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Hydrodynamic quantization of mechanical systems

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In the present paper we propose a method of quantization of mechanical systems that is applicable, inter alia, to systems acted on by nonconservative forces, such as dissipative systems. First, the correspondence between classical and quantum mechanics is considered and we make some important observations that can be used in defining the rules of quantization. Next, the quantization rules are presented and we give a few examples of the application of these rules to mechanical systems that are difficult to quantize within any other systemetic approach. Finally, it is shown that for any quantum-mechanical system there are stationary solutions of the evolution equations, and that the existence of these solutions is directly related to the proposed quantization rules.

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I. INTRODUCTION

It is well known that quantum mechanics corresponds with classical mechanics. Namely, it can be shown that the laws of classical mechanics result from the laws of quantum mechanics for the Planck's constant h tending to zero [1,2]. Thus in many cases it is also possible to obtain the quantum-mechanical evolution equations for a given physical system on the basis of the corresponding equations for its classical counterpart using, for example, one of the systematic approaches, known as methods of quantization, proposed by Dirac [3], Schrödinger [4], and, later, Feynman and Hibbs [5]. All of them are generally accepted, although according to some authors there are crucial objections to the suitability of the Dirac quantizations method, often referred to as the standard quantization, to connect the classical and quantum theories [6]. Unfortunately, however, none of them is directly applicable to systems that are acted on by nonconservative forces [6], and the quantization of such systems seems to be important for at least three basic reasons.

First, a typically quantum-mechanical type of behavior has been observed in some nonmicroscopic and therefore dissipative systems [6,7], so we should be able to use some quantum-mechanical evolution equations to make predictions regarding systems of this kind.

Second, it sometimes is convenient to use phenomenological friction forces in describing the evolution of microscopic systems. For instance, an electron in a plasma is subject to interactions that are similar to the interactions of a star in a star cluster [8]. Accordingly, it is possible to determine the effective force of dynamical friction acting on the electron by averaging all the effects of its interactions with the surrounding electrons and ions. Next, the trajectory of the particle can be determined from the Newton equation in which the effective force is used instead of all forces of interactions between the particle and the other bodies [8]. Of course this approach is applicable in the classical approximation, which is not always suitable for describing the evolution of electrons [1]. Thus we need some quantum-mechanical evolution equations that could be used in such cases.

Finally, if all of the laws of classical mechanics should result from the laws of quantum mechanics for h tending to zero, then for each classical-mechanical system of evolution equations there should be a quantum-mechanical one it follows from. In other words, there may be quantum-mechanical evolution equations that do not have their classical counterparts, but all of the classical-mechanical evolution equations, including the equations that correspond to the systems acted on by nonconservative forces, ought to have their quantum-mechanical analogs. Otherwise, the correspondence between classical and quantum mechanics would be incomplete.

Although the last point has not been expressedly recognized by the majority of authors, the other two have been taken into consideration and many attempts have been made to determine quantum-mechanical evolution equations for dissipative systems.

First of all, it has been observed that one can actually use the standard equations of quantum mechanics for describing a dissipative system if this system is considered as a part of a larger system, referred to as the universe, which additionally contains its environment [7,9]. Namely, in that case the universe can be regarded as a Hamiltonian system and the Heisenberg equations can be used for determining its evolution. Next, by averaging these equations with respect to the degrees of freedom corresponding to the dissipative system's environment, one can obtain the Heisenberg-Langevin equations for it [9,10]. Then, the Heisenberg-Langevin equations can be used, for example, to determine the evolution of mean values of the system's position and velocity, but it generally is difficult to reduce these equations to a single evolution equation for the system's wave function due to the lack of a linear evolution operator [9]. Nonetheless, Kostin has managed to derive the nonlinear Schrödinger-Langevin equation from the Heisenberg-Langevin equations for a Brownian particle interacting with a thermal environment [10].

