian convex roof—that is, the convex roof over pure Gaussian decompositions only—to any *bona fide* measure of bipartite entanglement defined for pure Gaussian states. The original motivation for the introduction of Gaussian EM's stems from the unfortunate fact that the optimization problem, Eq. (19), for the computation of the EOF of nonsymmetric two-mode Gaussian states has not yet been solved, and it stands as an open problem in the theory of entanglement [18]. However, the task can be somehow simplified by restricting to decompositions into pure Gaussian states only. The resulting measure, named the Gaussian EOF in Ref. [17], is an upper bound to the true EOF and coincides

with it for symmetric two-mode Gaussian states.

In general, we can define a Gaussian EM $G_E$ as follows. For any pure Gaussian state $\psi$ with CM $\boldsymbol{\sigma}^P$, one has

$$G_E(\boldsymbol{\sigma}^P) \equiv E(\psi), \tag{22}$$

where $E$ can be *any* proper measure of entanglement of pure states, defined as a monotonically increasing function of the entropy of entanglement (i.e., the von Neumann entropy of the reduced density matrix of one party).

For any mixed Gaussian state $\varrho$ with CM $\boldsymbol{\sigma}$, one has [17]

$$G_E(\boldsymbol{\sigma}) \equiv \inf_{\boldsymbol{\sigma}^P \leqslant \boldsymbol{\sigma}} G_E(\boldsymbol{\sigma}^P). \tag{23}$$

If the function $E$ is taken to be exactly the entropy of entanglement, then the corresponding Gaussian EM is known as *Gaussian EOF* [17]. In Ref. [37] the properties of the Gaussian EOF have been further investigated, and interesting connections with the capacity of bosonic Gaussian channels have been established.

In general, the definition, Eq. (23), involves an optimization over all pure Gaussian states with CM $\boldsymbol{\sigma}^P$ smaller than the CM $\boldsymbol{\sigma}$ of the mixed state whose entanglement one wishes to compute. Despite being a simpler optimization problem than that appearing in the definition, Eq. (19), of the true EOF (which, in CV systems, would imply considering decompositions over all, Gaussian *and* non-Gaussian pure states), the Gaussian EM's cannot be expressed in a simple closed form, not even in the simplest instance of (nonsymmetric) two-mode Gaussian states. It is the aim of the present paper to compute Gaussian EM's for two relevant classes of, *generally nonsymmetric*, two-mode Gaussian states—namely, the states of *extremal* (maximal and minimal) negativity at fixed global and local purities [19,20]—which will be reviewed in Sec. IV A. This will provide an insight into the problem of the ordering [38] of two-mode Gaussian states with respect to different measures of entanglement, leading to results somehow similar to those obtained for systems of two qubits [39], where in general the EOF and the negativity are found to be inequivalent.

Before moving on to the explicit computations, let us recall, as an important side remark, that any Gaussian EM is an entanglement monotone under Gaussian LOCC. The proof given in Sec. IV of Ref. [17] for the Gaussian EOF, in fact, automatically extends to every Gaussian EM constructed via the Gaussian convex roof of any proper measure $E$ of pure-state entanglement.

## IV. GAUSSIAN ENTANGLEMENT MEASURES FOR TWO-MODE GAUSSIAN STATES

The problem of evaluating Gaussian EM's for a generic two-mode Gaussian state has been solved in Ref. [17]. However, the explicit result contains so "cumbersome" expressions (involving the solutions of a fourth-order algebraic equation) that the authors of Ref. [17] considered them not particularly useful to be reported explicitly in their paper.

We recall here the computation procedure [17] that we will need in the following. For any two-mode Gaussian state with CM $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}_{sf}$ in standard form, Eq. (9), a generic Gaussian EM $G_E$ is given by the entanglement $E$ of the least entangled pure state with CM $\boldsymbol{\sigma}^P \leqslant \boldsymbol{\sigma}$. Denoting by $\gamma_q$ ($\gamma_p$) the $2\times 2$ submatrix obtained from $\boldsymbol{\sigma}$ by canceling the even (odd) rows and columns, we have, explicitly,

$$\gamma_q = \begin{pmatrix} a & c_+ \\ c_+ & b \end{pmatrix}, \quad \gamma_p = \begin{pmatrix} a & c_- \\ c_- & b \end{pmatrix}. \tag{24}$$

All the covariances relative to the "position" operators of the two modes are grouped in $\gamma_q$ and analogously for the "momentum" operators in $\gamma_p$. The total CM can then be written as a direct sum $\boldsymbol{\sigma} = \gamma_q \oplus \gamma_p$. Similarly, the CM of a generic pure two-mode Gaussian state in standard form (it has been proven that the CM of the optimal pure state has to be as well in the form with all diagonal two by two sub-blocks [17]) can be written as $\boldsymbol{\sigma}^P = \gamma_q^P \oplus \gamma_p^P$, where the global purity of the state imposes $(\gamma_p^P)^{-1} = \gamma_q^P \equiv \Gamma$. The pure states involved in the definition of the Gaussian EM must thus fulfill the condition

$$\gamma_p^{-1} \leqslant \Gamma \leqslant \gamma_q. \tag{25}$$

This problem is endowed with a nice geometric description [17]. Writing the matrix $\Gamma$ in the basis constituted by the identity matrix and the three Pauli matrices,

$$\Gamma = \begin{pmatrix} x_0 + x_3 & x_1 \\ x_1 & x_0 - x_3 \end{pmatrix}, \tag{26}$$

the expansion coefficients $(x_0, x_1, x_3)$ play the role of space-time coordinates in a three-dimensional Minkowski space. In this picture, for example, the rightmost inequality in Eq. (25) is satisfied by matrices $\Gamma$ lying on a cone, which is equivalent to the (backwards) light cone of $C_q$ in the Minkowski space, and similarly for the leftmost inequality. Indeed, one can show that, for the optimal pure state $\boldsymbol{\sigma}_{opt}^P$ realizing the mininum in Eq. (23), the two inequalities in Eq. (25) have to be simultaneously saturated [17]. From a geometrical point of view, the optimal $\Gamma$ has then to be found on the rim of the intersection of the forward and the backward cones of $\gamma_p^{-1}$ and $\gamma_q$, respectively. This is an ellipse, and one is left with the task of minimizing the entanglement $E$ of $\boldsymbol{\sigma}^P = \Gamma \oplus \Gamma^{-1}$ [see Eq. (22)] for $\Gamma$ lying on this ellipse [40].

