n-qubit-controlled phase gate with superconducting quantum-interference devices coupled to a resonator

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We present a way to realize an *n*-qubit controlled phase gate with superconducting quantum-interference devices (SQUIDs) by coupling them to a superconducting resonator. In this proposal, the two logical states of a qubit are represented by the two lowest levels of a SQUID. An intermediate level of each SQUID is utilized to facilitate coherent control and manipulation of quantum states of the qubits. It is interesting to note that an *n*-qubit controlled phase gate can be achieved with *n* SQUIDs by successively applying a $\pi/2$ Jaynes-Cummings pulse to each of the $n-1$ control SQUIDs before and after a π Jaynes-Cummings pulse on the target SQUID.

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I. INTRODUCTION AND MOTIVATION

Quantum computing has attracted much interest since it was clear that quantum computers are in principle able to solve hard computational problems much more efficiently than classical computers [1–3]. In the past decade, various physical systems have been considered for building up quantum-information processors. Among them, cavity QED analogs with solid-state systems are particularly appealing. Theoretically, it was predicted earlier that the strong-coupling limit, which is difficult to achieve with atoms in a microwave cavity, can readily be realized with superconducting charge qubits [4,5], superconducting flux qubits [6], or semiconducting quantum dots [7]. Recently, the strong-coupling cavity QED has been experimentally demonstrated with superconducting charge qubits and flux qubits [8,9] and semiconductor quantum dots embedded in a microcavity [10–12]. The results of these experiments make solid-state qubit cavity QED a very attractive approach to the quantum-information process.

It is known that a quantum-computing network can be decomposed into two-qubit gates and one-qubit rotations [13]. So far, a large number of theoretical proposals for realizing two-qubit gates have been presented with many physical systems. Moreover, two-qubit controlled-NOT (CNOT) gates or controlled phase gates have been experimentally demonstrated in cavity QED [14], ion traps [15], NMR [16], quantum dots [17], and superconducting charge qubits [18]. On the other hand, research on quantum computing has recently moved toward the physical realization of multiqubit quantum gates. Several schemes for realizing three-qubit Toffoli gates have been proposed with neutral atoms in an optical lattice [19] or hybrid atom-photon qubits via cavity QED [20]. In addition, experimental realization of a controlled phase gate in a three-qubit NMR quantum system has been reported recently [21].

In this paper, our goal is to present a way to realize an *n*-qubit controlled phase gate with superconducting quantum-interference devices (SQUIDs), coupled to a superconducting resonator. In this proposal, the two lowest levels $|0\rangle$ and $|1\rangle$ of each SQUID represent the two logical states of a qubit while a higher-energy level $|2\rangle$ of each SQUID is

used to facilitate coherent control and manipulation of quantum states of the qubits (Fig. 1). The method presented here essentially operates by having the resonator be (a) resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition of each SQUID sequentially, while (b) largely detuned from the $|1\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of each SQUID during the gate operation. We find that an *n*-qubit controlled phase gate can be achieved with *n* SQUIDs, by successively applying a $\pi/2$ Jaynes-Cummings pulse to each of the $n-1$ control SQUIDs before and after a π Jaynes-Cummings pulse on the remaining SQUID (the target qubit).

As shown below, this scheme has the following advantages. (i) No tunneling between the qubit levels $|0\rangle$ and $|1\rangle$ is required so that the storage time of each qubit can be made very long. (ii) No measurement on SQUIDs or photons is needed, therefore the operation is simplified. (iii) No auxiliary SQUIDs are needed, thus saving precious hardware resources. (iv) Coupling constants of each SQUID with the resonator could be nonidentical to accommodate inevitable nonuniformity in device parameters.

The motivation for this work is threefold. (i) SQUIDs have recently attracted much attention in the quantum-information community. Reasons for this are that SQUIDs are relatively easy to scale up and have been demonstrated to have relatively long decoherence times among solid-state qubits [9,22–27] and therefore have been considered as a promising candidate for building up superconducting quantum computers and information processors [28–37]. (ii) In the present proposal, the strong coupling between a SQUID and the resonator is analogous to atomic cavity quantum electrodynamics. Although many schemes for realizing two-qubit quantum gates with a variety of physical systems using cavity QED [5,32–34,38–45] or trapped ions [46–49] have been proposed, realization of a multiqubit controlled quantum gate based on cavity QED or trapped ions has not been thoroughly investigated. (iii) As is well known, multiqubit controlled quantum gates are of great importance to constructing quantum-computational networks, realizing quantum-error-correction protocols, and implementing quantum algorithms.

