Volume of separable states is super-doubly-exponentially small in the number of qubits

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In this paper we give sharp two-sided estimates of the volume of the set of separable states on N qubits. In particular, the magnitude of the "effective radius" of that set in the sense of volume is determined up to a factor which is a (small) power of N, and thus precisely on the scale of powers of its dimension. We also identify an ellipsoid that appears to optimally approximate the set of separable states. Additionally, one of the appendixes contains sharp estimates (by known methods) for the expected values of norms of the Gaussian unitary ensemble random matrices. We employ standard tools of classical convexity, high-dimensional probability, and geometry of Banach spaces.

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I. INTRODUCTION, NOTATION, AND THE MAIN RESULT

Entanglement is thought to be one of the key resources for quantum-information processing. Its presence in various theoretical or experimental contexts as well as its indispensability for various quantum computation or communication schemes have been of significant interest. In this paper we study the presence of entanglement throughout the space of density matrices of N spin-$\frac{1}{2}$ particles (qubits). We show that if N is large, then all but extremely few (as measured by the standard volume) of such matrices are entangled; see Eq. (2) below for the precise statement. We also identify an ellipsoid that appears to optimally approximate the set of separable matrices (i.e., not entangled); this may conceivably guide experiments aiming at avoiding separable states.

Let $\mathcal{H} := (\mathbb{C}^2)^{\otimes N}$ be the N-fold tensor power of $\mathbb{C}^2$ and denote by $d=2^N$ its dimension. We will investigate the structure of the set $\mathcal{D}=\mathcal{D}(\mathcal{H})$ of states on the algebra $\mathcal{B}(\mathcal{H})$ and, in particular, of its subset $\mathcal{S}=\mathcal{S}(\mathcal{H})$ consisting of (mixtures of) *separable* states. We recall that $\mathcal{D}$ is identified with the set of density matrices $\{\rho \in \mathcal{B}(\mathcal{H}) : \rho$ is positive semidefinite and $\operatorname{tr} \rho = 1\}$ and that, in our context,

$$\mathcal{S} = \operatorname{conv}\{\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N : \rho_j \in \mathcal{D}(\mathbb{C}^2), \quad j=1,2,\ldots,N\}.$$

We emphasize that separability of a state on $\mathcal{B}(\mathcal{H})$ is not an intrinsic property of the Hilbert space $\mathcal{H}$ or the algebra $\mathcal{B}(\mathcal{H})$; it *does* depend on the particular decomposition of $\mathcal{H}$ as a tensor product of (smaller) Hilbert spaces. Here we will work with a *fixed* decomposition. We will also suppress in our notation the dependence on N: absent a mention to the contrary, it should be assumed that $\mathcal{D},\mathcal{S}$, and related sets stem from the space $(\mathbb{C}^2)^{\otimes N}$ for a fixed (but *a priori* arbitrary) N.

The question of the size of $\mathcal{S}$ and, particularly, of its relative size as a subset of $\mathcal{D}$ was raised in [1] and further investigated, among others, in [2–6] (see also the survey [7]). One of the parameters that have been studied was the maximal size of homothetic images of $\mathcal{D}$ contained in $\mathcal{S}$. More precisely, one asks for which values of ϵ (say, with $\epsilon > 0$) we have

$$\epsilon\mathcal{D} + (1-\epsilon)I_d/d \subset \mathcal{S}, \tag{1}$$

where I_d stands here for the identity matrix in d dimensions; in the present context, I_d/d is referred to as "the maximally mixed state." Alternatively, one considers inclusions of type (1) with $\mathcal{D}$ replaced by the appropriate Euclidean (Hilbert-Schmidt) ball B. The bounds obtained until recently showed that, in both cases, the optimal (largest) value of ϵ is (asymptotically, as $d\to\infty$) of order contained between d^{-1} and $d^{-3/2}$. (A recent paper [8] improves the lower bound to $d^{-1.2925}$; cf. Appendix H.) While clarifying the situation somewhat, all these results leave open the question of the precise asymptotic order of these various "in-radii" of $\mathcal{S}$ on the power scale in $\dim \mathcal{S} = \dim \mathcal{D} = d^2-1$, as well as the issue of the "size" of $\mathcal{S}$ when measured by global invariants such as volume. In the latter direction we obtain here the following bounds, which comprise the main technical result of this paper: the ratio of the volumes of the set $\mathcal{S}$ of separable states and the set $\mathcal{D}$ of all states satisfies

$$\frac{c}{d^{1/2+\alpha}} \leqslant \left(\frac{\operatorname{vol}\mathcal{S}}{\operatorname{vol}\mathcal{D}}\right)^{1/\dim \mathcal{S}} \leqslant \frac{C(\ln d \ln\ln d)^{1/2}}{d^{1/2+\alpha}}, \tag{2}$$

where $c, C > 0$ are universal (notably independent of N) effectively computable numerical constants and $\alpha := \frac{1}{8}\log_2 \frac{27}{16} \approx 0.094\,361$ (or, equivalently, $\frac{1}{2}+\alpha = \frac{3}{8}\log_2 3$, or $d^{1/2+\alpha} = 27^{N/8}$; we recall that $d = \dim \mathcal{H} = 2^N$). Similarly, the "effective radius" of $\mathcal{S}$ in the sense of volume is precisely determined on the scale of powers of d [see Eq. (9)]. In what follows we shall present the main line of the argument leading to (2), relegating to Appendixes the discussion of some peripheral issues as well as the description of results and concepts from convexity and geometry of Banach spaces that are being used. We refer to [2,9,10,6] for a more professional exposition of the relevance of separability and entanglement to quantum computation in general and to NMR computing in particular.

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The upper estimate in (2) goes well beyond the suggestions about "ubiquity of entanglement" that have been put forward in the literature. The proportion $\mathrm{vol}\,\mathcal{S}/\mathrm{vol}\,\mathcal{D}$ of states that are separable, much more than being exponentially small in N, remains exponentially small even after being raised to the power $1/\dim \mathcal{S}$. Since complexity of a set can often be estimated using volumetric methods (see [11] for a modern exposition of this circle of ideas), the inequalities (2) go a long way toward the ability to compare complexities of $\mathcal{S}$ and of $\mathcal{D}$—even though the so-called Bures metric and the related volume may be more appropriate measures of size in the present context, see [12].

For comparison, we note that the results of [2,6] implied *lower* estimates on $(\mathrm{vol}\,\mathcal{S}/\mathrm{vol}\,\mathcal{D})^{1/\dim \mathcal{D}}$ which were of order $d^{-\beta}$ with, respectively, $\beta = \ln 10/\ln 4 \approx 1.660\,964$ and $\beta=1$. By contrast, no nontrivial *upper* estimates on the volume of $\mathcal{S}$ were apparently available prior to this work, except in very low dimensions. In the opposite direction, the expression on the right-hand side of (2) yields upper estimates on the ϵ's that may work in (1) and related inclusions. For example, we obtain in this way an upper bound on the radius of a Euclidean ball that may be contained in $\mathcal{S}$, which is tighter (roughly, by a factor of d^{α}) than the usually quoted and rather elementary $O(d^{-1})$ estimate; see Eq. (9) and Appendix H for more explicit statements in this regard and for more comments. Here we will just mention that our method does not exhibit—at least without any additional work—any *explicit* state that constitutes an obstruction to the inclusion $\epsilon B + (1-\epsilon) I_d/d \subset \mathcal{S}$ for $\epsilon = o(d^{-1})$, and that our results suggest that it may be more appropriate to relate $\mathcal{S}$ to an ellipsoid which is substantially different from the one induced by the Hilbert-Schmidt norm [see the paragraph containing Eq. (10) and Appendix H]. Finally, even for questions where (2) does not improve known bounds on ϵ, it yields additional information. For example, it shows that for $\epsilon \gg (\mathrm{vol}\,\mathcal{S}/\mathrm{vol}\,\mathcal{D})^{1/\dim \mathcal{D}}$, we have $\epsilon\rho + (1-\epsilon) I_d/d \notin \mathcal{S}$ not just for some very special states ρ, but for "nearly all" $\rho \in \mathcal{D}$; this may conceivably have bearing on entanglement production in experiments.