At this point, let us pause to briefly recall that any pure two-mode Gaussian state $\boldsymbol{\sigma}^P$ is locally equivalent to a two-mode squeezed state with squeezing parameter $r$, described by a CM

$$\boldsymbol{\sigma}_{sq}^P = \begin{pmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{pmatrix}. \tag{27}$$

The following statements are then equivalent: (i) $E$ is a monotonically increasing function of the entropy of entanglement, (ii) $E$ is a monotonically increasing function of the single-mode determinant $m \equiv \mathrm{Det}\boldsymbol{\alpha} \equiv \mathrm{Det}\boldsymbol{\beta}$ [see Eq. (8)],

(iii) $E$ is a monotonically decreasing function of the local purity $\mu_i \equiv \mu_1 \equiv \mu_2$ [see Eq. (7)], (iv) $E$ is a monotonically decreasing function of the smallest symplectic eigenvalue $\tilde{\nu}_-^P$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}^P$, and (v) $E$ is a monotonically increasing function of the squeezing parameter $r$. This chain of equivalences is immediately proven by simply recalling that a pure state is completely specified by its single-mode marginals and that for a single-mode Gaussian state there is a unique symplectic invariant (the determinant), so that all conceivable entropic quantities are monotonically increasing functions of this invariant [20]. In particular, statement (ii) allows us to minimize directly the single-mode determinant over the ellipse:

$$m = 1 + \frac{x_1^2}{\mathrm{Det}\Gamma}, \tag{28}$$

with $\Gamma$ given by Eq. (26).

To simplify the calculations, one can move to the plane of the ellipse with a Lorentz boost which preserves the relations between all the cones; one can then choose the transformation so that the ellipse degenerates into a circle (with fixed radius) and introduce polar coordinates on this circle. The calculation of the Gaussian EM for any two-mode Gaussian state is thus finally reduced to the minimization of $m$ from Eq. (28) at given standard-form covariances of $\boldsymbol{\sigma}$ as a function of the polar angle $\theta$ on the circle [40]. So far, this technique has been applied to the computation of the Gaussian EOF by minimizing Eq. (28) *numerically* [17] (see also [41]). In addition to that, as already mentioned, the Gaussian EOF has been exactly computed for symmetric states, and it has been proven that in this case the Gaussian EOF is the true EOF [16].

In this work we present analytical calculations of the Gaussian EM's for two relevant classes of nonsymmetric two-mode Gaussian states: the states of *extremal* negativities at fixed global and local purities [20], which will be introduced in the next subsection. We begin by writing the general expression of the single-mode determinant Eq. (28) in terms of the covariances of a generic two-mode state [see Eq. (9)] and of the polar angle $\theta$. After some tedious but straightforward algebra, one finds

$$\begin{aligned} m_\theta(a,b,c_+,c_-) = 1 + (\{c_+(ab-c_-^2) - c_- + \cos\theta\sqrt{[a-b(ab-c_-^2)][b-a(ab-c_-^2)]}\}^2) \\ \times \left\{ 2(ab-c_-^2)(a^2+b^2+2c_+c_+) - \frac{\cos\theta\{2abc_-^3 + (a^2+b^2)c_+c_-^2 + [(1-2b^2)a^2+b^2]c_- - ab(a^2+b^2-2)c_+\}}{\sqrt{[a-b(ab-c_-^2)][b-a(ab-c_-^2)]}} \right. \\ \left. + \sin\theta(a^2-b^2)\sqrt{1 - \frac{[c_+(ab-c_-^2)+c_-]^2}{[a-b(ab-c_-^2)][b-a(ab-c_-^2)]}} \right\}^{-1}, \end{aligned} \tag{29}$$

where we have assumed $c_+ \geqslant |c_-|$ without any loss of generality. This implies that, for any entangled state, $c_+ > 0$ and $c_- < 0$. The Gaussian EM [defined in terms of the function $E$ on pure states; see Eq. (22)] of a generic two-mode Gaussian state coincides then with the entanglement $E$ computed on the pure state with $m = m_{opt}$, with $m_{opt} \equiv \min_\theta(m_\theta)$. Accordingly, the symplectic eigenvalue $\tilde{\nu}_-$ of the partial transpose of the corresponding optimal pure-state CM $\boldsymbol{\sigma}_{opt}^P$, realizing the infimum in Eq. (23), would read [see Eq. (13)]

$$\tilde{\nu}_{-opt}^P \equiv \tilde{\nu}_-(\boldsymbol{\sigma}_{opt}^P) = \sqrt{m_{opt}} - \sqrt{m_{opt}-1}. \tag{30}$$

As an example, for the Gaussian EOF one has

$$G_{E_F}(\boldsymbol{\sigma}) = h(\tilde{\nu}_{-opt}^P(m_{opt})), \tag{31}$$

with $h(x)$ defined by Eq. (21).

Finding the minimum of Eq. (29) analytically for a generic state is a difficult task. Numerical investigations show that the equation $\partial_\theta m_\theta = 0$ can have from one to four physical solutions (in a period) corresponding to extremal points and the global minimum can be attained in any of them depending on the parameters of the CM $\boldsymbol{\sigma}$ under inspection. However, a closed solution can be found for two important classes of nonsymmetric two-mode Gaussian states, as we will now show.

### A. Parametrization of two-mode covariance matrices and definition of extremal states

We have shown in Refs. [19,20] that, at fixed global purity $\mu \equiv \mathrm{Tr}\varrho^2$ of the global state $\varrho$ and at fixed local purities $\mu_{1,2} \equiv \mathrm{Tr}\varrho_{1,2}^2$ of each of the two reduced single-mode states $\varrho_i = \mathrm{Tr}_{j\neq i}\varrho$, the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partial transpose of the CM $\boldsymbol{\sigma}$ of a generic two-mode Gaussian state (which qualifies its separability by the PPT criterion and quantifies its entanglement in terms of the negativities) is strictly bounded from above and from below. This entails the existence of two disjoint classes of extremal states, namely, the states of maximum negativity for fixed global and local purities (GMEMS) and the states of minimum negativity for fixed global and local purities (GLEMS) [20]. The negativities of the two *extremal* classes of Gaussian states, moreover, turn out to remain very close to each other for all the

possible assignments of the three purities, allowing for a reliable fexperimental estimate of the negativity of a generic two-mode Gaussian state in terms of the *average negativity* [19]. The latter is determined by knowledge of the three purities alone, which, in turn, may be experimentally measured in direct, possibly efficient, ways [42].