This paper is organized as follows. In Sec. II, we review the basic theory of a SQUID coupled to a single-mode resonator or driven by a classical microwave pulse. In Sec. III,
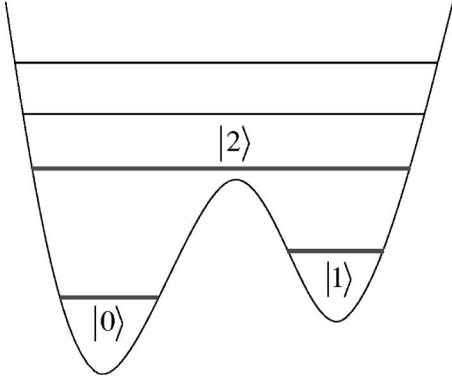
FIG. 1. Level diagram of a SQUID with the three lowest levels $|0\rangle$, $|1\rangle$, and $|2\rangle$ forming a Λ-type structure.

we show how to realize a three-qubit controlled phase gate with three SQUIDs in a resonator. We then discuss how the method can be generalized to implement an n-qubit controlled phase gate with n SQUIDs in a resonator. In Sec. IV, we give a discussion of the experimental issues. A concluding summary is given in Sec. V.

II. BASIC THEORY

The SQUIDs considered throughout this paper are rf SQUIDs each consisting of a Josephson tunnel junction enclosed by a superconducting loop (the size of a rf SQUID is on the order of 10–100 μm). The Hamiltonian for a rf SQUID (with junction capacitance C and loop inductance L) has the usual form [25]

$$H_s = \frac{Q^2}{2C} + \frac{(\Phi - \Phi_x)^2}{2L} - E_J \cos\left(2\pi\frac{\Phi}{\Phi_0}\right), \quad (1)$$

where Φ, the magnetic flux threading the ring, and Q, the total charge on the capacitor, are the conjugate variables of the system (with the commutation relation $[\Phi, Q] = i\hbar$), Φ_x is the static (or quasistatic) external magnetic flux applied to the ring, and $E_J \equiv I_c\Phi_0/2\pi$ is the Josephson coupling energy (I_c is the critical current of the junction and $\Phi_0 = h/2e$ is the flux quantum).

A. SQUID coupled to a single-mode resonator

Consider a SQUID coupled to a single-mode resonator. The SQUID is biased properly to have Λ-type three lowest levels, which are denoted by $|0\rangle$, $|1\rangle$, and $|2\rangle$, respectively (Fig. 1). Suppose that the coupling of $|0\rangle$, $|1\rangle$, and $|2\rangle$ with other levels of the SQUID via the resonator is negligible, which can be readily achieved by adjusting the level spacings of the SQUID. It can be shown that when the resonator is resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition while far off resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of the SQUID, the interaction Hamiltonian in the interaction picture, after making the rotating-wave approximation, can be written as [32]

$$H_I = \hbar(g a^\dagger |0\rangle\langle 2| + \text{H.c.}). \quad (2)$$

Here, $a^\dagger$ and a are the creation and annihilation operators of the resonator mode with frequency ν_r, and g is the coupling

constant between the resonator mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of the SQUID. For a superconducting one-dimensional standing-wave resonator, the expression of g is given by

$$\hbar g(x) = \frac{M_{sr}}{L}\sqrt{\frac{h\nu_r}{L_0 l}}\langle 0|\Phi|2\rangle \sin\left(\frac{2\pi}{\lambda}x\right), \quad (3)$$

where M_{sr} is the mutual inductance between the SQUID and the resonator, L_0 is the inductance per unit length of the resonator, l is the length of the resonator, ν_r is the frequency of the resonator mode with wavelength λ, and x is the center of the SQUID in the resonator.

The Hamiltonian (2) actually has the same form as the Jaynes-Cummings Hamiltonian of a two-level system resonant with a single-mode cavity field (a Hamiltonian well known in quantum optics). In the case when the resonator is initially in the photon-number state $|n\rangle$, the time evolution of the states of the system, governed by the Hamiltonian (2), is described by

$$|0\rangle|n\rangle \rightarrow \cos\sqrt{n}gt|0\rangle|n\rangle - i\sin\sqrt{n}gt|2\rangle|n-1\rangle,$$

$$|2\rangle|n\rangle \rightarrow -i\sin\sqrt{n+1}gt|0\rangle|n+1\rangle + \cos\sqrt{n+1}gt|2\rangle|n\rangle.$$

$$(4)$$

Note that the coupling strength g may vary with different SQUIDs due to nonuniform device parameters and/or nonexact placement of SQUIDs in the cavity. Therefore, hereafter we replace g by $g_1, g_2, \ldots,$ and g_n for SQUIDs 1, 2, …, and n, respectively.

B. SQUID driven by a classical microwave pulse

Now let us consider a SQUID driven by a classical microwave pulse with the magnetic component $\mathbf{B}_{\mu w}(\mathbf{r}, t) = \mathbf{B}_{\mu w}(\mathbf{r})\cos(2\pi\nu_{\mu w}t + \phi)$. Here, $\mathbf{B}_{\mu w}(\mathbf{r})$, $\nu_{\mu w}$, and ϕ are the magnetic field amplitude, frequency, and phase of the microwave pulse. It can be shown that if the microwave pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition but far off resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of the SQUID, then the interaction Hamiltonian in the interaction picture is given by

$$H_I = \frac{\hbar}{2}(\Omega_{12}e^{i\phi}|1\rangle\langle 2| + \text{H.c.}), \quad (5)$$

where Ω_{12} is the Rabi frequency of the pulse, which takes the following form [32]:

$$\Omega_{12}(t) = \frac{1}{L\hbar}\langle 1|\Phi|2\rangle\int_S \mathbf{B}_{\mu w}(\mathbf{r})\cdot d\mathbf{S}. \quad (6)$$