Unfortunately, the approach described above has some important disadvantages. For instance, in order to average the evolution equations for the universe over the environmental degrees of freedom one has to use a certain representation of the universe and a certain model of the interactions between the system in question and the environment, and the final evolution equations usually depend on the models used. Admittedly, it can be shown that virtually any realistic environment of the system can be represented by an infinite bath of harmonic oscillators [7], but it turns out that the resulting evolution equations for the dissipative system generally depend on the initial state of its environment [9]. Moreover, only the simplest models of interactions between the considered system and its environment lead to the Heisenberg-Langevin equations that can be solved, even if some approximations are made [7,9], so if the considered system is macroscopic, it is highly uncertain whether the model of the interactions that can be used in computations is realistic. Thus, as far as macroscopic dissipative systems are concerned, this approach is in fact phenomenological and very complicated, so the question arises if any simpler and therefore more practical method of quantization of dissipative systems can be proposed.

To give an answer to this question some authors have attempted to extrapolate the Dirac quantization method to the quantum-mechanical study of dissipative systems [11–14]. Yet, it turned out that the quantum-mechanical description of such systems could not not be obtained from either the Lagrangian or the Hamiltonian formalism due to the general difficulties and limitations in the use of the Dirac quantization rules for them [6,15–17]. Thus, in order to overcome the problems arising from the Dirac quantization, other quantization methods for dissipative systems have been proposed [6]. In particular, the Schrödinger-Langevin equation has been obtained by applying the Schrödinger method of quantization to the generalized Hamilton-Jacobi equation [16], by using the Nelson stochastic quantization procedure [18,19], and by applying the classical Wigner transformations to a Fokker-Planck equation [6]. Schrödinger-type equations for dissipative systems have also been derived by an extension of the Madelung model [20–22]. Although some of these approaches have been quite successful, they are rather limited in scope and have not been developed as general methods of quantization.

We propose a different quantization procedure applicable to all systems, dissipative as well as nondissipative. Our approach is based on the previous work in which we have derived the Schrödinger-Langevin equation from the hydrodynamiclike quantum-mechanical evolution equations for a dissipative system, obtained on the basis of some observations regarding the correspondence between classical and quantum mechanics that follow from the Madelung model [23]. In the present paper we clarify the argumentation and make additional important observations supporting it. Moreover, the approach is extended to multidimensional systems and we introduce well-defined quantization rules that can be applied to arbitrary Netwon equations in an automatic manner.

First, we observe that if the behavior of a mechanical system cannot be predicted exactly, the physical quantities corresponding to it should be represetned by random variables and we can only obtain the evolution equations for their probability densities and other corresponding probability functions. We find that in classical mechanics, under specific assumption regarding the initial state of the system, these probability functions satisfy evolution equations that correspond strictly to the quantum-mechanical equations resulting from the Madelung model.

Next, we consider the average energy of the system and notice that in quantum mechanics there is an additional contribution to it which is not taken into consideration in classical mechanics, and which is strictly related to the terms appearing in the quantum-mechanical evolution equations that are not present in the corresponding classical-mechanical equations. We use this observation to obtain hydrodynamiclike quantum-mechanical evolution equations for a one-dimensional dissipative system and we extend the approach to multidimensional systems. As a result of this, we are able to formulate general rules of quantization.

Finally, we give some examples of the application of the proposed quantization rules to dissipative systems. We also discuss the stationary states of a quantum system, since the appearance of these states turns out to be a direct consequence of the existence of the additional quantum-mechanical contribution to the system's average energy on which the proposed method of quantization is based.

II. CLASSICAL MECHANICS VS QUANTUM MECHANICS

A. Probability liquid

The correspondence between the laws of classical and quantum mechanics is usually not clearly visible because of the different language used in each of these theories. Namely, in classical mechanics the states of a physical system are represented by points in its phase space and its evolution is represented by a trajectory in that space, whereas in quantum mechanics the states of the system are represented by wave functions or density matrices, and its evolution is represented by a curve in the Hilbert space containing these objects. Still, it is possible to transform the laws of classical and quantum mechanics so that they refer to the same quantities, and then it is much easier to look for the correspondence between them.