Recalling these results, one can provide a very useful and insightful parametrization of the *entangled* two-mode Gaussian states in standard form (see also [43]). In fact, the coefficients appearing in Eq. (9) can be rewritten, in general, according to the following, useful parametrization:

$$a = s + d, \quad b = s - d, \tag{32}$$

$$c_{\pm} = \frac{1}{4\sqrt{s^2 - d^2}} \left\{ \sqrt{\left[ 4d^2 + \frac{1}{2}(g^2+1)(\lambda - 1) - (2d^2 + g)(\lambda + 1) \right]^2 - 4g^2} \right.$$
$$\left. \pm \sqrt{\left[ 4s^2 + \frac{1}{2}(g^2+1)(\lambda - 1) - (2d^2 + g)(\lambda + 1) \right]^2 - 4g^2} \right\}, \tag{33}$$

where the two local purities are regulated by the parameters $s$ and $d$, being $\mu_1 = (s+d)^{-1}$, $\mu_2 = (s-d)^{-1}$, and the global purity is $\mu = g^{-1}$. The coefficient $\lambda$ embodies the only remaining degree of freedom needed for the complete determination of the negativities, once the three purities have been fixed. It ranges from the minimum $\lambda = -1$ (corresponding to the GLEMS) to the maximum $\lambda = +1$ (corresponding to the GMEMS). Therefore, as it varies, $\lambda$ encompasses all possible entangled two-mode Gaussian states compatible with a given set of assigned values of the purities. The constraints that the parameters $s$, $d$, and $g$ must obey for Eq. (9) to denote a proper CM of a physical state are $s \geqslant 1$, $|d| \leqslant s - 1$, and

$$g \geqslant 2|d| + 1, \tag{34}$$

If the global purity is large enough so that inequality (34) is saturated, GMEMS and GLEMS coincide, the CM becomes independent of $\lambda$, and the two classes of extremal states coalesce into a unique class, completely determined by the marginals $s$ and $d$. We denote these states as GMEMMS [20]—that is, Gaussian two-mode states of maximal negativity at fixed local purities. Their CM is simply characterized by $c_{\pm} = \pm\sqrt{s^2 - (d+1)^2}$, where we have assumed, without any loss of generality, that $d \geqslant 0$ (corresponding to choose, for instance, mode 1 as the more mixed one: $\mu_1 \leqslant \mu_2$).

In general [19], a GMEMS ($\lambda = +1$) is entangled for

$$g < 2s - 1, \tag{35}$$

while a GLEMS ($\lambda = -1$) is entangled for smaller $g$: namely,

$$g < \sqrt{2(s^2 + d^2) - 1}. \tag{36}$$

To have a physical insight into these peculiar two-mode states, let us recall [20] that GMEMS are simply nonsymmetric thermal squeezed states, usually referred to as maximally entangled mixed states in CV systems. On the other hand, GLEMS are mixed states of partial minimum uncertainty, in the sense that the smallest symplectic eigenvalue of their CM is equal to 1, saturating the uncertainty inequality (11).

We are now equipped with the necessary tools, and in the next subsection we move on to compute Gaussian EM's for the two extremal classes of nonsymmetric two-mode Gaussian states, the GLEMS and GMEMS.

### B. Gaussian entanglement of minimum-negativity states (GLEMS)

We want to find the optimal pure state $\boldsymbol{\sigma}_{opt}^{P}$ entering in the definition Eq. (23) of the Gaussian EM. To do this, we have to minimize the single-mode determinant of $\boldsymbol{\sigma}_{opt}^{P}$, given by Eq. (29), over the angle $\theta$. It turns out that, for a generic GLEMS, the coefficient of $\sin\theta$ in the last line of Eq. (29) vanishes, and the expression of the single-mode determinant reduces to the simplified form

$$m_{\theta}^{\mathrm{GLEMS}} = 1 + \frac{[A\cos\theta + B]^2}{2(ab - c_-^2)[(g^2-1)\cos\theta + g^2 + 1]}, \tag{37}$$

with $A = c_+(ab - c_-^2) + c_-$, $B = c_+(ab - c_-^2) - c_-$, and $a, b, c_{\pm}$ are the covariances of GLEMS, obtained from Eqs. (32) and (33) setting $\lambda = -1$.

The only relevant solutions (excluding the unphysical and the trivial ones) of the equation $\partial_{\theta} m_{\theta} = 0$ are $\theta = \pi$ and

$$\theta = \pm\theta^{*} \equiv \arccos\left[ \frac{3 + g^2}{1 - g^2} - \frac{2c_-}{c_+(ab - c_-^2) + c_-} \right].$$

Studying the second derivative $\partial_{\theta}^2 m_{\theta}$ for $\theta = \pi$ one finds immediately that, for

$$g \geqslant \sqrt{-\frac{2c_+(ab - c_-^2) + c_-}{c_-}} \tag{38}$$

(remember that $c_- \leqslant 0$), the solution $\theta = \pi$ is a minimum. In this range of parameters, the other solution $\theta = \theta^{*}$ is unphysi-

cal (in fact $|\cos\theta^*| \geq 1$), so $m_{\theta=\pi}$ is the global minimum. When, instead, inequality (38) is violated, $m_\theta$ has a local maximum for $\theta=\pi$ and two minima appear at $\theta=\pm\theta^*$. The global minimum is attained in any of the two, given that, for GLEMS, $m_\theta$ is invariant under reflection with respect to the axis $\theta=\pi$. Collecting, substituting, and simplifying the obtained expressions, we arrive at the final result for the optimal $m$:

$$m_{opt}^{\mathrm{GLEMS}} = \begin{cases} 1, & g \geq \sqrt{2(s^2+d^2)-1} \quad \text{(separable state)}, \\ \dfrac{16s^2d^2}{(g^2-1)^2}, & \sqrt{\dfrac{(4s^2+1)d^2+s^2+4s\sqrt{(s^2+1)d^2+s^2}|d|}{d^2+s^2}} \leq g < \sqrt{2(s^2+d^2)-1}, \\ \dfrac{-g^4+2(2d^2+2s^2+1)g^2-(4d^2-1)(4s^2-1)-\sqrt{\delta}}{8g^2}, & 2|d|+1 \leq g < \sqrt{\dfrac{(4s^2+1)d^2+s^2+4s|d|\sqrt{(s^2+1)d^2+s^2}}{d^2+s^2}}. \end{cases} \tag{39}$$

Here $\delta \equiv (2d-g-1)(2d-g+1)(2d+g-1)(2d+g+1)(g-2s-1)(g-2s+1)(g+2s-1)(g+2s+1)$.

Immediate inspection crucially reveals that $m_{opt}^{\mathrm{GLEMS}}$ is *not* in general a function of the symplectic eigenvalue $\tilde{\nu}_-$ alone. Therefore, unfortunately, the Gaussian EM's and, in particular, the Gaussian EOF are not equivalent to the negativities for GLEMS. Further remarks will be given in the following, when the Gaussian EM's of GLEMS and GMEMS will be compared and their relationship with the negativities will be elucidated.