From the Hamiltonian (5), it is straightforward to see that a pulse of duration t results in the following rotation:

$$|1\rangle \rightarrow \cos\frac{\Omega_{12}}{2}t|1\rangle - ie^{-i\phi}\sin\frac{\Omega_{12}}{2}t|2\rangle,$$

$$|2\rangle \rightarrow -ie^{i\phi}\sin\frac{\Omega_{12}}{2}t|1\rangle + \cos\frac{\Omega_{12}}{2}t|2\rangle. \quad (7)$$

III. MULTIQUBIT CONTROLLED PHASE GATE WITH SQUIDS

In this section, for clarity, we will first give an explicit description of how to realize a three-qubit controlled phase gate with three SQUIDs coupled to a microwave resonator. We will then discuss how to extend the method to obtain an n-qubit controlled phase gate with a larger number n.

A. Three-qubit controlled phase gate

For three qubits, there are a total number of eight (2^3) computational basis states, denoted by $|000\rangle$, $|001\rangle$, ..., $|111\rangle$, respectively. A three-qubit controlled phase gate is described by

$$|000\rangle \rightarrow |000\rangle, \quad |100\rangle \rightarrow |100\rangle,$$

$$|001\rangle \rightarrow |001\rangle, \quad |101\rangle \rightarrow |101\rangle,$$

$$|010\rangle \rightarrow |010\rangle, \quad |110\rangle \rightarrow |110\rangle,$$

$$|011\rangle \rightarrow |011\rangle, \quad |111\rangle \rightarrow -|111\rangle, \quad (8)$$

which implies that if and only if the two control qubits (the first two qubits) are in the state $|1\rangle$, a phase flip happens to the state $|1\rangle$ of the target qubit (the last qubit) and nothing happens otherwise.

To realize this gate, consider SQUIDs (1,2,3) each having the Λ-type level configuration as depicted in Fig. 1. The transition between any two levels for each SQUID is initially far off resonant with the resonator (e.g., via prior adjustment of the level spacings) and the cavity mode is initially in the vacuum state $|0\rangle_c$.

The operations for realizing the three-qubit controlled phase gate are listed below.

Step (i). Apply a π microwave pulse ($\Omega_{12}\tau_{\mu w} = \pi$, where $\tau_{\mu w}$ is the pulse duration) with $\phi = -\pi/2$ to SQUID 1 [Fig. 2(a)]. The pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 1. After the pulse, the transformation $|1\rangle \rightarrow |2\rangle$ of SQUID 1 is obtained.

Step (ii). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 1 to resonance with the resonator for an interaction time $\tau_1 = \pi/(2g_1)$ [Fig. 2(b)], resulting in $|2\rangle_1|0\rangle_c \rightarrow -i|0\rangle_1|1\rangle_c$.

Step (iii). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 to resonance with the resonator for an interaction time $\tau_2 = \pi/(2g_2)$ [Fig. 2(c)]. As a result, the states $|0\rangle_2|0\rangle_c$, $|1\rangle_2|0\rangle_c$, and $|1\rangle_2|1\rangle_c$ remain unchanged, while the state $|0\rangle_2|1\rangle_c$ changes to $-i|2\rangle_2|0\rangle_c$.

Step (iv). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 3 to resonance with the resonator for an interaction time $\tau_3 = \pi/g_3$ [Fig. 2(d)], resulting in $|0\rangle_3|1\rangle_c \rightarrow -|0\rangle_3|1\rangle_c$ and no change for the states $|0\rangle_3|0\rangle_c$, $|1\rangle_3|0\rangle_c$, and $|1\rangle_3|1\rangle_c$.

Step (v). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 2 to resonance with the resonator for an interaction time $\tau_2 = \pi/(2g_2)$ [Fig. 2(c)]. As a result, the states $|0\rangle_2|0\rangle_c$, $|1\rangle_2|0\rangle_c$, and $|1\rangle_2|1\rangle_c$ remain unchanged, while the state $|2\rangle_2|0\rangle_c$ becomes $-i|0\rangle_2|1\rangle_c$.

Step (vi). Bring the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID 1 to resonance with the resonator for an interaction time $\tau_1 = \pi/(2g_1)$ [Fig. 2(b)], resulting in $|0\rangle_1|1\rangle_c \rightarrow -i|2\rangle_1|0\rangle_c$.

Step (vii). Apply a π microwave pulse with $\phi = \pi/2$ to SQUID 1 [Fig. 2(a)]. The pulse is resonant with the $|1\rangle \leftrightarrow |2\rangle$ transition of SQUID 1. After the pulse, the transformation $|2\rangle \rightarrow |1\rangle$ of SQUID 1 is achieved.