The main difficulty in comparing the laws of classical and quantum mechanics stems from the assumption made in the latter that the behavior of a physical system cannot be predicted exactly even if its initial state is known as precisely as physically possible. In other words, if we have two systems that are as similar as physically possible, so they can be considered as physically equivalent, each of them will behave in a slightly different way and for each of them the results of measurements of various physical quantities will generally be different. Thus the results of these measurements in a set of physically equivalent systems may be considered as *random*. Nonetheless, if the same measurement is repeated in a great number of physically equivalent systems, a definite average pattern of results should appear. Therefore, although the behavior of physical systems is not exactly predictable, they are subject to *statistical regularity* [24]. Consequently, physical quantities that in classical mechanics are represented by real-valued functions of time ought to be treated as time-dependent random variables.

Note, however, that the existence of the basic indeterminacy in the physical world assumed in quantum mechanics is not the only possible reason for which we may be unable to predict the results of various experiments with certainty. For instance, it is also possible that we do not have enough data about the conditions of the experiment, or we do not know all the casual forces at work [24]. In either case we can treat the outcomes of the experiments as random and we can assume that they are statistically regular. Accordingly, in either case we ought to treat physical quantities as random variables.

Consider, for example, the position of a system in its configurational space. If we believe that the position of the system at any instant of time t can be predicted exactly, we can introduce the function $\mathbf{x}(t)$, the values of which represent the positions of the system at different instants of time, and we can try to find evolution equations allowing us to determine this function. On the other hand, if we expect that, whatever the reason, the positions of the system at different instants of time cannot be predicted exactly, but the system is subject to statistical regularity, we should introduce a family of random variables $\tilde{\mathbf{x}}(t)$ such that for given t the probability distribution function of the random variable $\tilde{\mathbf{x}}(t)$, $P(\mathbf{x},t)$, will allow us to compute the probability of finding the system in any given area of its configurational space at time t. In such a case, if $P(\mathbf{x},t_0)$ for certain t_0 is known, we may be able to determine the probability distribution function $P(\mathbf{x},t)$ for $t \geqslant t_0$ with the help of some evolution equations.

Usually, we can assume that $P(\mathbf{x},t)$ is a smooth function of $\mathbf{x}$. Then, by differentiating it we can obtain the configurational probability density $\rho(\mathbf{x},t)$, which for an n-dimensional system is given by

$$\rho(\mathbf{x},t) = \partial_1 \partial_2 \cdots \partial_n P(\mathbf{x},t), \tag{1}$$

where ∂_j denotes the differentiation with respect to the jth Cartesian configurational coordinate of the system. Formally, the symbol $\rho(\mathbf{x},t)$ denotes a family of probability densities indexed by t. Still, since the set of all values of t is continuous, we can also assume $\rho(\mathbf{x},t)$ to be a smooth function of t which for any given value of this variable is a probability density with respect to $\mathbf{x}$. Consequently, $\rho(\mathbf{x},t)$ may formally be treated as the density of a fictitious n-dimensional liquid referred to as the *probability liquid* [1,22,23,27]. Also, it can be shown that $\rho(\mathbf{x},t)$ satisfies the continuity equation [28]:

$$\partial_t \rho(\mathbf{x},t) = -\sum_{\alpha=1}^{n} \partial_\alpha [\rho(\mathbf{x},t) v_\alpha(\mathbf{x},t)], \tag{2}$$

where ∂_t denotes the differentiation with respect to time, and $v_\alpha(\mathbf{x},t)$ are components of the n-dimensional velocity field of the probability liquid $\mathbf{v}(\mathbf{x},t)$, which may be referred to as the *probability velocity field* of the system.

It turns out that, under specific additional assumptions, we are able to determine a closed system of evolution equations for the configurational probability density and the probability velocity field of the system using the laws of either classical, or quantum mechanics [23].

B. Probability liquid in classical mechanics

In classical mechanics it is usually assumed that the position of the system at any instant of time can be predicted exactly, so there is a function of time the values of which represent the positions of the system at different instants of time. For a one-dimensional system consisting of one body of mass m it is a real-valued function and we denote it by $q(t)$. Of course, if the body is acted on by a potential force $F(x)$, where x denotes the single configurational coordinate of the system, the function $q(t)$ ought to satisfy the following Newton equation of motion:

$$\frac{d^2 q}{dt^2} = \frac{F(q)}{m}, \tag{3}$$

which allows us to determine $q(t)$ provided that its value and the value of its first derivative at certain time t_0 are known.