### C. Gaussian entanglement of maximum-negativity states (GMEMS)

The minimization of $m_\theta$ from Eq. (29) can be carried out in a simpler way in the case of GMEMS, whose covariances can be retrieved from Eq. (33) setting $\lambda=1$. First of all, one can notice that, when expressed as a function of the Minkowski coordinates $(x_0,x_1,x_3)$, corresponding to the submatrix $\Gamma$, Eq. (26), of the pure state $\boldsymbol{\sigma}^P=\Gamma\oplus\Gamma^{-1}$ entering in the optimization problem, Eq. (23), the single-mode determinant $m$ of $\boldsymbol{\sigma}^P$ is globally minimized for $x_3=0$. In fact, from Eq. (28), $m$ is minimal, with respect to $x_3$, when $\mathrm{Det}\Gamma=x_0^2-x_1^2-x_3^2$ is maximal. Next, one can show that for GMEMS there always exists a matrix $\Gamma$, with $x_3=0$, which is a simultaneous solution of the two matrix equations obtained by imposing the saturation of the two sides of inequality (25). As a consequence of the above discussion, this matrix would denote the optimal pure state $\boldsymbol{\sigma}_{opt}^P$. Solving the system of equations $\mathrm{Det}(\gamma_q-\Gamma)=\mathrm{Det}(\Gamma-\gamma_p^{-1})=0$, where the matrices involved are explicitly defined combining Eqs. (24) and (33) with $\lambda=1$, one finds the following two solutions for the coordinates $x_0$ and $x_1$:

$$x_0^{\pm} = \frac{(g+1)s \pm \sqrt{[(g-1)^2-4d^2](-d^2+s^2-g)}}{2(d^2+g)},$$

$$x_1^{\pm} = \frac{(g+1)\sqrt{-d^2+s^2-g} \pm s\sqrt{(g-1)^2-4d^2}}{2(d^2+g)}. \tag{40}$$

The corresponding pure state $\boldsymbol{\sigma}^{P\pm}=\Gamma^{\pm}\oplus\Gamma^{\pm^{-1}}$ turns out to be, in both cases, a two-mode squeezed state described by a CM of the form, Eq. (27), with $\cosh(2r)=x_0^{\pm}$. Because the single-mode determinant $m=\cosh^2(2r)$ for these states, the optimal $m$ for GMEMS is simply equal to $(x_0^-)^2$. Summarizing,

$$m_{opt}^{\mathrm{GMEMS}} = \begin{cases} 1, & g \geq 2s-1 \text{ (separable state)}, \\ \dfrac{\{(g+1)s-\sqrt{[(g-1)^2-4d^2](-d^2+s^2-g)}\}^2}{4(d^2+g)^2}, & 2|d|+1 \leq g < 2s-1. \end{cases} \tag{41}$$

Once again, also for the class of GMEMS the Gaussian EM's are not simple functions of the symplectic eigenvalue $\tilde{\nu}_-$ alone. Consequently, they provide a quantification of CV entanglement of GMEMS inequivalent to the one determined by the negativities. Furthermore, we will now show how these results raise the problem of the ordering of two-mode

Gaussian states according to their degree of entanglement, as quantified by different families of entanglement measures.

## V. EXTREMAL ORDERING OF TWO-MODE GAUSSIAN STATES

Entanglement is a physical quantity. It has a definite mathematical origin within the framework of quantum mechanics, and its conceptual meaning in the end stems from and is rooted in the existence of the superposition principle. Further, entanglement has a fundamental operative interpretation as that resource that in principle enables information processing and communication in better-than-classical realizations [8]. One would then expect that, picking two states $\varrho_A$ and $\varrho_B$ out of a certain (subset of) Hilbert space, the question—is $\varrho_A$ more entangled than $\varrho_B$?—should have a unique, well-defined answer, independent of the measure that one chooses to quantify entanglement. But contrary to the common expectations, this is generally *not* the case for mixed states. Different measures of entanglement will in general induce different, inequivalent *orderings* on the set of entangled states belonging to a given Hilbert space [38], as they usually measure different aspects of quantum correlations existing in generic mixed states.

In the context of CV systems, when one restricts oneself to symmetric, two-mode Gaussian states, which include all pure states, the known computable measures of entanglement all correctly induce *the same* ordering on the set of entangled states. We will now show that, indeed, this nice feature is not preserved moving to mixed, nonsymmetric two-mode Gaussian states. We aim at comparing Gaussian EM's and negativities on the two extremal classes of two-mode Gaussian states [20], introducing thus the concept of *extremal ordering*. At fixed global and local purities, the negativity of GMEMS (which is the maximal one) is obviously always greater than the negativity of GLEMS (which is the minimal one). If for the same values of purities the Gaussian EM's of GMEMS are larger than those of GLEMS, we will say that the extremal ordering is preserved. Otherwise, the extremal ordering is inverted. In this latter case, which is clearly the most intriguing, the states of minimal negativities are more entangled, with respect to Gaussian EM's, than the states of maximal negativities, and the inequivalence of the orderings, induced by the two different families of entanglement measures, becomes manifest.

The problem can be easily stated. By comparing $m_{opt}^{\mathrm{GLEMS}}$ from Eq. (39) and $m_{opt}^{\mathrm{GMEMS}}$ from Eq. (41), one has that in the range of global and local purities or, equivalently, of parameters $\{s,d,g\}$, such that

$$m_{opt}^{\mathrm{GMEMS}} \geqslant m_{opt}^{\mathrm{GLEMS}}, \tag{42}$$

the extremal ordering is preserved. When inequality (42) is violated, the extremal ordering is inverted. The boundary between the two regions, which can be found imposing the equality $m_{opt}^{\mathrm{GMEMS}}=m_{opt}^{\mathrm{GLEMS}}$, yields the range of global and local purities such that the corresponding GMEMS and GLEMS, despite having different negativities, have equal Gaussian EM's. This boundary surface can be found numerically, and the result is shown in the three-dimensional plot of Fig. 1.