The states of the whole system after each step of the above operations are summarized below:

$$
\begin{array}{l}
|100\rangle|0\rangle_c \\
|101\rangle|0\rangle_c \\
|110\rangle|0\rangle_c \\
|111\rangle|0\rangle_c
\end{array}
\xrightarrow{\text{Step(i)}}
\begin{array}{l}
|200\rangle|0\rangle_c \\
|201\rangle|0\rangle_c \\
|210\rangle|0\rangle_c \\
|211\rangle|0\rangle_c
\end{array}
\xrightarrow{\text{Step(ii)}}
\begin{array}{l}
-i|000\rangle|1\rangle_c \\
-i|001\rangle|1\rangle_c \\
-i|010\rangle|1\rangle_c \\
-i|011\rangle|1\rangle_c
\end{array}
\xrightarrow{\text{Step(iii)}}
\begin{array}{l}
-|020\rangle|0\rangle_c \\
-|021\rangle|0\rangle_c \\
-i|010\rangle|1\rangle_c \\
-i|011\rangle|1\rangle_c
\end{array}
\xrightarrow{\text{Step(iv)}}
\begin{array}{l}
-|020\rangle|0\rangle_c \\
-|021\rangle|0\rangle_c \\
i|010\rangle|1\rangle_c \\
-i|011\rangle|1\rangle_c
\end{array}
\xrightarrow{\text{Step(v)}}
\begin{array}{l}
i|000\rangle|1\rangle_c \\
i|001\rangle|1\rangle_c \\
i|010\rangle|1\rangle_c \\
-i|011\rangle|1\rangle_c
\end{array}
$$

$$
\xrightarrow{\text{Step(vi)}}
\begin{array}{l}
|200\rangle|0\rangle_c \\
|201\rangle|0\rangle_c \\
|210\rangle|0\rangle_c \\
-|211\rangle|0\rangle_c
\end{array}
\xrightarrow{\text{Step(vii)}}
\begin{array}{l}
|100\rangle|0\rangle_c \\
|101\rangle|0\rangle_c \\
|110\rangle|0\rangle_c \\
-|111\rangle|0\rangle_c
\end{array}, \quad (9)
$$

where $|ijk\rangle$ is the abbreviation of the state $|i\rangle_1|j\rangle_2|k\rangle_3$ of SQUIDs (1,2,3) with $i,j,k \in \{0,1,2\}$.

On the other hand, it is obvious that the following states of the system:

$$|000\rangle|0\rangle_c, \quad |001\rangle|0\rangle_c, \quad |010\rangle|0\rangle_c, \quad |011\rangle|0\rangle_c \quad (10)$$

remain unchanged during the operation. This is because (a) the state $|0\rangle$ of SQUID 1 was not affected by the applied microwave pulse, since the $|0\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of SQUID 1 are far off resonant with the applied microwave pulse; and (b) no photon was emitted to the resonator during the Step (ii) operation, when SQUID 1 is initially in the state $|0\rangle$. Hence, it can be concluded from Eq. (9) that the three-qubit controlled phase gate (8) was achieved with three SQUIDs (i.e., the control SQUIDs 1 and 2, as well as the target SQUID 3) after the above process.
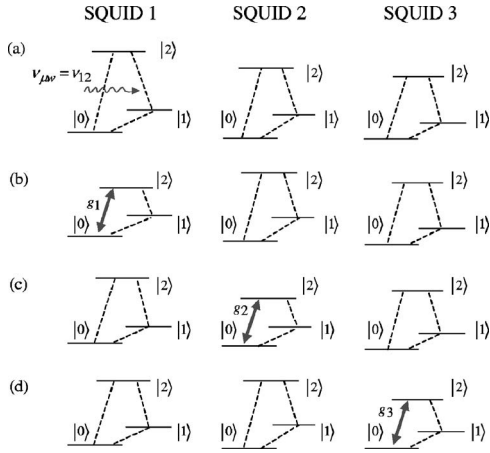
FIG. 2. Illustration for the change of the level structure (reduced) of three SQUIDs (1, 2, 3) during a three-qubit controlled phase gate performance. In (a), (b), (c), and (d), figures from left to right represent the level structures for SQUIDs 1, 2, and 3, respectively; the nonidentical level spacings of the SQUIDs could be caused by nonuniform device parameters. g_1, g_2, and g_3 are the resonant coupling constants between the resonator mode and the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUIDs 1, 2, and 3, respectively. The difference for g_1, g_2, and g_3 is due to device parameter nonuniformity or nonexact placement of each SQUID. $\nu_{\mu w}$ is the frequency of the applied microwave pulse while ν_{12} is the $|1\rangle \leftrightarrow |2\rangle$ transition frequency for SQUID 1. In (a), the level spacings of SQUID 1 is set to be much different from that of SQUIDs 2 and 3, such that SQUIDs 2 and 3 are decoupled from the applied pulse. The transition between any two levels linked by a dashed line is far off resonant with the resonator mode.