Suppose, however, that for some reason we do not know these values. Assume instead that we are given the system's configurational probability density $\rho(x,t_0)$ and probability velocity field $v(x,t_0)$. Obviously in that case we are unable to determine $q(t)$. Nevertheless, we can say something about $\rho(x,t)$ and $v(x,t)$ for $t \geqslant t_0$.

To show this we observe that the probability distribution function $P(x,t_0)$, the derivative of which is $\rho(x,t_0)$, can be approximated by the following staircase function:

$$P(\{x_k\}_N, x, t_0) = \sum_{k=1}^{N+1} P_k \chi(x - x_k), \tag{4}$$

where the symbols x_k denote N fixed points in the system's configurational space,

$$P_k = \begin{cases} P(x_1,t_0), & k=1 \\ P(x_k,t_0) - P(x_{k-1},t_0), & 1 < k \leqslant N \\ 1 - P(x_N,t_0), & k = N+1 \end{cases}, \tag{5}$$

and the function $\chi(x)$ is equal to zero for $x<0$ and equal to 1 for $x \geqslant 0$. By differentiating $P(\{x_k\}_N, x, t_0)$ with respect to x we obtain the associated probability density

$$\rho(\{x_k\}_N, x, t_0) = \sum_{k=1}^{N+1} P_k \delta(x - x_k), \tag{6}$$

where $\delta(x)$ denotes the well-known Dirac delta distribution. This probability density can be treated as a special representation of $\rho(x,t_0)$ and may be referred to as the *reduced* configurational probability density of the system. Its definition obviously depends on the selection of N points x_k in the system's configurational space which is denoted explicitly by the symbol $\{x_k\}_N$.

Now, we notice that in principle each of the x_k used in defining $P(\{x_k\}_N, x, t_0)$ can also be used as $q(t_0)$ in determining the function $q(t)$ from Eq. (3). Yet, for this purpose we also need to know the velocity v_k that should be used along with x_k as the initial value of the derivative of $q(t)$. If we know it, we can use Eq. (3) to determine the function of time $q_k(t)$ such that

$$q_k(t_0) = x_k, \qquad \left.\frac{dq_k}{dt}\right|_{t_0} = v_k, \tag{7}$$

and we have

$$\rho(\{x_k\}_N,x,t) = \sum_{k=1}^{N+1} P_k \delta(x - q_k(t)) \tag{8}$$

for $t \geqslant t_0$.

To determine the velocities v_k we differentiate $\rho(\{x_k\}_N,x,t)$ with respect to time and get

$$\partial_t \rho(\{x_k\}_N,x,t) = -\partial_x j^{(1)}(\{x_k\}_N,x,t), \tag{9}$$

where the current $j^{(1)}(\{x_k\}_N,x,t)$ is given by

$$j^{(1)}(\{x_k\}_N,x,t) = \sum_{k=1}^{N+1} P_k \frac{dq_k}{dt} \delta(x - q_k(t)). \tag{10}$$

Next, we use the observation that $\rho(x,t)$ ought to satisfy the continuity equation:

$$\partial_t \rho(x,t) = -\partial_x [\rho(x,t) v(x,t)], \tag{11}$$

and require that

$$j^{(1)}(\{x_k\}_N,x,t_0) = \rho(\{x_k\}_N,x,t_0) v(x,t_0), \tag{12}$$

which means that $v_k = v(x_k,t_0)$, because of Eqs. (6) and (7). Thus, formally, given $\rho(x,t_0)$ and $v(x,t_0)$, we can obtain $\rho(\{x_k\}_N,x,t)$ and $j^{(1)}(\{x_k\}_N,x,t)$ for any N and for any selection of the points x_k in the system's configurational space. Still, it is quite difficult to do this by determining individual functions $q_k(t)$, so we need to use some evolution equations for this purpose.