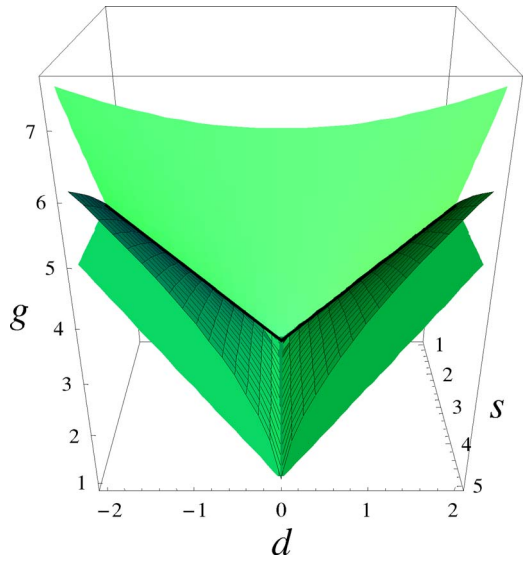


FIG. 1. (Color online) Comparison between the ordering induced by Gaussian EM's on the classes of states with extremal (maximal and minimal) negativities. This *extremal ordering* of the set of entangled two-mode Gaussian states is studied in the space of the CM parameters $\{s,d,g\}$, related to the global and local purities by the relations $\mu_1=(s+d)^{-1}$, $\mu_2=(s-d)^{-1}$, and $\mu=g^{-1}$. The intermediate, meshed surface is constituted by those global and local mixednesses such that the Gaussian EM's give equal values for the corresponding GMEMS (states of maximal negativities) and GLEMS (states of minimal negativities). Below this surface, the extremal ordering is inverted (GMEMS have fewer Gaussian EM's than GLEMS). Above it, the extremal ordering is preserved (GMEMS have more Gaussian EM's than GLEMS). However, it must be noted that this does not exclude that the individual orderings induced by the negativities and by the Gaussian EM's on a pair of nonextremal states may still be inverted in this region. Above the uppermost, lighter surface, GLEMS are separable states, so that the extremal ordering is trivially preserved. Below the lowermost, darker surface, no physical two-mode Gaussian states can exist. All the quantities plotted are dimensionless.

One can see, as a crucial result, that a region where the extremal ordering is inverted does indeed exist. The Gaussian EM's and the negativities are thus definitely *not* equivalent for the quantification of entanglement in nonsymmetric two-mode Gaussian states. The interpretation of this result is quite puzzling. On the one hand, one could think that the ordering induced by the negativities is a natural one, due to the fact that such measures of entanglement are directly inspired by the necessary and sufficient PPT criterion for separability. Thus, one would expect that the ordering induced by the negativities should be preserved by any *bona fide* measure of entanglement, especially if one considers that the extremal states GLEMS and GMEMS have a clear physical interpretation [20]. Therefore, as the Gaussian EOF is an upper bound to the true EOF, one could be tempted to take this result as evidence that the Gaussian EOF overestimates the true EOF, at least for GLEMS, and that, moreover, the true EOF of GLEMS should be lower than the true EOF of GMEMS, at fixed values of the purities. If this were the case, the true EOF would not coincide with the Gaussian EOF,

whose evaluation would consequently necessarily involve a decomposition over non-Gaussian states. However, this is only a qualitative and speculative argument: proving or disproving that the Gaussian EOF is the true EOF for any two-mode Gaussian state is still an open question under lively debate [18].

On the other hand, one could take the simplest discrete-variable instance, constituted by a two-qubit system, as a test case for comparison. There, although for pure states the negativity coincides with the concurrence, an entanglement monotone equivalent to the EOF for all states of two qubits [44], the two measures cease to be equivalent for mixed states, and the orderings they induce on the set of entangled states can be different [39]. This analogy seems to support again the stand that, in the arena of mixed states, a unique measure of entanglement is a *chimera* and cannot really be expected, due to the different operative meanings and physical processes (in the cases when it has been possible to identify them) that are associated to each definition: one could think, for instance, of the operative difference existing between the definitions of distillable entanglement and entanglement cost. In other words, from this point of view, each inequivalent measure of entanglement introduced for mixed states should capture physically distinct aspects of quantum correlations existing in these states. Then, joining this kind of outlook, one could hope that the Gaussian EM's might still be considered as proper measures of CV entanglement, especially if one were able to prove the conjecture that the Gaussian EOF is the true EOF for a broader class of Gaussian states beyond the symmetric ones. One could then live with the existence of inverted orderings of entangled states and see it as a not so annoying problem.

Whatever be the case, we have shown that two different families of measures of CV entanglement can induce different orderings on the set of two-mode entangled states. This is more clearly illustrated in Fig. 2, where we keep fixed one of the local mixednesses and we classify, in the space of the other local mixedness and of the global mixedness, the different regions related to entanglement and extremal ordering of two-mode Gaussian states, improving and completing a similar diagram previously introduced in Ref. [19] to describe separability in the space of purities.

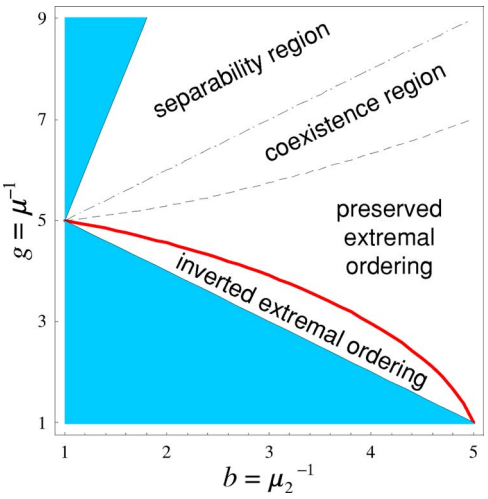


FIG. 2. (Color online) Summary of entanglement properties of two-mode Gaussian states in the projected space of the local mixedness $b=\mu_2^{-1}$ of mode 2 and of the global mixedness $g=\mu^{-1}$, while the local mixedness of mode 1 is kept fixed at a reference value $a=\mu_1^{-1}=5$. Below the thick curve, obtained imposing the equality in inequality (42), the Gaussian EM's yield GLEMS more entangled than GMEMS, at fixed purities: the extremal ordering is thus inverted. Above the thick curve, the extremal ordering is preserved. In the coexistence region (see Ref. [19]), GMEMS are entangled while GLEMS are separable. The boundaries of this region are given by Eq. (36) (dashed line) and Eq. (35) (dash-dotted line). In the separability region, GMEMS are separable too, so all two-mode Gaussian states whose purities lie in that region are not entangled. The shaded regions cannot contain any phisical two-mode Gaussian state. All the quantities plotted are dimensionless.

## VI. GAUSSIAN MEASURES OF ENTANGLEMENT VERSUS NEGATIVITIES

In this section we wish to give a more direct comparison of the two families of entanglement measures for two-mode Gaussian states. In particular, we are interested in finding the maximum and minimum values of one of the two measures if the other is kept fixed. A very similar analysis has been performed by Verstraete *et al.* [39] in their comparative analysis of the negativity and the concurrence for states of two-qubit systems.