B. *n*-qubit controlled phase gate

A quantum-controlled phase gate of n qubits $(1, 2, \ldots, n)$ is defined by the following transformation:

$$|i_1 i_2 \cdots i_n\rangle \rightarrow (-1)^{i_1 \times i_2 \times \cdots \times i_n} |i_1 i_2 \cdots i_l \cdots i_n\rangle, \quad (11)$$

where the subscript l represents qubit l, $i_l \in \{0, 1\}$, and $|i_1 i_2 \cdots i_l \cdots i_n\rangle$ is the n-qubit computational basis state. For n qubits, there are a total number of 2^n computational basis states, which form a set of complete orthogonal bases in a 2^n-dimensional Hilbert space of the n qubits. Equation (11) shows that only when the $n-1$ control qubits (the first $n-1$ qubits) are all in the state $|1\rangle$ does the state $|1\rangle$ of the target qubit (the last qubit) undergo a phase flip, i.e., $|11 \cdots 1\rangle \rightarrow -|11 \cdots 1\rangle$, while nothing happens to all the other $2^n - 1$ computational basis states. In the following, we will discuss how this gate can be achieved with n SQUIDs coupled to a resonator.

The n SQUIDs are labeled by $1, 2, \ldots,$ and n. The first $n-1$ SQUIDs $(1, 2, \ldots, n-1)$ act as control qubits while the SQUID n is the target qubit. Suppose that the n SQUIDs $(1, 2, \ldots, n)$ are initially decoupled from the resonator (initially in the vacuum state). Examining the above operations for the three-qubit controlled phase gate carefully, we find that the n-qubit controlled phase gate (11) can be obtained with the resonator mode returning to the original vacuum state, by the sequence of operators

$$U = U_1^+ \otimes \left(\prod_{l=n-1}^{1} U_{lr} \right) \otimes U_{nr}^2 \otimes \left(\prod_{l=1}^{n-1} U_{lr} \right) \otimes U_1, \quad (12)$$

where $\prod_{i=1}^{k} U_{ir} \equiv U_{kr} U_{(k-1)r} \cdots U_{2r} U_{1r}$; U_1^+ and U_1 denote the operators on SQUID 1 with the following matrices;

$$U_1^+ U_1 = I, \quad U_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (13)$$

in the basis states $|1\rangle_1 = (0, 1)^T$ and $|2\rangle_1 = (1, 0)^T$; U_{lr} is the joint operator on the SQUID l and the resonator mode ($l = 1, 2, \ldots, n-1$), represented by the matrix

$$U_{lr} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad (14)$$

in the basis states $|0\rangle_l |1\rangle_c = (0, 1)^T$ and $|2\rangle_l |0\rangle_c = (1, 0)^T$; and U_{nr} is the joint operator on the SQUID n and the resonator mode with the same matrix as the one described by Eq. (14) [in the basis states $|0\rangle_n |1\rangle_c = (0, 1)^T$ and $|2\rangle_n |0\rangle_c = (1, 0)^T$].

From the description in the previous subsection, we see the following.

(i) U_1 (U_1^+) denotes the application of a π microwave pulse ($\Omega_{12} \tau_{\mu w} = \pi$, where $\tau_{\mu w}$ is the pulse duration) with $\phi = -\pi/2(\pi/2)$ and $\nu_{\mu w} = \nu_{12}$ (the $|1\rangle \leftrightarrow |2\rangle$ transition frequency of SQUID 1) to SQUID 1.

(ii) U_{lr} corresponds to the operation of bringing the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID l ($l = 1, 2 \ldots, n-1$) to resonance with the resonator for an interaction time $\tau_l = \pi/(2g_l)$ (i.e., a $\pi/2$ Jaynes-Cummings pulse). and

(iii) U_{nr}^2 indicates two $\pi/2$ Jaynes-Cummings pulses resonant with the $|0\rangle \leftrightarrow |2\rangle$ transition of SQUID n, which are, when combined together, equivalent to a π Jaynes-Cummings pulse (i.e., $\tau_n = \pi/g_n$).

We remark that all other basis states of the system involved in each step of the above operations, which form a complete set of orthonormal states together with the basis states described above, are not affected by (a) setting the microwave pulse to be largely detuned from the $|0\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of SQUID 1 and (b) setting the $|1\rangle \leftrightarrow |2\rangle$ transition and the $|0\rangle \leftrightarrow |1\rangle$ transition of each SQUID far off resonant from the resonator mode.

The method presented here for realizing the n-qubit controlled phase gate (11) is an extension of the three-qubit version (8) described in the previous subsection. This can be seen as follows. From Eq. (9), one can see that the three-qubit controlled phase gate (8) was realized essentially through the operation of step (iv). This operation led to a phase shift on the state of SQUID 3 (i.e., $|0\rangle \rightarrow -|0\rangle$ and $|1\rangle \rightarrow |1\rangle$) with the aid of the photon, when the two control SQUIDs 1 and 2 are initially in the basis state $|11\rangle$. But, when the two control SQUIDs 1 and 2 are initially in the other basis states, this operation results in no change to the state of SQUID 3, due to the fact that no photon was left in the resonator mode after the step (iii). Similarly, the realization of the n-qubit controlled phase gate (11) described above is mainly based on the operation described by U_{nr}^2, which causes a phase shift on the state of SQUID n (the target) with the assistance of the photon when the $n-1$ con-

trol SQUIDs $(1, 2, \ldots, n-1)$ are initially in the basis state $|11\cdots1\rangle$. Note that only for this initial basis state would the photon originally created by the operation $U_{1r}U_1$ remain in the resonator mode after the operation $\Pi_{l=2}^{n-1}U_{lr}$. In addition, similar to the operations of steps (v)–(vii) for the three-qubit controlled phase gate (8), the operation described by $U_1^+ \otimes (\Pi_{l=n-1}^1 U_{lr})$ has the photon emitted back into the resonator mode from the system of SQUIDs $(2, 3, \ldots, n-1)$ and finally absorbed by SQUID 1. Hence, the resonator mode returns to the original vacuum state and the n SQUIDs $(1, 2, \ldots, n)$ are back to the initial states except a phase flip to the n-qubit basis state $|11\cdots1\rangle$.