Since it is conceivable that the inequalities (2) may be of interest not just asymptotically, but also for some specific "moderately large" values of N, we put some effort into obtaining reasonable (but certainly not optimal) values of the numerical constants. Our main argument gives $c=1/4$ and shows that (2) holds with $4[N \log_2(4N)]^{1/2} = 4[\log_2 d \log_2(4 \log_2 d)]^{1/2}$ in the numerator of its third member. A slightly more precise (and more tedious) calculation yields $c = \sqrt{e/8\pi} \approx 0.328\,87$; see the comments following (11) and Appendix E. It is also easy to follow the argument and to obtain somewhat sharper estimates for specific values of N, which may be of interest, e.g., in the context of a threshold of 23 mentioned in [6]. Such improvements are sketched in Appendix G leading to a nontrivial (i.e., <1) bound on $(\mathrm{vol}\,\mathcal{S}/\mathrm{vol}\,\mathcal{D})^{1/\dim \mathcal{S}}$ starting with $N=6$ (by contrast, $4[N \log_2(4N)]^{1/2}/d^{1/2+\alpha} < 1$ if and only if $N \geq 8$). Likewise, tighter bounds can be obtained if one is only interested in very large N; for example, one may have $c = c_N \to e^{3/4}/\sqrt{2\pi} \approx 0.844\,56$ and $C = C_N \to e^{1/4}\sqrt{2/\ln 2} \approx 2.1811$ as $N \to \infty$; see Appendix E. Finally, our methods allow analyzing separable states on tensor products involving spaces $\mathbb{C}^k$ with $k>2$, leading to nontrivial but not definitive results; some remarks to that effect are presented in Appendix I.

II. SYMMETRIZATIONS AND THE VOLUME RADII

Instead of working directly with $\mathcal{D}$ and $\mathcal{S}$, we shall consider their respective symmetrizations

$$\Delta \coloneqq \mathrm{conv}(-\mathcal{D} \cup \mathcal{D}), \quad \Sigma \coloneqq \mathrm{conv}(-\mathcal{S} \cup \mathcal{S}), \tag{3}$$

where all sets are thought of as being contained in the real d^2-dimensional vector space of self-adjoint elements of $\mathcal{B}(\mathcal{H})$ (further identifiable with $\mathcal{M}_d^{sa}$, the space of $d \times d$ complex Hermitian matrices). We do that because, first, the geometry of *symmetric* convex sets is much better understood than that of the general ones and, secondly, the specific symmetric sets Δ and Σ are familiar objects in geometry of Banach spaces, which allows us to refer to known concepts and results. In Appendix D we indicate how one can treat directly $\mathcal{D}$ and $\mathcal{S}$ without passing to symmetrizations; however, this yields only a very small improvement in the constants c, C in (2) at the price of obscuring somewhat the argument.

It is readily verified that Δ consists exactly of those (self-adjoint) elements of $\mathcal{B}(\mathcal{H})$ whose *trace class norm* is ≤ 1. Equivalently, Δ is the unit ball of the space $\mathcal{C}_1^d \coloneqq (\mathcal{M}_d^{sa}, \|\cdot\|_1)$, where, for $p \in [1,\infty)$, $\|A\|_p \coloneqq (\mathrm{tr}(A^\dagger A)^{p/2})^{1/p}$ is the Schatten–von Neumann p-norm of the matrix A. A similar argument shows that Σ is the unit ball of the Nth projective tensor power of $\mathcal{C}_1^2$ (in the sense of the Banach space theory; see Appendix B). We shall denote the corresponding norm on $\mathcal{M}_d^{sa}$ by $\|\cdot\|_\pi$. For future reference, we note that in the above notation $\|\cdot\|_\infty$ corresponds to $\|\cdot\|_{op}$, the usual norm of a matrix as an operator on the Euclidean space. Let us also point out that while in this paper we focus on ($\mathbb{R}$-linear) spaces of Hermitian matrices and self-adjoint operators, the Schatten–von Neumann classes $\mathcal{C}_p^d$ are most often defined in the literature to include all (be it real or complex) scalar matrices and not just the Hermitian ones.

The plan of the rest of the argument is as follows. First, using classical general results from convexity, we relate the volumes of Δ and Σ to those of $\mathcal{D}$ and $\mathcal{S}$. Next, we obtain two-sided estimates for $\mathrm{vol}\,\Delta$ and $\mathrm{vol}\,\Sigma$, which are most conveniently described using the following concept: if K is a subset of an n-dimensional Euclidean space with the unit ball B, we call $(\mathrm{vol}\,K/\mathrm{vol}\,B)^{1/n}$ the *volume radius* of K. (As hinted earlier, in the present context the Euclidean structure is determined by the 2-norm defined above, also often called the Hilbert-Schmidt norm or the Frobenius norm, and the inner product is $\langle u,v\rangle = \mathrm{tr}\,uv$.) Equivalently, the volume radius of K is the radius of a Euclidean ball whose volume is equal to that of K. Our approach will determine the volume radius of Σ up to a factor which is a power of $\ln d$, in particular *precisely* on the scale of powers of d; this is the principal result of the present paper. The corresponding problem for Δ, the unit ball in the trace class norm, is much better understood. Indeed, two-sided estimates for the volume radius of Δ involving a rather large (but universal, i.e., independent of N) constant follow from an early paper [13]. Moreover, explicit formulas for the volume of $\mathcal{D}$ involving multiple integrals can be produced; see [14] for an analysis of a closely related problem, which can be routinely modified to yield similar expressions for $\mathcal{D}$. After a preliminary version of the present paper was circulated, the author learned that this circle of ideas has led to a closed formula for the volume of $\mathcal{D}$ in recent work [15]; see Appendix E for more details and [16] for related results concerning the Bures volume. (Undoubtedly, formulas for the volume of Δ may be similarly obtained.) The unified argument for estimating the volume radii that is presented in this paper allows to deduce (from known facts and with very little extra work) the value of the volume radius of $\mathcal{D}$ up to a factor of 2.

For the first point, i.e., comparing the volumes of convex sets and their symmetrizations, we use a 1958 result of Rogers and Shephard [17] (see Appendix C for more details and background) to deduce that

$$\frac{2}{\sqrt{d}}\mathrm{vol}\,\mathcal{D} \leqslant \mathrm{vol}\,\Delta \leqslant \frac{2}{\sqrt{d}}\frac{2^n}{n+1}\mathrm{vol}\,\mathcal{D}, \tag{4}$$

where $n := \dim \mathcal{D} = d^2-1$. [The factor $2/\sqrt{d}$ appears because it is the distance between the hyperplanes containing $\mathcal{D}$ and $-\mathcal{D}$; note that strictly speaking we should be writing and to refer to n and $(n+1)$-dimensional volume respectively.] Similarly

$$\frac{2}{\sqrt{d}}\mathrm{vol}\,\mathcal{S} \leqslant \mathrm{vol}\,\Sigma \leqslant \frac{2}{\sqrt{d}}\frac{2^n}{n+1}\mathrm{vol}\,\mathcal{S}. \tag{5}$$

Combining Eqs. (4) and (5) we obtain

$$\left(\frac{2^n}{n+1}\right)^{-1}\frac{\mathrm{vol}\,\Sigma}{\mathrm{vol}\,\Delta} \leqslant \frac{\mathrm{vol}\,\mathcal{S}}{\mathrm{vol}\,\mathcal{D}} \leqslant \frac{2^n}{n+1}\frac{\mathrm{vol}\,\Sigma}{\mathrm{vol}\,\Delta}. \tag{6}$$

Given that the proper homogeneity is achieved by raising the volume ratios to the power $1/n$ [or $1/(n+1)$], we see that one may replace $\mathcal{D}$ and $\mathcal{S}$ in (2) by Δ and Σ with the accuracy of the estimates affected at most by a factor of 2.

It remains to estimate vol Δ and vol Σ; this will be accomplished by separately estimating their volume radii, in other words, by comparing each of these bodies with the d^2-dimensional Euclidean ball B_{HS} (the unit ball with respect to the Hilbert-Schmidt norm; we shall also denote by S_{HS} the corresponding d^2-1-dimensional sphere).

III. ESTIMATING THE VOLUME RADII: THE URYSOHN INEQUALITY AND RANDOM MATRICES

We first consider the set Δ. We claim that its volume radius satisfies

$$1/\sqrt{d} \leqslant (\mathrm{vol}\,\Delta/\mathrm{vol}\,B_{HS})^{1/d^2} \leqslant 2/\sqrt{d}. \tag{7}$$