Here it is useful to perform the comparison directly between the symplectic eigenvalue $\tilde{\nu}_-(\boldsymbol{\sigma})$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$ of a generic two-mode Gaussian state with CM $\boldsymbol{\sigma}$ and the symplectic eigenvalue $\tilde{\nu}_-(\boldsymbol{\sigma}_{opt}^P)$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}_{opt}^P$ of the optimal pure state with CM $\boldsymbol{\sigma}_{opt}^P$, which minimizes Eq. (23). In fact, the negativities are all monotonically decreasing functions of $\tilde{\nu}_-(\boldsymbol{\sigma})$, while the Gaussian EM's are all monotonically decreasing functions of $\tilde{\nu}_-(\boldsymbol{\sigma}_{opt}^P)$.

To start with, let us recall once more that for pure states and for mixed symmetric states (in the set of two-mode Gaussian states), the two quantities coincide. For nonsymmetric states, one can immediately prove the bound

$$\tilde{\nu}_-(\boldsymbol{\sigma}_{opt}^P) \leqslant \tilde{\nu}_-(\boldsymbol{\sigma}). \tag{43}$$

In fact, from Eq. (23), $\boldsymbol{\sigma}_{opt}^P \leqslant \boldsymbol{\sigma}$ [17]. For positive matrices, $A \geqslant B$ implies $a_k \geqslant b_k$, where the $a_k$s($b_k$s) denote the ordered symplectic eigenvalues of $A(B)$ [45]. Because the ordering $A \geqslant B$ is preserved under partial transposition, inequality (43) holds true. This fact induces a characterization of symmetric states, which saturate inequality (43), as the two-mode Gaussian states with *minimal* Gaussian EM's at fixed negativities.

It is then natural to raise the question whether an upper bound on the Gaussian EM's at fixed negativities exists as well. It seems hard to address this question directly, as one lacks a closed expression for the Gaussian EM's of generic states. But we can promptly give partial answers if we restrict to the classes of GLEMS and GMEMS, for which the Gaussian EM's have been explicitly computed in the previous section.

Let us begin with the GLEMS. We can compute the squared symplectic eigenvalue $\tilde{\nu}_-^2(\boldsymbol{\sigma}^{\mathrm{GLEMS}})=\{4(s^2+d^2)-g^2-1-\sqrt{[4(s^2+d^2)-g^2-1]^2-4g^2}\}/2$. Next, we can reparametrize the CM [obtained by Eq. (33) with $\lambda=-1$] to make $\tilde{\nu}_-$ appear explicitly: namely, $g=\sqrt{\tilde{\nu}_-^2[4(s^2+d^2)-1-\tilde{\nu}_-^2]/(1+\tilde{\nu}_-^2)}$. At this point, one can study the piecewise function $m_{opt}^{\mathrm{GLEMS}}$ from Eq. (39) and find out that it is a convex function of $d$ in the whole space of parameters corresponding to entangled states. Hence, $m_{opt}^{\mathrm{GLEMS}}$ and, thus, the Gaussian EM are maximized at the boundary $|d|=(2\tilde{\nu}_-s-\tilde{\nu}_-^2-1)/2$, resulting from the saturation of inequality (34). The states maximizing Gaussian EM's at fixed negativities, if we restrict to the class of GLEMS, have then to be found in the subclass of GMEMMS [states of maximal negativity for fixed marginals [20], defined after inequality (34)], depending on the parameter $s$ and on the eigenvalue $\tilde{\nu}_-$ itself, which completely determines the negativity). For these states,

$$m_{opt}^{\mathrm{GMEMMS}}(s,\tilde{\nu}_-)=\left(\frac{2s}{1-\tilde{\nu}_-^2+2\tilde{\nu}_-s}\right)^2. \tag{44}$$

The further optimization over $s$ is straightforward because $m_{opt}^{\mathrm{GMEMMS}}$ is an increasing function of $s$, so its global maximum is attained for $s\to\infty$. In this limit, one has simply

$$m_{\max}^{\mathrm{GMEMMS}}(\tilde{\nu}_-)=\frac{1}{\tilde{\nu}_-^2}. \tag{45}$$

From Eq. (30), one thus finds that for all GLEMS the following bound holds:

$$\tilde{\nu}_-(\boldsymbol{\sigma}_{opt}^P)\geqslant\frac{1}{\tilde{\nu}_-(\boldsymbol{\sigma})}[1-\sqrt{1-\tilde{\nu}_-^2(\boldsymbol{\sigma})}]. \tag{46}$$

One can of course perform a similar analysis for GMEMS. But after analogous reasonings and computations, what one finds is exactly the same result. This is not so surprising, keeping in mind that GMEMS, GLEMS, and all two-mode Gaussian states with generic $s$ and $d$ but with global mixedness $g$ saturating inequality (34) collapse into the same family of two-mode Gaussian states, the GMEMMS, completely determined by the local single-mode properties (they can be viewed as a generalization of the pure two-mode states: the symmetric GMEMMS are in fact pure). Hence, the bound of inequality (46), limiting the Gaussian EM's from above at fixed negativities, must hold for all GMEMS as well.

At this point, it is tempting to conjecture that inequality (46) holds for all two-mode Gaussian states. Unfortunately, the lack of a closed, simple expression for the Gaussian EM of a generic state makes the proof of this conjecture impossible, at the present time. However, one can show, by analytical power-series expansions of Eq. (29), truncated to the leading order in the infinitesimal increments, that, for any infinitesimal variation of the parameters of a generic CM around the limiting values characterizing GMEMMS, the Gaussian EM's of the resulting states lie always below the boundary imposed by the corresponding GMEMMS with the same $\tilde{\nu}_-$. In this sense, the GMEMMS are, at least, a *local* maximum for the Gaussian EM versus negativity problem.

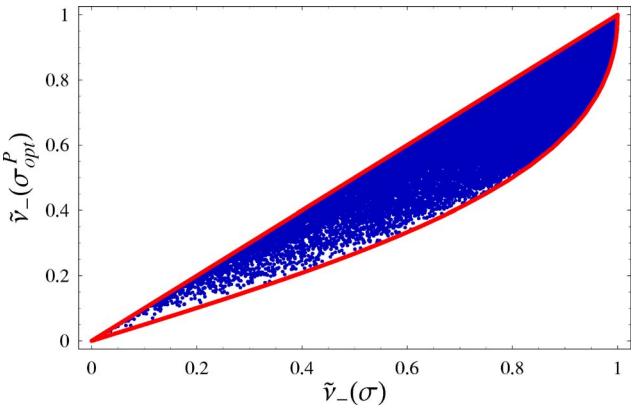


FIG. 3. (Color online) Comparison between Gaussian EM's and negativities for two-mode Gaussian states. On the horizontal axis we plot the symplectic eigenvalue $\tilde{\nu}_-(\boldsymbol{\sigma})$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}$ of a generic two-mode Gaussian state with CM $\boldsymbol{\sigma}$. On the vertical axis we plot the symplectic eigenvalue $\tilde{\nu}_-(\boldsymbol{\sigma}_{opt}^P)$ of the partially transposed CM $\tilde{\boldsymbol{\sigma}}_{opt}^P$ of the optimal pure state with CM $\boldsymbol{\sigma}_{opt}^P$, which minimizes Eq. (23). The negativities are all monotonically decreasing functions of $\tilde{\nu}_-(\boldsymbol{\sigma})$, while the Gaussian EM's are all monotonically decreasing functions of $\tilde{\nu}_-(\boldsymbol{\sigma}_{opt}^P)$. The equations of the two boundary curves are obtained from the saturation of inequality (43) (upper bound) and inequality (46) (lower bound), respectively. The dots represent 50 000 randomly generated CM's of two-mode Gaussian states. Of up to $1\times10^6$ random CM's, none has been found to lie below the lower solid-line curve, enforcing the conjecture that it be an absolute boundary for all two-mode Gaussian states. All the quantities plotted are dimensionless.