Before closing this section, some points may need to be addressed here.

(a) The irrelevant SQUIDs in each step of the operation need to be decoupled from the resonator or pulse during the resonator- or pulse-SQUID interaction. The resonator mode needs to be not excited during the application of the microwave pulse. In addition, the coupling of the levels $|0\rangle$, $|1\rangle$, and $|2\rangle$ with the other levels should be negligible for each SQUID. In principle, these conditions can be satisfied by adjusting the level spacings of the SQUIDs. Note that for a SQUID, the level spacings can be changed readily by varying the external flux Φ_x or the critical current I_c (e.g., for variable-barrier rf SQUIDs) [50].

(b) It is not necessary to have a single-mode resonator since for a multimode resonator one can choose one mode to interact with the SQUIDs while have all other modes well decoupled from the three lowest levels of the SQUIDs (e.g., with proper device parameters).

(c) The method presented here is applicable to a one-dimensional (1D), 2D, or 3D microwave resonator or cavity as long as the conditions described above can be met.

(d) As is well known, an n-qubit controlled-NOT gate (known as the Toffoli gate for $n=3$) can be obtained by combining the n-qubit controlled phase gate (11) with two single-qubit Hadamard gates, which are performed on the target qubit before and after the n-qubit controlled phase gate (11) respectively (see Fig. 3). Each of the single-qubit Hadamard transformations $|0\rangle \rightarrow (1/\sqrt{2})(|0\rangle+|1\rangle)$ and $|1\rangle \rightarrow (1/\sqrt{2})(|0\rangle -|1\rangle)$ can be done with a $\pi/2$ microwave pulse resonant with the $|0\rangle \leftrightarrow |1\rangle$ transition of the target SQUID qubit. When combining with the above quantum-controlled phase gate operations, one obtains the n-SQUID qubit CNOT gate.

(e) The present method provides a simple way to implement a multiqubit CNOT gate with SQUIDs coupled to a resonator. To see this, let us consider the simple case of $n=3$. It is known that construction of a Toffoli gate requires at least *six* two-qubit CNOT gates and *ten* single-qubit gates (i.e., two Hadamard, one phase, and seven $\pi/8$ gates) [51]. Note that a two-qubit CNOT gate consists of a two-qubit controlled phase gate and two single-qubit Hadamard gates as described above. Therefore, using the conventional gate-constructing technique, at least 28 steps of operations will be necessary to realize a Toffoli gate, assuming that the realization of a single-qubit gate or a two-qubit controlled phase gate requires one-step operation only. However, as discussed above, the present method only needs nine steps of operations. That is, seven steps of operations for the three-qubit controlled
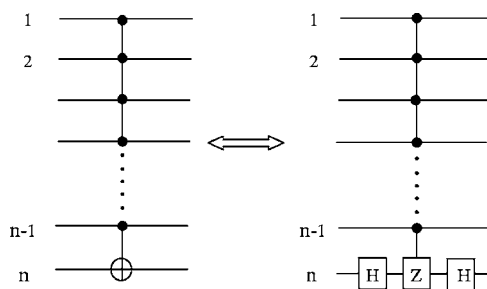

FIG. 3. Relationship between an n-qubit controlled-NOT gate and an n-qubit controlled phase gate. For the circuit on the left side, the element denoted by $\oplus$ corresponds to a controlled-NOT gate (with $n-1$ controls on the filled circles); if the $n-1$ controls are all in the state $|1\rangle$, then the state at $\oplus$ is bit flipped. On the other hand, for the circuit on the right side, the element Z represents a Pauli rotation σ_z, i.e., a phase flip operation (with $n-1$ controls on the filled circles); namely, if the $n-1$ control qubits are all in the state $|1\rangle$, then the state $|1\rangle$ at Z is phase flipped as $|1\rangle \rightarrow -|1\rangle$ while nothing happens to the state $|0\rangle$ at Z. In addition, the element containing H corresponds to a Hadamard transformation described by $|0\rangle \rightarrow (1/\sqrt{2})(|0\rangle+|1\rangle)$ while $|1\rangle \rightarrow (1/\sqrt{2})(|0\rangle-|1\rangle)$.

phase gate (8) plus two steps of operations for the two Hadamard gates. Finally, it is obvious that the simplicity of the present method in constructing an n-qubit CNOT gate may become more apparent with the increment of n, when compared with the use of the conventional gate-decomposition protocol.