To show this, we note first the "trivial" inclusions $B_{HS}/\sqrt{d} \subset \Delta \subset B_{HS}$, which just reflect the inequalities $\|\cdot\|_2 \leqslant \|\cdot\|_1 \leqslant \sqrt{d}\|\cdot\|_2$ between the trace class and the Hilbert-Schmidt norms. The first inclusion implies the lower estimate on the volume radius in (7). The upper bound is less obvious, but it may be shown by the following rather general argument. The first step is the classical Urysohn inequality, which in our context asserts that

$$\left(\frac{\mathrm{vol}\,\Delta}{\mathrm{vol}\,B_{HS}}\right)^{1/d^2} \leqslant \int_{S_{HS}} \|A\|_{op} dA =: \mu_d, \tag{8}$$

where the integration is performed with respect to the normalized Lebesgue measure on the Hilbert-Schmidt sphere. (For clarity and to indicate flexibility of the approach we shall present a general statement and a short proof in Appendix A.) The quantity μ_d is most easily handled by passing to an integral with respect to the standard Gaussian measure, which reduces the problem to finding expected value of the norm of the random Gaussian matrix $G=G(\omega)\in\mathcal{M}_d^{sa}$, usually called the Gaussian unitary ensemble (GUE). It is well known that $\mathbb{E}\|G\|_{op}=\gamma_{d^2}\mu_d$, where $\gamma_k := \sqrt{2}\Gamma((k+1)/2)/\Gamma(k/2)$ for $k\in\mathbb{N}$ (this equality holds for any one-homogeneous measurable function in place of the norm $\|\cdot\|_{op}$), and it is easy to check that $\sqrt{k-1}<\gamma_k<\sqrt{k}$ for all k. In other words, $\mu_d \sim \mathbb{E}\|G\|_{op}/d$ for large d. On the other hand, it is a well-known strengthening of Wigner's semicircle law that $\mathbb{E}\|G\|_{op}/\sqrt{d}\to 2$ as $d\to\infty$. This shows the second inequality in (7) with 2 replaced by $2+o(1)$. We sketch the argument that gives the exact number 2 in Appendix F (it follows from known facts, but appears to have been overlooked in the random matrix literature), yet we will not dwell on it as it intervenes only in the lower estimate in (2) and, in any case, the constants—neither in (7) nor in our final results—are not meant to be optimal. Indeed, (7) combined with (4) implies that the volume radius of $\mathcal{D}$ is between $\frac{1}{2}d^{-1/2}$ and $2d^{-1/2}$, while the formulas from [15] allow one to deduce that it is equivalent to $(1/e^{1/4})d^{-1/2}$ as $d\to\infty$.

We now pass to the analysis of the volume radius of Σ. We shall show that

$$1/d^{1+\alpha} \leqslant (\mathrm{vol}\,\Sigma/\mathrm{vol}\,B_{HS})^{1/d^2} \leqslant C\sqrt{\ln d \ln\ln d}/d^{1+\alpha}, \tag{9}$$

where α is the same as in (2). Our main result (2) follows then by combining (7), (9), and (6). [To be precise, one obtains *a priori* $1/d^2$ in the exponent, but a more careful analysis of lower-order factors such as $2/\sqrt{d}$ and $1/(n+1)$ appearing in Eqs. (4)–(6) allows one to replace d^2 by $\dim\mathcal{S}=d^2-1$ without any loss in the constants. We include a general statement to this effect in Appendix C.]

Before proceeding, let us compare (9) with the results of [6], which estimate from *below* the in-radius of $\mathcal{S}$ in the Hilbert-Schmidt metric by a quantity that is of order of $d^{-\eta}$, where $\eta=3/2$. The easy *upper* bound on that radius is the in-radius of $\mathcal{D}$, which equals $1/\sqrt{d(d-1)}=O(d^{-1})$. The second inequality in (9) yields (for large N) a better upper estimate that roughly corresponds to $\eta=1+\alpha\approx 1.094\,361$; we elaborate on these and related issues in Appendix H. (A recent paper [8] gives a *lower* bound with $\eta=\ln 6/\ln 4 \approx 1.292\,481$. By building on the approach of the present paper it is possible to show that this last exponent is optimal; the details will be reported elsewhere [18].)

IV. BALANCING THE SET Σ: THE LÖWNER ELLIPSOID

If we repeat the reasoning that led to (the right-hand side inequality in) (7) by directly substituting Σ for Δ, we will arrive at the Gaussian expectation of the norm in the *injective* (see Appendix B) tensor power of the space $\mathcal{C}_\infty^2$ and end up with an upper estimate for the volume radius of Σ containing d in the denominator [as opposed to $d^{1+\alpha}$ asserted in (9)]. To make the argument more optimal it is necessary to replace the sets Σ by their affine images which are more "balanced." This original "lack of balance" is responsible for the appearance of the mysterious number α in the exponents.

Consider first the sets in question when $N=1$. Then $\mathcal{S}(\mathbb{C}^2)$ and $\mathcal{D}(\mathbb{C}^2)$ (trivially) coincide. As is well known, $\mathcal{D}(\mathbb{C}^2)$ is the "Bloch ball," which geometrically is a (solid) Euclidean ball of radius $1/\sqrt{2}$ centered at $I_2/2$. Its boundary is the Bloch sphere $\mathcal{T}=\mathcal{T}(\mathbb{C}^2)$, consisting exactly of pure states on $\mathcal{B}(\mathbb{C}^2)$ (further identifiable with rank-1 projections on $\mathbb{C}^2$). Consequently, $\Sigma(\mathbb{C}^2)=\Delta(\mathbb{C}^2)$ is a four-dimensional cylinder whose base is the Bloch ball and whose axis is the segment $[-I_2/2,I_2/2]$ of Euclidean length $\sqrt{2}$. For definiteness, let us identify $\mathcal{M}_2^{sa}$ with $\mathbb{R}^4$ via the usual basis $\{I_2/\sqrt{2},\sigma_x/\sqrt{2},\sigma_y/\sqrt{2},\sigma_z/\sqrt{2}\}$, where σ_x, σ_y, and σ_z are the Pauli matrices (the factors $1/\sqrt{2}$ make this basis orthonormal in the Hilbert-Schmidt sense). Let now A be a linear map on $\mathcal{M}_2^{sa}$ which is diagonal in that basis and whose action is defined by $AI_2=I_2/\sqrt{2}$, $A\sigma_i=\sqrt{3/2}\sigma_i$ for $i=x,y,z$. Set $\tilde{\Sigma}=\tilde{\Sigma}(\mathbb{C}^2):=A\Sigma$; the important properties of A and $\tilde{\Sigma}$ are as follows.

(i) The image of the Bloch sphere $A\mathcal{T}=:\tilde{\mathcal{T}}=\tilde{\mathcal{T}}(\mathbb{C}^2)$ is geometrically a two-dimensional sphere of radius $\sqrt{3}/2$ centered at $I_2/\sqrt{8}$ and, as the Bloch sphere itself, it is contained in the *unit* Euclidean sphere S_{HS}; this implies that $\tilde{\Sigma}\subset B_{HS}$.

(ii) $\det A=\sqrt{27/16}$ and so $\mathrm{vol}\,\tilde{\Sigma}=\sqrt{27/16}\mathrm{vol}\,\Sigma$.

(iii) Vertices of any regular tetrahedron inscribed in $\tilde{\mathcal{T}}$ form an orthonormal basis in $\mathcal{M}_2^{sa}$.

The geometric property of the set $\tilde{\Sigma}$, which arguably is the reason for its relevance, is that the ellipsoid of smallest volume containing it (the so-called Löwner ellipsoid of $\tilde{\Sigma}$) is the Euclidean ball. An equivalent and perhaps more natural point of view would be to compare Σ with its own Löwner ellipsoid. This is in turn equivalent to replacing the Hilbert-Schmidt inner product $\langle u,v\rangle=\mathrm{tr}\,uv$ with

$$\mathrm{tr}[(Au)(Av)] = (3\mathrm{tr}\,uv - \mathrm{tr}\,u\mathrm{tr}\,v)/2. \qquad (10)$$

It is likely that this nonisotropic inner product, its tensor powers, and objects associated with them play an important role in the theory. In particular, we obtain this way ellipsoids which, for large N, are essentially equivalent—from the volumetric point of view—to $\mathcal{S}$ or Σ, and which still enjoy certain permanence relations with respect to the action of the unitary group. (See more on this in Appendix H).

If $N>1$, we set $\tilde{\Sigma}=\tilde{\Sigma}((\mathbb{C}^2)^{\otimes N}):=A^{\otimes N}\Sigma$. Since $\det A^{\otimes N}=(\det A)^{N\cdot 4^{N-1}}=[(27/16)^{N/8}]^{d^2}=(2^{\alpha N})^{d^2}=(d^\alpha)^{d^2}$, we deduce that (9) is equivalent to

$$1/d \leq (\mathrm{vol}\,\tilde{\Sigma}/\mathrm{vol}\,B_{HS})^{1/d^2} \leq C\sqrt{\ln d \ln\ln d}/d. \qquad (11)$$

For the lower estimate in (11) we shall produce a simple (and, at the first sight, not very optimal) geometric argument. Let u_1,u_2,u_3,u_4 be vertices of any regular tetrahedron inscribed in $\tilde{\mathcal{T}}(\mathbb{C}^2)$. By the property (iii) above, $(u_j)_{j=1}^4$ is an orthonormal basis of $\mathcal{M}_2^{sa}$. Accordingly, the set $\tilde{U}:=\{u_{j_1}\otimes u_{j_2}\otimes\cdots\otimes u_{j_N}\}$, where each j_i ranges over $\{1,2,3,4\}$, is an orthonormal basis of $\mathcal{M}_d^{sa}$. The first inequality in (11) follows now from the inclusions $\tilde{U},-\tilde{U}\subset\tilde{\Sigma}$, and $B_{HS}/d\subset\mathrm{conv}(-\tilde{U}\cup\tilde{U})$ (the latter is a consequence of the orthogonality of elements of $\tilde{U}$).