Furthermore, extensive numerical investigations of up to $1\times10^6$ CM's of randomly generated two-mode Gaussian states provide confirmatory evidence that GMEMMS attain indeed the *global* maximum (see Fig. 3). We can thus quite confidently conjecture, however, at the moment, without a complete formal proof of the statement, that GMEMMS, in the limit of infinite average local mixedness ($s\to\infty$), are the states of maximal Gaussian EM's at fixed negativities, among *all* two-mode Gaussian states.

A direct comparison between the two prototypical representatives of the two families of entanglement measures—respectively, the Gaussian EOF $G_{E_F}$ and the logarithmic negativity $E_{\mathcal{N}}$—is plotted in Fig. 4. For any fixed value of $E_{\mathcal{N}}$, inequality (43) provides in fact a rigorous lower bound on $G_{E_F}$, namely,

$$G_{E_F}\geqslant h[\exp(-E_{\mathcal{N}})], \tag{47}$$

while inequality (46) provides the conjectured lower bound

$$G_{E_F}\leqslant h\{\exp(E_{\mathcal{N}})[1-\sqrt{1-\exp(-2E_{\mathcal{N}})}]\}, \tag{48}$$

where we exploited Eqs. (18) and (31) and $h[x]$ is given by Eq. (21).

The existence of lower and upper bounds on the Gaussian EM's at fixed negativities (the latter strictly proven only for extremal states) limits to some extent the inequivalence arising between the two families of entanglement measures for nonsymmetric two-mode Gaussian states.

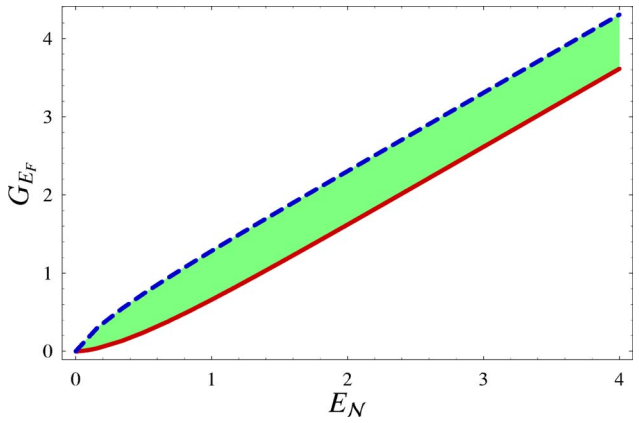


FIG. 4. (Color online) Comparison between the Gaussian entanglement of formation $G_{E_F}$ and the logarithmic negativity $E_{\mathcal{N}}$ for two-mode Gaussian states. Symmetric states accomodate on the lower boundary (solid line), determined by the saturation of inequality (47). GMEMMS with infinite, average local mixedness, lie on the dashed line, whose defining equation is obtained from the saturation of inequality (48). All GMEMS and GLEMS lie below the dashed line. The latter is conjectured, with strong numerical support, to be the upper boundary for the Gaussian EOF of all two-mode Gaussian states, at fixed negativity. All the quantities plotted are dimensionless.

## VII. SUMMARY AND OUTLOOK

In this work we focused on the simplest conceivable states of a bipartite CV system: two-mode Gaussian states. We have shown that, even in this simple instance, the theory of quantum entanglement hides several subtleties and reveals some surprising aspects. In particular, we have studied the relations existing between different computable measures of entanglement, showing how the negativities (including the standard logarithmic negativity) and the Gaussian convex-roof extended measures (Gaussian EM's, including the Gaussian entanglement of formation [17]) are inequivalent entanglement quantificators for nonsymmetric two-mode Gaussian states. We have computed Gaussian EM's explicitly for the two classes of two-mode Gaussian states having extremal (maximal and minimal) negativities at fixed purities [20]. We have highlighted how, in a certain range of values of the global and local purities, the ordering on the set of entangled states, as induced by the Gaussian EM's, is inverted with respect to that induced by the negativities. The question whether a certain Gaussian state is more entangled than another, thus, has no definite answer, not even when only extremal states are considered, as the answer comes to depend on the measure of entanglement one chooses. Extended comments on the possible meanings and consequences of the existence of inequivalente orderings of entangled states have been given in Secs. V and VI. Furthermore, we have proven the existence of a lower bound holding for the Gaussian EM's at fixed negativities and that this bound is saturated by two-mode symmetric Gaussian states. Finally, we have provided some strong numerical evidence and partial analytical proofs restricted to extremal states that an upper bound on the Gaussian EM's at fixed negativities exists as well and is saturated by states of maximal negativity for given marginals in the limit of infinite average local mixedness.

We believe that our results will raise renewed interest in the problem of the quantification of entanglement in CV systems, which seemed fairly well understood in the special instance of two-mode Gaussian states. Moreover, we hope that the present work may constitute a first step toward the solution of more general problems concerning the entanglement of Gaussian states, such as the computation of the entanglement of formation for generic two-mode Gaussian states [18] and the proof of its identity with the Gaussian EOF in a larger class of Gaussian states beyond the symmetric instance. On the other hand, the explicit expressions, computed in the present work, now available for the Gaussian EOF of GMEMS and GLEMS, might serve as well as a basis to find an explicit counterexample to the conjecture that the decomposition over all pure Gaussian states, in the definition of the EOF, is the optimal one for all two-mode Gaussian states.

Finally, the results collected in the present work might prove useful as well in the task of quantifying multipartite entanglement of Gaussian states. For instance, we should mention here that any two-mode reduction of a pure three-mode Gaussian state is a GLEMS, as a consequence of the Schmidt decomposition operated at the CM level [46]. Therefore, thanks to the results that we have derived here, its Gaussian EOF can be explicitly computed and can be compared with the entropy of entanglement between one reference mode and the remaining two in the global state. One has then available the tools and can apply them to investigate the sharing structure of multipartite CV entanglement of three-mode and, more generally, multimode Gaussian states [47].