IV. DISCUSSION

In this section we discuss issues that are important to experimental implementation. Without loss of generality, let us consider performing an n-qubit controlled phase gate with n identical SQUIDs $(1, 2, \ldots, n)$ at locations where the $\mathbf{B}_r$ fields are the same (e.g., antinodes of the cavity field). In this case, we have $g_l = g(l=1, 2, \ldots, n)$. For the method to work, the total operation time $\tau = 2n(\tau_r + \tau_a) + 2\tau_{\mu w}$ $[\tau_r = \pi/(2g)]$ should be much shorter than the energy relaxation time γ_2^{-1} of the level $|2\rangle$, and the lifetime of the resonator mode $\kappa^{-1} = Q/2\pi\nu_r$, where Q is the (loaded) quality factor of the resonator. Here, τ_a is the typical time required for adjusting the level spacings of a single SQUID. In addition, direct coupling between SQUIDs needs to be negligible since this interaction is not intended.

In principle, these requirements can be realized, since (i) τ_r can be reduced by increasing the coupling constant g, (ii) τ_a can be shortened by rapid adjustment of the level spacings of the SQUIDs, (iii) κ^{-1} can be increased by employing a high-Q resonator so that the resonator dissipation is negligible during the operation, (iv) the SQUIDs can be designed so that the energy relaxation time γ_2^{-1} of the level $|2\rangle$ is sufficiently long, and (v) direct interaction between SQUIDs is negligible as long as the following condition can be met:

$$H_{s\text{-}s} \ll H_{s\text{-}r}, \; H_{s\text{-}\mu w}, \tag{15}$$

where $H_{s\text{-}s}$ is the interaction energy between the two nearest SQUIDs, $H_{s\text{-}r}$ is the SQUID-resonator interaction energy,

TABLE I. Parameters for a SQUID resonator. β_L is the SQUID's potential shape parameter, R is the SQUID's effective damping resistance, and S is the surface bounded by the loop of the SQUID with width a and length b. γ_2^{-1} (γ_1^{-1}) is the energy relaxation time of the level $|2\rangle$ ($|1\rangle$). ν_{02} (ν_{12}) is the $|0\rangle \leftrightarrow |2\rangle$ ($|1\rangle \leftrightarrow |2\rangle$) transition frequency. $\phi_{ij} \equiv \langle i|\Phi|j\rangle/\Phi_0$ is the magnetic dipole coupling matrix element between levels $|i\rangle$ and $|j\rangle$ ($i=0,1;j=2$). l is the length of the quasi-one-dimensional CPW resonator, λ is the wavelength of the resonator mode with frequency, ν_r, d is the gap between the center conductor and the adjacent ground plane, w is the width of the center conductor, t is the width of each ground plane, L_0 is the inductance per unit length of the waveguide, and ε_e is the effective relative dielectric constant.

SQUID				
$C=135$ fF $L=240$ pH	$\beta_L=1.13$	$\phi_x=0.4991\Phi_0$	$R=20\ M\Omega$	$S=200\times100\ \mu m^2$
$\nu_{02}\sim11.4$ GHz $\nu_{12}\sim5.8$ GHz	$\phi_{01}=6.0\times10^{-3}$	$\phi_{02}=3.2\times10^{-2}$	$\phi_{12}=2.6\times10^{-2}$	$\gamma_2^{-1}\sim3.2\ \mu s$ $\gamma_1^{-1}\sim0.16$ ms
Resonator				
$\nu_r=11.4$ GHz	$\lambda\sim10.5$ mm	$l=1.5\lambda$	$d\sim45\ \mu m$	$w\sim20\ \mu m$
$t\gg d$	$\varepsilon_e\sim6.3$	$L_0\sim0.65$ pH/μm	$Q_r\sim3\times10^3$	$\kappa^{-1}\sim41.7$ ns

and $H_{s\text{-}\mu w}$ is the SQUID-microwave interaction energy.

It is straightforward to see that the condition (15) can be realized if

$$\zeta = \frac{ML_l^{-1}L_{l+1}^{-1}\max\{\phi_{ij}^{(l)}\phi_{ij}^{(l+1)}\}\Phi_0^2}{\min\{\hbar g_l, \hbar\Omega_{12}\}} \ll 1. \quad (16)$$

Here, M is the mutual inductance between the two nearest SQUIDs l and $l+1 (l=1,2,\ldots,n-1)$, $\phi_{ij}^{(l)}(\phi_{ij}^{(l+1)}) \equiv \langle i|\Phi|j\rangle/\Phi_0$ is the magnetic dipole coupling matrix element between levels $|i\rangle$ and $|j\rangle$ of SQUID $l(l+1)$, and $ij=01, 02$, or 12.