The above argument may appear rather *ad hoc*, and so it may be instructive to rephrase it in the language of geometry of Banach spaces. Let A_1 be a linear map from $\mathbb{R}^4$ to $\mathcal{M}_2^{sa}$ which sends the standard unit vector basis onto vertices of any regular tetrahedron inscribed in $\mathcal{T}(\mathbb{C}^2)$. By construction, A_1 is a contraction from ℓ_1^4 to $\mathcal{C}_1^2$ and so its Nth tensor power $A_1^{\otimes N}$ induces a contraction between the respective projective tensor powers of ℓ_1^4 and $\mathcal{C}_1^2$ (where ℓ_1^k denotes $\mathbb{R}^k$ endowed with the norm $\|(x_j)\|=\Sigma|x_j|$). As the projective tensor product of ℓ_1-spaces is again an ℓ_1-space, it follows that $\Sigma((\mathbb{C}^2)^{\otimes N})$ contains the image under $A_1^{\otimes N}$ of the unit ball of $\ell_1^{d^2}$, and hence the image of the Euclidean ball of radius $1/d$. In particular, $\mathrm{vol}\,\Sigma/\mathrm{vol}(A_1^{\otimes N}B_{HS})\geq(1/d)^{d^2}$. On the other hand, one verifies [directly, or by noticing that $A=|A_1^{-1}|=(A_1^{-1\dagger}A_1^{-1})^{1/2}$] that $\mathrm{vol}(A_1^{\otimes N}B_{HS})=[(16/27)^{N/8}]^{d^2}\mathrm{vol}\,B_{HS}$, which substituted into the preceding estimate gives exactly the first inequality in (11).

We note in passing that using for the lower bound on the volume of Σ the (larger) volume of the image of the $\ell_1^{d^2}$ ball $[\mathrm{conv}(-\tilde{U}\cup\tilde{U})]$ would only result in a slightly better constant c in (2) (specifically, the value $c=\sqrt{e/8\pi}$ that was mentioned earlier). This is because the volume radius of the unit ball in ℓ_1^m is roughly the same as that of the inscribed Euclidean ball, the ratio between the two is $\sqrt{2e/\pi}[1-O(1/m)]$. This property is behind many striking phenomena discovered in the asymptotic theory of finite dimensional normed spaces, and is closely related to our upper estimates for vol Σ and vol $\mathcal{S}$, to which we pass now.

V. MAJORIZING THE VOLUME OF Σ AND SUPREMA OF GAUSSIAN PROCESSES

To prove the upper estimate in (11), we shall again use the Urysohn inequality. Analogously to (8) and to the reasoning that followed it [see also (A1)], we get

$$\left(\frac{\mathrm{vol}\,\tilde{\Sigma}}{\mathrm{vol}\,B_{HS}}\right)^{1/d^2} \leq \int_{S_{HS}} \max_{X\in\tilde{\Sigma}} \mathrm{tr}(XA)dA = \gamma_{d^2}^{-1}\mathbb{E}\max_{X\in\tilde{\Sigma}}\mathrm{tr}(XG) \qquad (12)$$

and so it remains to show that the above expectation is $O(\sqrt{\ln d \ln\ln d})$. The expression under the expectation can be thought of as a maximum of a Gaussian process indexed by

$\tilde{\Sigma}$ (this just means the family $\{\mathrm{tr}[XG(\omega)]:X\in\tilde{\Sigma}\}$ of jointly Gaussian random variables). There are several methods of differing sophistication which can be used to estimate the expectation of such a maximum. The two leading ones are the Fernique-Talagrand majorizing measure theorem, which gives the correct asymptotic order, but is usually difficult to apply, and the Dudley majoration (by the metric entropy integral), which is almost as precise and usually easier to handle; see [19] for a comprehensive exposition. We shall employ here an even simpler "one-level-discretization" method which, in our context, yields approximately the same result as the Dudley majoration, and which we now describe in elementary language.

Let μ be the standard Gaussian measure on $\mathbb{R}^m$ [i.e., the one given by the density $(2\pi)^{-m/2}\exp(-|x|^2/2)$, where $|\cdot|$ is the corresponding Euclidean norm] and let $F\subset\mathbb{R}^m$ be a finite set contained in a ball of radius R. Then

$$\int_{\mathbb{R}^m}\max_{y\in F}\langle y,x\rangle d\mu(x)\leqslant R\sqrt{2\ln(\mathcal{C}F)},\tag{13}$$

where $\mathcal{C}$ stands for the cardinality of a set. [The estimate above is usually quoted with a different numerical constant appearing in place of 2; see [20], Proposition 1.1.3., for an elegant proof of the present version.] The idea now is to construct a finite set $\mathcal{F}\subset\tilde{\Sigma}$ such that $\mathrm{conv}\,\mathcal{F}\supset r\tilde{\Sigma}$ for an appropriate $r\in(0,1)$; it will then follow that

$$\mathbb{E}\max_{X\in\tilde{\Sigma}}\mathrm{tr}(XG)\leqslant r^{-1}\sqrt{2\ln(\mathcal{C}\mathcal{F})}.\tag{14}$$

[We note that the maxima of the type appearing in (12), (13), or (14) do not change if we replace the underlying (closed) set F by its convex hull or, conversely, by its extreme points.] Specifically, F will be a "sufficiently dense" subset of the set of extreme points of $\tilde{\Sigma}$, i.e., of $-\tilde{\mathrm{T}}\cup\tilde{\mathcal{T}}$, where

$$\tilde{\mathcal{T}}=\tilde{\mathcal{T}}((\mathbb{C}^2)^{\otimes N}):=\{A\rho_1\otimes A\rho_2\otimes\cdots\otimes A\rho_N\}$$

and where each ρ_j is a pure state on $\mathcal{B}(\mathbb{C}^2)$ (i.e., an element of the Bloch sphere). In other words, $\tilde{\mathcal{T}}$ is a tensor product of N copies of $\tilde{\mathcal{T}}(\mathbb{C}^2)=A\mathcal{T}(\mathbb{C}^2)$ which, as we noted earlier, is geometrically a two-dimensional sphere of radius $\sqrt{3}/2$ contained in the unit sphere of the four-dimensional Euclidean space.

We start by constructing an appropriate dense subset (usually called a *net*) in each copy of $\tilde{\mathcal{T}}(\mathbb{C}^2)$ and then consider tensor products of those nets. To facilitate references to existing literature we first look at the *unit* Euclidean ball S^2 rather than $\tilde{\mathcal{T}}(\mathbb{C}^2)$. Let $\delta\in(0,\sqrt{2})$ and let $\mathcal{N}$ be a δ-net of S^2, i.e., a subset such that the union of balls of radius δ centered at points of $\mathcal{N}$ covers S^2. An elementary argument shows that $\mathrm{conv}\,\mathcal{N}$ contains then a ball of radius $(1-\delta^2/2)$ centered at the origin. If now $\tilde{\mathcal{N}}\subset\tilde{\mathcal{T}}(\mathbb{C}^2)$ is the appropriate dilation of $\mathcal{N}$ (i.e., with the ratio $\sqrt{3}/2$), then $\mathrm{conv}\,\tilde{\mathcal{N}}$ contains a ball of radius $(1-\delta^2/2)\sqrt{3}/2$ [in the three-dimensional affine space containing $\tilde{\mathcal{T}}(\mathbb{C}^2)$] with the same center as that of $\tilde{\mathcal{T}}(\mathbb{C}^2)$. It follows that $\mathrm{conv}(-\tilde{\mathcal{N}}\cup\tilde{\mathcal{N}})\supset(1-\delta^2/2)\tilde{\Sigma}(\mathbb{C}^2)$ and, consequently, if we set $\mathcal{F}:=(-\tilde{\mathcal{N}}\cup\tilde{\mathcal{N}})^{\otimes N}=\tilde{\mathcal{N}}^{\otimes N}\cup(-\tilde{\mathcal{N}}^{\otimes N})$, then $\mathrm{conv}\,\mathcal{F}\supset(1-\delta^2/2)^N\tilde{\Sigma}((\mathbb{C}^2)^{\otimes N})$.