## ACKNOWLEDGMENTS

We are grateful to O. Krüger and M. M. Wolf for fruitful discussions and for providing us with supplementary material on the computation of the Gaussian entanglement of formation. Financial support from CNR-Coherentia, INFN, and MIUR is acknowledged.

[1] *Quantum Information Theory with Continuous Variables*, edited by S. L. Braunstein, and A. K. Pati (Kluwer, Dordrecht, 2002).
[2] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 531 (2005).
[3] H. P. Yuen and A. Kim, Phys. Lett. A **241**, 135 (1998); K. Bencheikh, T. Symul, A. Jankovic, and J. A. Levenson, J. Mod. Opt. **48**, 1903 (2001); F. Grosshans, G. Van Assche, J. Wenger, R. Brouri, N. J. Cerf, and P. Grangier, Nature (London) **421**, 238 (2003).
[4] L. Vaidman, Phys. Rev. A **49**, 1473 (1994); S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).
[5] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998); W. P. Bowen, N. Treps, B. C. Buchler, R. Schnabel, T. C. Ralph, Hans-A. Bachor, T. Symul, and P. K. Lam, Phys. Rev. A **67**, 032302 (2003); T. C. Zhang, K. W. Goh, C. W. Chou, P. Lodahl, and H. J. Kimble, *ibid.* **67**, 033802 (2003).
[6] N. Takei, H. Yonezawa, T. Aoki, and A. Furusawa, Phys. Rev. Lett. **94**, 220502 (2005).
[7] P. van Loock, and S. L. Braunstein, Phys. Rev. Lett. **84**, 3482 (2000); H. Yonezawa, T. Aoki, and A. Furusawa, Nature (London) **431**, 430 (2004).
[8] G. Adesso and F. Illuminati, Phys Rev. Lett., e-print quant-ph/0412125 (to be published).
[9] P. van Loock, and S. L. Braunstein, Phys. Rev. A **61**, 010302(R) (2000); X. Jia, X. Su, Q. Pan, J. Gao, C. Xie, and K. Peng, Phys. Rev. Lett. **93**, 250503 (2004).
[10] M. Ban, J. Opt. B: Quantum Semiclassical Opt. **1**, L9 (1999); S. L. Braunstein, and H. J. Kimble, Phys. Rev. A **61**, 042302 (2000); X. Li, Q. Pan, J. Jing, J. Zhang, C. Xie, and K. Peng, Phys. Rev. Lett. **88**, 047904 (2002); J. Mizuno, K. Wakui, A. Furusawa, and M. Sasaki, Phys. Rev. A **71**, 012304 (2005).
[11] B. Julsgaard, J. Sherson, J. I. Cirac, J. Fiurášek, and E. S. Polzik, Nature (London) **432**, 482 (2004).
[12] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. **82**, 1784 (1999); T. C. Ralph, A. Gilchrist, G. J. Milburn, W. J. Munro, and S. Glancy, Phys. Rev. A **68**, 042319 (2003).
[13] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[14] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[15] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[16] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. **91**, 107901 (2003).
[17] M. M. Wolf, G. Giedke, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. A **69**, 052320 (2004).
[18] E-print quant-ph/0504166 (unpublished), edited by O. Krüger and R. F. Werner; http://www.imaph.tu-bs.de/qi/problems
[19] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004).
[20] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318, 2004.
[21] A. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Clarendon Press, Oxford, 1997).
[22] R. Simon, E. C. G. Sudarshan, and N. Mukunda, Phys. Rev. A **36**, 3868 (1987).
[23] R. Simon, N. Mukunda, and B. Dutta, Phys. Rev. A **49**, 1567 (1994)
[24] J. Williamson, Am. J. Math. **58**, 141 (1936); R. Simon, S. Chaturvedi, and V. Srininivasan, J. Math. Phys. **40**, 3632 (1999).
[25] M. G. A. Paris, F. Illuminati, A. Serafini, and S. De Siena, Phys. Rev. A **68**, 012314 (2003).
[26] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[27] A. Serafini, F. Illuminati, and S. De Siena, J. Phys. B **37**, L21 (2004).
[28] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996).
[29] R. F. Werner and M. M. Wolf, Phys. Rev. Lett. **86**, 3658 (2001).
[30] A. Serafini, G. Adesso, and F. Illuminati, Phys. Rev. A **71**, 032349 (2005).
[31] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998).
[32] J. Eisert, Ph.D. thesis, University of Potsdam, Potsdam, 2001.
[33] K. Audenaert, M. B. Plenio, and J. Eisert, Phys. Lett. A **241**, 135 (1998);
[34] M. B. Plenio, Phys. Rev. Lett. **95**, 090503 (2005).
[35] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[36] P. M. Hayden, M. Horodecki, and B. M. Terhal, J. Phys. A **34**, 6891 (2001).
[37] J. Eisert and M. M. Wolf, e-print quant-ph/0505151 (unpublished).
[38] J. Eisert and M. Plenio, J. Mod. Opt. **46**, 145 (1999); S. Virmani and M. B. Plenio, Phys. Lett. A **268**, 31 (2000).
[39] F. Verstraete, K. M. R. Audenaert, J. Dehaene, and B. De Moor, J. Phys. A **34**, 10327 (2001).
[40] The geometric picture describing the optimal two-mode state which enters in the determination of the Gaussian EM is introduced in Ref. [17]. A more detailed discussion, including the explicit expression of the Lorentz boost needed to move into the plane of the ellipse, can be found in O. Krüger (unpublished), available online at the URL: http://www.imaph.tu-bs.de/qi/papers.html.
[41] J. I. Cirac, G. Giedke, O. Krüger, R. F. Werner, and M. M. Wolf (unpublished), available online at the URL: http://www.imaph.tu-bs.de/qi/papers.html.
[42] J. Fiurášek and N. J. Cerf, Phys. Rev. Lett. **93**, 063601 (2004); J. Wenger, J. Fiurášek, R. Tualle-Brouri, N. J. Cerf, and P. Grangier, Phys. Rev. A **70**, 053812 (2004).
[43] G. Adesso, A. Serafini, and F. Illuminati, Open Syst. Inf. Dyn. **12**, 189 (2005).
[44] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[45] G. Giedke, J. Eisert, J. I. Cirac, and M. B. Plenio, Quantum Inf. Comput. **3**, 211 (2003).
[46] A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001).
[47] G. Adesso and F. Illuminati, e-print quant-ph/0410050v3 (unpublished).