For the sake of definitiveness, let us consider the experimental feasibility of realizing a three-qubit controlled phase gate using SQUIDs with the parameters listed in Table I. Note that SQUIDs with these parameters are readily available at the present time [22,23,52]. With the choice of these parameters, the SQUIDs have the desired three-level structure as depicted in Fig. 1. For a superconducting one dimensional standing-wave coplanar waveguide (CPW) resonator with the parameters listed in Table I and SQUIDs placed along the cavity axis (Fig. 4), one has $M_{sr}\sim100$ pH. When each SQUID is located at one of the antinodes of the resonator mode (Fig. 4), a simple calculation shows $g\sim7.5\times10^9\ s^{-1}$, resulting in $\tau_r\approx0.2$ ns. With the choice of $\tau_{\mu w}\sim\tau_a\sim\tau_r$, one has $\tau\sim2.8$ ns, which is much smaller than
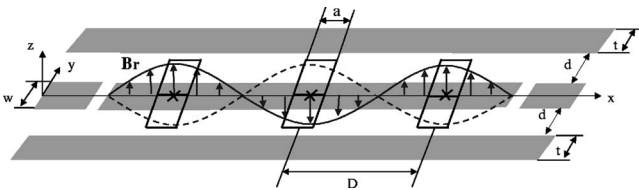

FIG. 4. Sketch of the setup for three SQUIDs (1, 2, 3) and a standing-wave quasi-one-dimensional CPW resonator (not drawn to scale). Each SQUID is placed in the plane of the resonator between the two lateral ground planes (i.e., the $x-y$ plane) and at an antinode of the $\mathbf{B}_r$ field. The two curved lines represent the standing-wave $\mathbf{B}_r$ field, which is in the z direction.

$\gamma_2^{-1}\sim3.2\ \mu s$ and $\kappa^{-1}\sim41.7$ ns for a resonator with $Q_r=3\times10^3$. Note that superconducting CPW resonators with higher quality factors have been demonstrated by recent cavity QED experiments with superconducting charge qubits [8].

For a resonator with $\nu_r=11.4$ GHz, the wavelength of the resonator mode is $\lambda\sim10.5$ mm. When each SQUID is placed at an antinode of the $\mathbf{B}_r$ field (Fig. 4), one has $D\sim5.3$ mm, where D is the distance between the two nearest SQUIDs. A simple estimate gives $M<0.1$ aH, resulting in $\zeta<10^{-8}$ for the parameters considered above. Thus, the condition of negligible direct coupling between SQUIDs is very well satisfied.

The above analysis demonstrates that the realization of a three-qubit controlled phase gate is possible using SQUIDs and a resonator. We remark that a quantum-controlled phase gate with a larger number of qubits can in principle be obtained by increasing the length of the resonator though the conditions of $\tau\ll\gamma_2^{-1},\kappa^{-1}$ becomes increasingly difficult to satisfy.

We emphasize that the primary purpose of this work is to provide a different approach to implement a multiqubit quantum-controlled phase gate with SQUIDs. However, we note that (a) when coupled to a cavity mode, many physical qubit systems (such as atoms, quantum dots, and superconducting charge qubits) have the same type of qubit-cavity interaction described by the Hamiltonian (2), and (b) the condition, i.e., the $|0\rangle\leftrightarrow|2\rangle$ transition being resonant while the transition between any other two levels is far off resonant with the resonator, can always be obtained via the adjustment of the level spacings (e.g., for quantum dots and atoms, the level spacings can be changed via adjusting the external electric field [53]). Therefore, it is straightforward to show that the method can be generalized to realize the multiqubit controlled phase gate in other types of qubit systems with the Λ-type three-level configuration within cavity QED.

V. CONCLUSION

Before concluding, we should point out that the idea of tuning the individual qubits in and out of resonance with the

cavity mode was previously proposed for superconducting charge qubits [54]. Rather, our scheme is for a different system and it differs in the details of both the qubits and the coupling structure. In our case, we consider a system consisting of flux qubits (SQUIDs) coupled to a microwave resonator and cavity, while the system described in [54] comprises charge qubits and an *LC* oscillator mode in the circuit. In addition, the idea of performing a phase shift via the assistance of the cavity photon was proposed earlier for realizing a *two-qubit* quantum-controlled phase gate with trapped ions [46]. However, to the best of our knowledge, our scheme is the first to demonstrate that a quantum-controlled phase gate *with a large number of qubits* can in principle be achieved within cavity QED, which is of great importance.

In summary, we have presented a method to realize a multiqubit controlled phase gate with SQUIDs coupled to a microwave resonator, which operates essentially by exchanging a single photon between the control SQUIDs and the resonator mode before and after a phase shift is performed on the target SQUID. The method has the following advantages. (i) Only one SQUID interacts with the microwave pulse. (ii) No auxiliary SQUID or measurement is needed during the entire operation; thus the hardware resources are significantly reduced and the operation is greatly simplified. (iii) As

tunneling between the qubit levels $|0\rangle$ and $|1\rangle$ is not required during the operation, decay from the level $|1\rangle$ can be made negligibly small during the operation (via prior adjustment of the potential barrier between the qubit levels $|0\rangle$ and $|1\rangle$ [50]) and therefore the storage time of each qubit can be made much longer. (iv) The coupling constants of SQUIDs with the resonator could be different, which makes the present proposal much easier to implement since neither identical SQUIDs nor exact placement of SQUIDs is needed. (v) The method can in principle be applied to obtain an *n*-qubit controlled phase gate with a large number *n*. Finally, as discussed above, the present method is quite general, and can be applied to implement a multiqubit-controlled phase gate for other types of physical qubit systems with the Λ-type three-level structure within cavity QED.

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