To be able to apply (14), it remains to find a reasonable bound on $\mathcal{C}\mathcal{F}=2(\mathcal{C}\tilde{\mathcal{N}})^N=2(\mathcal{C}\mathcal{N})^N$. A standard argument comparing areas of caps and that of the entire sphere shows that one may have a δ-net of S^2 of cardinality $<16/\delta^2$. [This bound is by far not optimal; the asymptotically—as $\delta\to0$—correct order for cardinalities of efficient δ-nets of S^2 is $(2/\sqrt{3})^3\pi/\delta^2$, but we could not find an easy reference with a substantially better formula; cf. [21,22].] This leads to an estimate $\mathcal{C}\mathcal{F}<2(16/\delta^2)^N$, which in combination with (14) gives

$$\mathbb{E}\max_{X\in\tilde{\Sigma}}\mathrm{tr}(XG)\leqslant(1-\delta^2/2)^{-N}\sqrt{2\ln[2(16/\delta^2)^N]}.\tag{15}$$

Optimizing the expression on the right-hand side over $\delta\in(0,\sqrt{2})$ yields a quantity that is of order $\sqrt{2N\ln N}$ for large N [choose, for example, $\delta=(N\ln 2N)^{-1/2}$], as required to complete the proof of (9) [and hence of (2)]. Moreover, substituting the obtained bound into (12) and verifying numerically small values of N yields

$$\left(\frac{\mathrm{vol}\,\tilde{\Sigma}}{\mathrm{vol}\,B_{HS}}\right)^{1/d^2}\leqslant\frac{\sqrt{4N\log_2(4N)}}{d}=\frac{\sqrt{4\log_2 d\log_2(4\log_2 d)}}{d}\tag{16}$$

(note that the inequality is trivial for $N=2$), which implies that (2) holds with the third member of the form $4\sqrt{N\log_2(4N)}/d^{1/2+\alpha}$. This may be somewhat improved for small to moderate values of N by using estimates on cardinalities of nets of S^2 listed in [22]; see Appendix G.

VI. CONCLUSIONS

We considered the set of separable states on N qubits and determined its effective radius in the sense of volume precisely on the scale of powers of $d=2^N$ (the dimension of the underlying Hilbert space). In particular, we have found that, as measured by the usual Euclidean volume, the proportion of the set of all states occupied by separable states is, up to factors of lower order, $3^{-3Nd^2/8}$ or $d^{-(1/2+\alpha)d^2}$, where $\alpha=\frac{1}{8}\log_2\frac{27}{16}\approx0.094\,361$. This implies an upper bound of roughly $d^{-1-\alpha}$ on the radius of a Euclidean (or Hilbert-Schmidt) ball contained in the set of separable states. We also identify a nonstandard scalar product on the space of Hermitian matrices, which is particularly well adapted to the study of the set of separable states: ellipsoids related to that product appear to optimally approximate the set of separable states (see Appendix H). This may conceivably guide experiments aiming at producing highly entangled states: generate density matrices by some random procedure which favors those matrices whose "nonstandard" inner product norm is large. Our approach combines standard techniques of discrete geometry, classical convexity, geometry of Banach spaces, random matrix theory and theory of Gaussian processes. We include a review of some of these techniques and sketch their several additional applications in the Appendixes.

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APPENDIX A: THE URYSOHN INEQUALITY

If K is a convex body in the m-dimensional Euclidean space which contains 0 in its interior, then

$$\left(\frac{\mathrm{vol}\,K}{\mathrm{vol}\,B}\right)^{1/m} = \left(\int_{S^{m-1}} \|x\|_K^{-m} dx\right)^{1/m} \geqslant \int_{S^{m-1}} \|x\|_K^{-1} dx \geqslant \left(\int_{S^{m-1}} \|x\|_K dx\right)^{-1},$$

where $B=B^m$ is the Euclidean ball, $\|x\|_K$ is the gauge of K (the norm for which K is the unit ball if K is 0 symmetric—which is the case in the main text) and the integration is performed with respect to the normalized Lebesgue measure on the sphere S^{m-1}. If K is 0-symmetric, this may be combined with the Santaló inequality [23] which asserts that then

$$\frac{\mathrm{vol}\,K}{\mathrm{vol}\,B}\frac{\mathrm{vol}\,K^\circ}{\mathrm{vol}\,B} \leqslant 1,$$

where $K^\circ := \{x : \langle x, y\rangle \leqslant 1 \text{ for all } y \in K\}$ is the polar body of K, to obtain $(\mathrm{vol}\,K^\circ/\mathrm{vol}\,B)^{1/m} \leqslant \int_{S^{m-1}} \|x\|_K dx$, and the Urysohn inequality

$$(\mathrm{vol}\,K/\mathrm{vol}\,B)^{1/m} \leqslant \int_{S^{m-1}} \|x\|_{K^\circ} dx = \int_{S^{m-1}} \max_{y\in K}\langle x, y\rangle dx \tag{A1}$$

follows by exchanging the roles of K and K°. Moreover, since the Santaló inequality holds also for not-necessarily-symmetric sets after an appropriate translation, the above inequality holds for any (measurable) bounded set K. Indeed, the integral on the right equals $1/2$ of the *mean width* of K, a well-known classical geometric parameter of a set in the Euclidean space, which does not change if K is replaced by its translation. It is primarily the mean widths of various sets—and not directly volume—that are being majorized throughout this paper.

This is not the most elementary proof of the Urysohn inequality, but one that offers a lot of flexibility. For example, the repeated applications of the Hölder inequality in the first chain of inequalities above can be modified to yield as the last expression $(\int_{S^{m-1}} \|x\|_K^p dx)^{-1/p}$ for an arbitrary $p>0$ and, letting $p\to 0$, the geometric mean $\exp(-\int_{S^{m-1}} \ln\|x\|_K dx)$. Similar inequalities also hold if $p\in[-n,0)$, and the case $p=-n$ is of course the strongest statement of such nature, the Santaló inequality itself. We also take this opportunity to point out that the Santaló inequality and the so-called reverse Santaló inequality [24] together imply that the volume radius of a convex set and its polar are roughly (i.e., up to universal multiplicative constants) reciprocal.

The application of the Urysohn inequality in (8) uses implicitly the elementary fact that, for any Hermitian matrix A,

$$\|A\|_{\Delta^\circ} = \max_{Y\in\Delta} \mathrm{tr}\, AY = \max_{\rho\in\mathcal{D}} |\mathrm{tr}\, A\rho| = \|A\|_{op}. \tag{A2}$$

This is just a restatement of the fact that the trace class norm $\|\cdot\|_1$ and the operator norm $\|\cdot\|_{op}$ are dual with respect to the trace duality. For the sets $\Delta(\mathbb{C}^k)$, the Urysohn inequality gives the correct order of the volume radius, cf. (7). However, this is not always the case, even for rather regular convex bodies. For example, if K is the unit ball of ℓ_1^m, then its volume radius—as we have already mentioned—exceeds the radius of the inscribed Euclidean ball by less than $\sqrt{2e/\pi}$, while the upper bound obtained from the Urysohn inequality contains a parasitic factor which is of order $\sqrt{\ln m}$. It is thus conceivable that the logarithmic factors in (2) can be replaced by universal numerical constants. On the other hand, if we *do* use the Urysohn inequality to establish an upper bound for the volume radius of $\tilde{\Sigma}$ [cf. (11), (12), and (16)], then our estimates cannot be substantially improved. Indeed, since our argument showed that $\tilde{\Sigma}$ contained a rotation of the unit ball of $\ell_1^{d^2}$, the previous remark implies that the $\sqrt{N}=\sqrt{\log_2 d}$ factor cannot then be avoided, and so it is only the $\sqrt{\ln\ln d}$ factor that can possibly be eliminated by more careful majorizing of $\mathbb{E}\|G\|_{\tilde{\Sigma}^\circ}$.

Finally, we mention that there exist general volume estimates for convex hulls of finite sets which are asymptotically more precise than the one we derive from the Urysohn inequality (see [25] and its references). However, these estimates are equivalent to the ones presented here in the relevant range of parameters and, moreover, their formulations available in the literature contain unspecified numerical constants, which would make the corresponding bounds difficult to apply for specific values of N.