To find the evolution equations for $\rho(\{x_k\}_N,x,t)$ and $j^{(1)}(\{x_k\}_N,x,t)$ we differentiate the current with respect to time and get

$$\partial_t j^{(1)}(\{x_k\}_N,x,t) = \frac{F(x)}{m} \rho(\{x_k\}_N,x,t) - \partial_x j^{(2)}(\{x_k\}_N,x,t), \tag{13}$$

where the quantity $j^{(2)}(\{x_k\}_N,x,t)$ is given by

$$j^{(2)}(\{x_k\}_N,x,t) = \sum_{k=1}^{N+1} P_k \left(\frac{dq_k}{dt}\right)^2 \delta(x - q_k(t)), \tag{14}$$

which follows from Eq. (3). By differentiating $j^{(2)}(\{x_k\}_N,x,t)$ with respect to time we obtain an equation similar to Eq. (14), in which there is the quantity $j^{(3)}(\{x_k\}_N,x,t)$ defined in analogy with $j^{(2)}(\{x_k\}_N,x,t)$ and so on. Generally, for $n > 1$, we have

$$\partial_t j^{(n)}(\{x_k\}_N,x,t) = \frac{F(x)}{m} j^{(n-1)}(\{x_k\}_N,x,t) - \partial_x j^{(n+1)}(\{x_k\}_N,x,t), \tag{15}$$

where

$$j^{(n)}(\{x_k\}_N,x,t) = \sum_{k=1}^{N+1} P_k \left(\frac{dq_k}{dt}\right)^n \delta(x - q_k(t)). \tag{16}$$

Thus we obtain an infinite system of evolution equations for the system's reduced configurational probability density $\rho(\{x_k\}_N,x,t)$ and the family of functions $j^{(n)}(\{x_k\}_N,x,t)$, $n=1, 2, \ldots$, so generally in order to determine, for example, $\rho(\{x_k\}_N,x,t)$ we need to know $\rho(\{x_k\}_N,x,t_0)$, as well as $j^{(1)}(\{x_k\}_N,x,t_0)$, $j^{(2)}(\{x_k\}_N,x,t_0)$, etc.

Having obtained Eqs. (9), (13), and (15) we can use the remark that they are satisfied for any N and for any selection of the points x_k in the system's configurational space, which implies that they should also be satisfied by $\rho(x,t)$ and a family of smooth functions $j^{(n)}(x,t)$ corresponding to $j^{(n)}(\{x_k\}_N,x,t)$, respectively. Consequently, since Eq. (9) is the counterpart of the continuity equation, Eq. (11), we can identify the current $j^{(1)}(x,t)$ as the product of $\rho(x,t)$ and $v(x,t)$. Thus we have

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = \frac{F(x)}{m} - \frac{1}{\rho(x,t)} \partial_x [j^{(2)}(x,t) - \rho(x,t) v^2(x,t)], \tag{17}$$

which follows from Eq. (13) and from the continuity equation, Eq. (11).

Of course, to obtain a closed system of evolution equations for $\rho(x,t)$ and $v(x,t)$ we need to eliminate the function $j^{(2)}(x,t)$ from Eq. (17) and it turns out that it can only be done if

$$j^{(n)}(x,t) = \rho(x,t) v^n(x,t), \tag{18}$$

for any natural n [23]. Then, we get

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = \frac{F(x)}{m}, \tag{19}$$

and all of the evolution equations for $j^{(n)}(x,t)$ are automatically satisfied. Accordingly, given $\rho(x,t_0)$ and $v(x,t_0)$, we can determine $\rho(x,t)$ and $v(x,t)$ for $t \geqslant t_0$ with the help of Eqs. (11) and (19).

C. Representation of states

In classical mechanics we can only obtain a closed system of evolution equations for the configurational probability density and probability velocity field of a one-dimensional physical system if Eq. (18) is satisfied for any natural n. Otherwise, we have to take the whole infinite system of evolution equations for the functions $j^{(n)}(x,t)$ into consideration, which in many cases is simply impractical. Therefore the assumed satisfaction of Eq. (18) for any n seems to be quite important, so we ought to explain the actual meaning of this assumption. However, for this purpose we first need to extend the notion of states to the systems the behavior of which cannot be predicted exactly, although they are subject to statistical regularity.

If the position, let alone the velocity, of the system cannot be predicted exactly, there is no use in representing the system's states by phase-space points. Still, it is convenient to think that the system can assume various states and it is helpful to use specific mathematical objects for representing them.