APPENDIX B: PROJECTIVE TENSOR PRODUCTS OF NORMED SPACES

If X and Y are (say, finite-dimensional) normed spaces, their tensor product $X\otimes Y$ may be endowed with the projective tensor product norm $\|\cdot\|_\pi$ defined by

$$\|\tau\|_\pi := \inf\left\{\sum_{j=1}^m \|x_j\|\cdot\|y_j\| : \sum_{j=1}^m x_j\otimes y_j = \tau\right\}.$$

The resulting normed space is usually denoted $X\hat{\otimes}Y$ or $X\otimes_\pi Y$. If B_X and B_Y are unit balls of X and Y respectively, it follows that the unit ball of $X\hat{\otimes}Y$ coincides with

$$\mathrm{conv}\{x\otimes y : x\in B_X, y\in B_Y\} = \mathrm{conv}\{x\otimes y : x\in \mathrm{ext}\,B_X, y\in \mathrm{ext}\,B_Y\},$$

where $\mathrm{ext}(K)$ denotes the set of extreme points of K. If

$X=Y=\mathcal{C}_1^2$, the analysis is further simplified by the fact that the set of extreme points of $\Delta(\mathbb{C}^2)$, the unit ball of $\mathcal{C}_1^2$, is of the form $-\mathcal{T}\cup\mathcal{T}$, where $\mathcal{T}$ is the set of pure states on $\mathcal{B}(\mathbb{C}^2)$. The fact that the set $\Sigma((\mathbb{C}^2)^{\otimes 2})$ is the unit ball of $\mathcal{C}_1^2\hat{\otimes}\mathcal{C}_1^2$ follows directly from these identifications. Projective tensor products of more than two spaces are defined analogously (or by induction), and one similarly checks that the unit ball of the Nth projective tensor power of $\mathcal{C}_1^2$ is $\Sigma((\mathbb{C}^2)^{\otimes N})$. Tensor products involving spaces $\mathbb{C}^k$ with $k>2$ may be treated in the same way. For example, the symmetrization of the set of separable states on $\mathcal{B}(\mathbb{C}^{k_1}\otimes\mathbb{C}^{k_2}\otimes\cdots\otimes\mathbb{C}^{k_m})$ can be identified with the unit ball in $\mathcal{C}_1^{k_1}\hat{\otimes}\mathcal{C}_1^{k_2}\hat{\otimes}\cdots\hat{\otimes}\mathcal{C}_1^{k_m}$. The problem of the relative size of the set of separable states on $\mathcal{B}((\mathbb{C}^D)^{\otimes N})$, or N qudits, was investigated in [5]. While a more definitive treatment of the higher-dimensional case will be presented elsewhere [18], we offer some comments on the topic in Appendix I.

On the more elementary level, the projective tensor square of a Euclidean space $\mathbb{C}^k\hat{\otimes}\mathbb{C}^k$ can be identified with $(\mathcal{M}_k,\|\cdot\|_1)$, and the contractively complemented subspace of its Hermitian elements is, in our notation, $\mathcal{C}_1^k$. We also mention in passing that the dual space to $X\hat{\otimes}Y$ can be identified with the so-called *injective* tensor product of the duals X^* and Y^*, and so Σ° can be thought of as a unit ball in the injective tensor power of the self-adjoint part of $\mathcal{B}(\mathbb{C}^2)$ (i.e., of the space of 2×2 Hermitian matrices endowed with the operator norm). While making this identification explicit does not seem to help our analysis at the present level of depth, we mention in passing that various existing criteria for detecting entanglement use separation theorems for convex sets which are based on a form of this duality.

APPENDIX C: THE ROGERS-SHEPHARD RESULTS ON SYMMETRIZATIONS OF CONVEX SETS

Let $W\subset\mathbb{R}^{n+1}$ be an n-dimensional convex set and denote by h the distance from the affine hyperplane H spanned by W to the origin. Let Ω be the symmetrization of W, i.e., $\Omega:=\mathrm{conv}(-W\cup W)$. It was shown in [17] that then

$$2h\,\mathrm{vol}\,W\leq\mathrm{vol}\,\Omega\leq 2h\frac{2^n}{n+1}\mathrm{vol}\,W, \tag{C1}$$

whereby vol W and vol Ω we mean the n- and the $(n+1)$-dimensional volume respectively. To explain the factors appearing in (C1) we note that the inequalities become equalities if W is centrally symmetric for the first one (this is simple) and if W is a simplex for the second (this is the heart of the Rogers-Shephard result).

To further clarify the first inequality in (C1) (used in the upper estimates on the volume of separable states, which is the main point of this paper) we point out that it is actually a simple consequence of a much older theorem of Brunn-Minkowski, and more specifically of the following corollary of that theorem.

Let K be an $(n+1)$-dimensional convex body, u a vector in the ambient space containing K, and H a hyperplane in that space. Then the function $t\to\mathrm{vol}[K\cap(tu+H)]^{1/n}$ (the n-dimensional volume) is concave on its support.

If we apply the above fact with $K=\Omega$ and u a unit vector perpendicular to H, then the function $\phi(t):=\mathrm{vol}[\Omega\cap(tu+H)]$, being even on $[-h,h]$, must attain its maximum at 0 and minimum at h and $-h$. The first inequality in (C1) follows then from the Cavalieri principle. A version of the second inequality, which would be sufficient for our purposes, follows similarly from the estimate $\phi(0)\leq 2^{-n}\binom{2n}{n}\mathrm{vol}\,W$, which is the main result of [26].

We used the inequality (C1) to conclude that the volume radii of the sets $\mathcal{S}$ and $\mathcal{D}$ differ from those of their respective symmetrizations at most by a factor of 2. This actually requires some care since the volume radii of a set and its symmetrization are calculated by comparing with Euclidean balls of different dimensions, and then raising to different powers. However, the "equivalence up to a factor of 2" does hold unless the body Ω is rather unbalanced. In the notation of (C1) and denoting $\kappa_n:=\mathrm{vol}\,B^n/\mathrm{vol}\,B^{n+1}$ we have (i) if $(\mathrm{vol}\,\Omega/\mathrm{vol}\,B^{n+1})^{1/(n+1)}\leq a$ and $a/h\leq 2\kappa_n$, then $(\mathrm{vol}\,W/\mathrm{vol}\,B^n)^{1/n}\leq a$; (ii) if $(\mathrm{vol}\,\Omega/\mathrm{vol}\,B^{n+1})^{1/(n+1)}\geq b$ and $h/b\leq(n+1)/2\kappa_n$, then $(\mathrm{vol}\,W/\mathrm{vol}\,B^n)^{1/n}\geq b/2$.

To clarify the conditions in (i) and (ii) we note that a good approximation for $\kappa_n=\pi^{-1/2}\Gamma((n+1)/2+1)/\Gamma((n/2)+1)$ is $\sqrt{(n+\frac{3}{2})/2\pi}$, and that the conditions (i) and (ii) are satisfied if, respectively, $a/h\leq\sqrt{2(n+1)/\pi}$ and $h/b\leq\sqrt{\pi n/2}$. Thus (i) and (ii) apply as long as, roughly, the ratio between the volume radii involved and h stays between $1/\sqrt{n}$ and $\sqrt{n}$.

APPENDIX D: WORKING DIRECTLY WITH NONSYMMETRIC SETS

Similar but slightly more complicated arguments may be used to obtain *upper* estimates for the volumes of the nonsymmetric sets $\mathcal{D}$ and $\mathcal{S}$ by studying directly these sets and not their symmetrizations Σ and Δ. [In principle, this could help to avoid the parasitic factors $2^n/(n+1)$—where $n=d^2-1$—when passing from Σ,Δ to $\mathcal{S},\mathcal{D}$.] In both cases it is convenient to pass to a translate of the set in question obtained by subtracting the appropriate multiple of the maximally mixed state I_d/d, and to consider the translates as subsets of H_0, the (d^2-1)-dimensional space of Hermitian matrices with vanishing trace.

For the set $\mathcal{D}$ (translated by I_d/d), the quantity which replaces $\|\cdot\|_{op}$ in the analog of (8) is $\lambda_1(\cdot)$, the largest eigenvalue of a matrix. [This is because in the analog of (A2) we need to consider $\max_{\rho\in\mathcal{D}}\mathrm{tr}\,A\rho$ rather than $\max_{\rho\in\mathcal{D}}|\mathrm{tr}\,A\rho|$.] The largest eigenvalue is of course dominated by the norm, and since the (random Gaussian) trace 0 matrix G_0 can be represented as a conditional expectation of the general Gaussian matrix G, it follows—by the convexity of the norm or of the largest eigenvalue—that $\mathbb{E}\lambda_1(G_0)\leq\mathbb{E}\|G\|_{op}\leq 2\sqrt{d}$ which, after some work, leads to an upper estimate for the volume radius of $\mathcal{D}$ identical to that of Δ obtained in (7). (To fully justify the steps above one needs to appeal to Appendixes A and F. Also, the concentration of measure phenomenon implies that there is practically no loss when replacing $\mathbb{E}\lambda_1(\cdot)$ by $\mathbb{E}\|\cdot\|_{\mathrm{op}}$.)

For the set $\mathcal{S}$, we pass first to the face $\tilde{\mathcal{S}}$ of the rescaled set $\tilde{\Sigma}$ that corresponds to $\mathcal{S}$, and then subtract $I_d/d^{3/2}$ (the difference with respect to the case of $\mathcal{D}$ is due to the rescaling). Next, we "approximate" the translate by sets built from the points corresponding to elements of $\mathcal{F}$. There are several differences between this setting and that described in the main text, but they can be accounted for fairly easily and, moreover, they are negligible for large N. We now briefly describe the necessary modifications.

The good news is that the new points are not on the *unit* sphere since the component in the direction of I_d was subtracted, but this improves our estimate on vol S only by a factor $1-O(d^{-1})$. A somewhat more substantial *loss* comes from the fact that—due to the rescaling—the width of $\tilde{\Sigma}$ in the direction of I_d is different from that of Σ by a factor of $2^{N/2}=d^{1/2}$; this affects the relationships between volumes of these bodies and those of $\tilde{\mathcal{S}}$ and $\mathcal{S}$, and consequently our estimates, by the same factor. However, since we are in dimension d^2-1, the loss in the *volume radius* is a not so significant factor $1+O(\ln d/d)$. The next issue that needs to be analyzed is that while we knew that conv $\mathcal{F}\supset(1-\delta^2/2)^N\tilde{\Sigma}$, it is not *a priori* clear that a similar inclusion holds for the face $\tilde{\mathcal{S}}$ (or, more precisely, for its translate $\tilde{\mathcal{S}}-I_d/d^{3/2}$). While for a general convex set $K\subset H_1$ the relationship between K and its symmetrization conv$(-K\cup K)$ may be more involved, using the fact that $\tilde{\mathcal{S}}$ is a convex hull of points contained in a *sphere* centered at $I_d/d^{3/2}$ we can infer that conv$\mathcal{F}\supset(1-\kappa)\tilde{\Sigma}$ implies conv$(\mathcal{F}\cap\tilde{\mathcal{S}})\supset I_d/d^{3/2}+(1-2\kappa)(\tilde{\mathcal{S}}-I_d/d^{3/2})$, and the difference between the factors $1-\kappa$ and $1-2\kappa$ is asymptotically insignificant. Similarly insignificant is replacing in the estimates $\mathcal{CF}$ by that of its "positive half." Finally, as was explained at the end of Appendix C, the reduction of the dimension of the problem (from d^2 to d^2-1) has no bearing on the estimates.

While the above argument allows to avoid symmetrizations while estimating the volume radii of $\mathcal{D}$ and $\mathcal{S}$ from above, we still get (essentially) the same majorants as in (7) and (9) A lower bound on the volume radius of $\mathcal{S}$ can be obtained by noticing that $\mathcal{S}$ contains a simplex spanned by the elements of $(A^{\otimes N})^{-1}\tilde{U}$. However, since for the simplex the relevant Rogers-Shephard inequalities become equalities, there is again no significant improvement. Finally, while there are various direct ways to establish lower bounds on the volume radius of $\mathcal{D}$ [also needed to derive (2)], the approach via symmetrizations appears to be by far the simplest.

APPENDIX E: THE EXACT EXPRESSIONS ON THE VOLUME OF $\mathcal{D}$ AND THE CONSTANTS IN (2) AS $N\to\infty$

Recently, the author learned that a closed formula for the volume of $\mathcal{D}$ was derived in a very recent work [15]. While we were able to calculate the volume radius of $\mathcal{D}$ to within a factor of 2 by the same methods that were employed to analyze $\mathcal{S}$ and with very little extra work, it is instructive to compare the so obtained estimates to those that can be deduced from the exact formula, which in our notation reads

$$\text{vol}\,\mathcal{D}(\mathbb{C}^d)=\sqrt{d}(2\pi)^{d(d-1)/2}\frac{\prod_{j=1}^{d}\Gamma(j)}{\Gamma(d^2)}. \qquad \text{(E1)}$$

A tedious but routine calculation based on the Stirling formula shows that the volume radius of $\mathcal{D}(\mathbb{C}^d)$ behaves as $(1/e^{1/4})d^{-1/2}[1+O(d^{-1})]$ as $d\to\infty$.

We now recall the refinements related to the volume radii of $\mathcal{S}$ and Σ suggested in the main text. First, we had the argument that gave $c=\sqrt{e/8\pi}$ in (2), based on using the volume of the unit ball in $\ell_1^{d^2}$, i.e., $2^{d^2}/(d^2)!$, as a lower bound for vol $\tilde{\Sigma}$ [see the comments following (11)]. Next, we noted that, for large N, the expressions in (15) can be majorized by a quantity that is of order $\sqrt{2N\ln N}$. Combining these with the improvement related to $\mathcal{D}$ we are led to an asymptotic version of (2) with $c_N\to e^{3/4}/\sqrt{2\pi}\approx0.844\,561$ and $C_N\to e^{1/4}\sqrt{2/\ln 2}\approx2.1811$.

APPENDIX F: NORMS OF GUE MATRICES AND THE CONSTANT 2 IN (7)

It has been known for some time (in fact in a much more general setting) that if $G=G(\omega)$ is the random matrix distributed according to the standard Gaussian measure on $\mathcal{M}_d^{sa}$ (usually called the Gaussian unitary ensemble or GUE), then, for large d, $\|G\|_{op}$ is, with high probability, close to $2\sqrt{d}$. We sketch here a derivation, from known facts, of the arguably elegant inequality $\mathbb{E}\|G\|_{op}<2\sqrt{d}$, valid for any d, which appears to have been overlooked in the random matrix theory literature. Similar inequalities are known for Gaussian matrices all whose entries are independent or for real symmetric matrices (also known as the GOE ensemble; however, in the latter case the precise inequality seems to have been established only for the largest eigenvalue, and not for the norm), see [27]. Analogous inequalities with the expected value replaced by the median can probably be deduced—at least for large d—from [28,29].

Our starting point is the recurrence formula for the (even) moments $a_p=a_p(d):=d^{-1}\mathbb{E}\,\text{tr}[(G/2)^{2p}]$, $p\in\mathbb{N}$, derived, e.g., in [30] [see also [31], formulas (6) through (9), for a similar argument and a related estimate]

$$a_p=\frac{2p-1}{2p+2}\left(a_{p-1}+\frac{p(p-1)}{4d^2}\frac{2p-3}{2p}a_{p-2}\right),$$

with $a_0=1$ and $a_1=1/4$. From these one easily derives by induction

$$a_p\leqslant\frac{1}{2^{2p}(p+1)}\binom{2p}{p}\prod_{j=1}^{p}\left(1+\frac{j(j-1)}{4d^2}\right).$$

[This estimate is actually asymptotically precise for $p=o(d)$.] Next, using successively the Stirling formula to majorize the binomial coefficient, the inequalities $1+x\leqslant e^x$ and $\Sigma_{j=1}^{p}j(j-1)\leqslant p^3/3$ to estimate the product, and denoting $t=pd^{-2/3}$, we arrive at

$$\mathbb{E}\text{tr}[(G/2)^{2p}]=da_p\leqslant d\frac{e^{p^3/12d^2}}{\sqrt{\pi}p^{3/2}}=\frac{e^{t^3/12}}{\sqrt{\pi}t^{3/2}}.$$

Hence

$$\frac{1}{2}\mathbb{E}\|G\|_{op} < \{\mathbb{E}\mathrm{tr}[(G/2)^{2p}]\}^{1/2p} \leqslant \left[\left(\frac{e^{t^3/6}}{\pi t^3}\right)^{1/4t}\right]^{1/d^{2/3}}.$$

This is valid for $t>0$, at least if the corresponding value of $p=td^{2/3}$ is an integer. The minimal value of the expression in brackets over $t>0$ is attained at $t\approx 1.383\,19$ and is approximately $0.738\,542\approx \exp(-0.303\,077)<e^{-0.3}$. Since for sufficiently large d the interval corresponding to values which are $<e^{-0.3}$ contains an element of $d^{-2/3}\mathbb{N}$, we deduce that for such d we have $\mathbb{E}\|G\|_{op}<2e^{-0.3d^{-2/3}}$. A more careful checking shows that in fact the inequality $\mathbb{E}\|G\|_{op}<2-0.6d^{-2/3}$ holds for *all* values of d [in fact, by the above argument, the same upper estimate is valid for $(\mathbb{E}\|G\|_{op}^r)^{1/r}$ with, say, $r=2$ or $r=d^{2/3}$].

Going back to the issue of having the precise constant 2 in inequality (7), let us note that the other source of difficulty, namely, the fact that the parameter γ_k is only asymptotically of order $\sqrt{k}$ but not equal to $\sqrt{k}$, introduces an error that is of smaller order than our "margin of safety." As pointed out earlier, we have $\gamma_k>\sqrt{k-1}$ and so $\gamma_k/\sqrt{k}>\sqrt{(1-1/k)}\approx 1-1/2k$. The relevant value of k is d^2, leading to the relative error of order $d^{-2}/2$, as opposed to the margin of safety of $0.3d^{-2/3}$ (note also that 4 is the smallest value of d that is of interest).

APPENDIX G: UPPER ESTIMATES ON vol $\mathcal{S}$ FOR SMALL TO MODERATE N

We now indicate how one may use the explicit efficient nets of the sphere S^2 listed in [22] to majorize the volume radius of $\mathcal{S}=\mathcal{S}((\mathbb{C}^2)^{\otimes N})$ if N is not too large. As a demonstration, we will derive bounds for the volume of the set of separable states on eight qubits (one may say, a qubyte).

The site [22] lists, for $m\in\{4,\ldots,130\}$, sets $\mathcal{N}_m$ of m points in S^2 such that every point of S^2 is within $\epsilon=\epsilon_m$ (measured in degrees) from one of the points of $\mathcal{N}_m$, with the dependence $m\to\epsilon_m$ "putatively optimal" (and very likely nearly optimal). Noting that $\delta=2\sin\epsilon/2$, we verify numerically that in most of the interesting range the putatively optimal value δ_m verifies $m\delta_m^2\approx 5$ (more precisely, $5.1\pm 1\%$, still not far from the asymptotic value $(2/\sqrt{3})^3\pi\approx 4.8368$ that we mentioned earlier). Since in the present context the bound in (15) becomes $(1-\delta_m^2/2)^{-8}\sqrt{2\ln(2m^8)}=(\cos\epsilon_m)^{-8}\sqrt{2\ln(2m^8)}$, substituting $m=5/\delta_m^2$ leads to a function $\phi(\delta)=(1-\delta^2/2)^{-8}\sqrt{2\ln[2(5/\delta^2)^8]}$, which attains its minimum very near $\delta=.15$. This suggests that the optimal value of m should be around 222. This is beyond the range of the tables from [22], but using the largest available $m=130$ and the corresponding $\epsilon_{130}=11.316\,562\,5°$ yields an upper bound of 10.417 406, which is less than 2% larger than the majorant that would presumably be given by $m=222$. Plugging in the obtained bound into (12) and using the fact that $(\mathrm{vol}\,\tilde{\Sigma}/\mathrm{vol}\,\Sigma)^{1/d^2}=d^{\alpha}=(27/16)^{N/8}$ we are led to

$$\begin{aligned} v &:= (\mathrm{vol}\,\Sigma/\mathrm{vol}\,B_{HS})^{1/d^2} \leqslant (16/27)10.417\,406/\gamma_{d^2} \\ &< 0.024\,114\,46. \end{aligned}$$

Taking into account (5) and substituting the explicit expression for vol B_{HS} we obtain

$$\mathrm{vol}\,\mathcal{S} \leqslant \frac{\sqrt{d}}{2}\frac{\pi^{d^2/2}v^{d^2}}{\Gamma(d^2/2+1)}.$$

Finally, using the closed formula for vol $\mathcal{D}$ from (E1) we get

$$\frac{\mathrm{vol}\,\mathcal{S}}{\mathrm{vol}\,\mathcal{D}} \leqslant \frac{(2\pi)^{d/2}v^{d^2}\Gamma(d^2)}{2^{d^2/2+1}\Gamma(d^2/2+1)\prod_{j=1}^{d}\Gamma(j)} < 8.6\times 10^{-19996},$$

which (modulo rounding errors) is equivalent to a much less impressive bound $(\mathrm{vol}\,\mathcal{S}/\mathrm{vol}\,\mathcal{D})^{1/\dim\mathcal{D}}\leqslant 0.495\,34$.

The smallest N for which an argument such as the above gives a nontrivial upper bound which appears to be 6; we get then $(\mathrm{vol}\,\mathcal{S}/\mathrm{vol}\,\mathcal{D})^{1/\dim\mathcal{D}}<0.95$. We refer to [32] and its references for extensive (largely numerical) treatment of the case $N=2$.

APPENDIX H: THE IN-RADII OF $\mathcal{S}$ AND Σ

The papers [2,6] estimate from *below* the in-radius of $\mathcal{S}$ in the Hilbert-Schmidt metric by a quantity that is of the order of $d^{-\eta}$, where $\eta=\ln 20/\ln 4\approx 2.160\,964$ and 3/2 respectively. The "trivial" upper bound on that radius is the in-radius of $\mathcal{D}$, which—by a rather elementary and well-known argument—equals $1/\sqrt{d(d-1)}=O(d^{-1})$. By comparing volumes we see that the second inequality in (9) yields an asymptotically better upper estimate that (up to logarithmic factors) corresponds to $\eta=1+\alpha\approx 1.094\,361$. This follows by taking into account (5) or by observing that, by simple geometric considerations, the Euclidean in-radius of Σ is at least as large as that of $\mathcal{S}$ (the latter considered in the hyperplane H_1 of trace one matrices). By tinkering with the argument it is possible to remove the logarithmic factors and, indeed, to obtain an upper bound on the in-radius of Σ (and hence of $\mathcal{S}$) which is $o(d^{-1-\alpha})$, but to improve the exponent new ideas appear to be necessary.

Let us also note that our argument yields as well a lower bound $6^{-N/2}$ on the in-radius of Σ which corresponds to $\eta=\ln 6/\ln 4\approx 1.2925$, and so is stronger than those that can be formally derived from [2] or [6]. To see this it is enough to combine the "trivial" lower estimate $d^{-1}=2^{-N}$ on the in-radius of the set $\tilde{\Sigma}$ defined in what follows [cf. the paragraph following (11)] with the known value $(3/2)^{N/2}$ of the norm of the related map $A^{\otimes N}$. (The recent paper [8] shows that the lower bound $6^{-N/2}$ on the in-radius works also for $\mathcal{S}$. As mentioned earlier, by building on the approach of the present paper it is possible to show that this bound is optimal up to factors of lower order; the details will be reported elsewhere [18].)

While our calculations narrow down the potential range of the in-radii of $\mathcal{S}$ and Σ, and while further progress along the same lines is possible, it seems likely that to determine the exact asymptotic behavior of these quantities a more careful calculation involving, e.g., spherical harmonics may be necessary. In another direction, as we have already noted, it may be more natural to consider in this context the inner product norm, which is different from the Hilbert-Schmidt norm and induced by the inner product $(u,v)\to(3\,\mathrm{tr}\,uv-\mathrm{tr}\,u\,\mathrm{tr}\,v)/2$ on each factor $\mathcal{M}_2^{sa}$. The (solid) ellipsoid $\mathcal{E}$, which is the unit ball with respect to the corresponding inner product norm on $\mathcal{M}_d^{sa}=(\mathcal{M}_2^{sa})^{\otimes N}$, verifies $\mathcal{E}/d\subset\Sigma\subset\mathcal{E}$ (this is just a restatement of $B_{HS}/d\subset\tilde{\Sigma}\subset B_{HS}$) and these inclusions are essentially optimal. On the one hand, all extreme points of Σ (which are, up to a sign, pure separable states) belong to the boundary of $\mathcal{E}$. On the other hand, it follows from, say, (11) that the volume radius of Σ is, on the power scale, the same as that of $\mathcal{E}/d$. This means that—from the volumetric point of view—Σ and $\mathcal{E}/d$ are essentially equivalent. It would be of interest to determine the precise in-radius of $\mathcal{S}$ with respect to the inner product norm induced by $\mathcal{E}$ or, equivalently, the largest ϵ such that $I_d/d+\epsilon(\mathcal{E}\cap H_0)\subset\mathcal{S}$, where $H_0:=\{A\in\mathcal{M}_d^{sa}:\mathrm{tr}\,A=0\}$. (It is conceivable that this radius is of the order d^{-1}.)

APPENDIX I: SEPARABLE STATES ON *N* QUDITS

This topic has been studied, e.g., in [5]. Most of the elements of our analysis can be generalized to tensor products involving spaces $\mathbb{C}^D$ with $D>2$, leading to nontrivial but not definitive results. As a demonstration, let $D\geqslant 3$ and consider the family of spaces $(\mathbb{C}^D)^{\otimes N}$. We shall employ analogous notation to that of the main text, in particular $d=\dim(\mathbb{C}^D)^{\otimes N}=D^N$. The set of pure states on $\mathcal{B}(\mathbb{C}^D)$ coincides (up to rescaling by a factor $\sqrt{2}$) with the projective space $\mathbb{C}P^{D-1}$ whose real dimension is $2D-2$ and which admits, for $\delta>0$, δ-nets of cardinality not exceeding $(C'/\delta)^{2D-2}$, where C' is a universal constant. This leads to a bound on $(\mathrm{vol}\,\Sigma/\mathrm{vol}\,B_{HS})^{1/d^2}$ which is of the order $(1-\delta)^{-N}\sqrt{ND\ln(C'/\delta)}/d$. Choosing, say, $\delta=1/N$ and using the same bound on vol $\mathcal{D}$ as earlier combined with the "easy" part of (5) we obtain

$$(\mathrm{vol}\,\mathcal{S}/\mathrm{vol}\,\mathcal{D})^{1/\dim\mathcal{D}}=O(\sqrt{ND\ln N}/d^{1/2}).$$

Since $d=D^N$, this leads to a nontrivial bound even for $N=2$ if D is large enough. It is also possible to improve slightly the exponent of d by working (as we did for $D=2$) with a more balanced affine image of Σ. The resulting improvement $\alpha=\alpha_D$ decreases as D increases; for example, $\alpha_3=8\log 2/9\log 3-1/2\approx 0.0608264$ and, for large D, $\alpha_D\sim(2D\ln D)^{-1}$. However, showing optimality of the so obtained exponents requires (for $D>2$) new ideas, which—together with details of the argument hinted above—will be presented elsewhere [18].

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