In principle to differentiate the system's states from each other we can use the measurements of various physical quantities. Yet, for a system that behaves in a more or less random way the physical quantities should be represented by random variables, as we have already stated in Sec. II A, and single measurement of the value of a random variable does not really mean anything. To the random variable important are the moments of its probability distribution [24], and they can be used in defining the states of the system. Namely, we can say that the state of the system has changed over time if any moment of the probability distribution of any random variable representing a physical quantity has changed. For example, if the average square of the system's position at the time t_0 is different to the average square of the system's position at the time t, we can say that the states of the system at the instants of time t_0 and t are different. Accordingly, the states of the system should be represented by mathematical objects allowing us to compute the moments of probability distributions of random variables that represent physical quantities.

For the system discussed in Sec. II B these objects are joint probability densities of the random variables $\tilde{x}(t)$ and $\tilde{u}(t)$ representing the system's position and velocity, respectively. Namely, in classical mechanics for each instant of time t there is a joint probability of $\tilde{x}(t)$ and $\tilde{u}(t)$ that we denote by $s(x,u,t)$. Of course, for any t the probability density $s(x,u,t)$ belongs to the set Ω of all non-negative real-valued functions $p(x,u)$, defined for x from the system's configurational space and real u, and satisfying the normalization condition $\int p(x,u)dxdu=1$, where the integrations with respect to x and u are carried out over the whole system's configurational space and the whole real axis, respectively. Therefore the system may be thought of as being in different states, represented by different elements of Ω, at different instants of time, represented by different values of the parameter t. Accordingly, the evolution of $s(x,u,t)$, which in fact is equivalent to the evolution of the system, may be seen as a curve, or a trajectory, in Ω, and the evolution equations allow us to determine this trajectory for given initial probability density $s(x,u,t_0)=p_0(x,u)$. For this reason $s(x,u,t)$ can be referred to as the *state probability density* of the system.

It turns out that the state probability density of the system considered in Sec. II B can be determined relatively easily. Namely, there is an easy-to-establish relationship between the functions $j^{(n)}(x,t)$ introduced in Sec. II B and moments of the joint probability distribution of $\tilde{x}(t)$ and $\tilde{u}(t)$, i.e.,

$$\langle x^k u^n\rangle(t) = \int dx x^k j^{(n)}(x,t), \tag{20}$$

which follows from Eqs. (10) and (16). Thus we can define the following characteristic function:

$$M_u(iq,x,t) = \sum_{n=0}^{\infty} \frac{(iq)^n}{n!} j^{(n)}(x,t), \tag{21}$$

and, subsequently, we can obtain $s(x,u,t)$ using the standard formula [24]:

$$s(x,u,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \exp(-qu)M_u(iq,x,t). \tag{22}$$

It is straightforward to verify that $s(x,u,t)$ given by the right-hand side of Eq. (22) has all of the required properties and allows us to compute any moment of the probability distribution of any physical quantity being a function of the system's position and velocity.

Now, let us recall that the configurational probability density and probability velocity field of the system, $\rho(x,t)$ and $v(x,t)$, satisfy Eqs. (11) and (19) if the condition given by Eq. (18) holds for any n. Then, we can express the characteristic function $M_u(iq,x,t)$ in terms of $\rho(x,t)$ and $v(x,t)$, i.e.,

$$M_u(iq,x,t) = \exp[iqv(x,t)]\rho(x,t), \tag{23}$$

so the system's state probability density is given by

$$s(x,u,t) = \rho(x,t)\delta(u - v(x,t)). \tag{24}$$

Thus we see that if Eq. (18) is satisfied for any n, the state represented by $s(x,u,t)$ is special. Namely, in this state the system's velocity is not known exactly only because it depends on the system's position that is really random. Then, according to the discussion in Appendix A, the system's state probability density $s(x,u,t)$ is a *semipure* joint probability density of $\tilde{x}(t)$ and $\tilde{u}(t)$. Nevertheless, in that case $s(x,u,t)$ satisfies the Liouville equation, which is also shown in Appendix A.

It follows from the previous paragraph that the satisfaction of Eq. (18) for any n is equivalent to the requirement that the system's state probability density $s(x,u,t)$ be semipure for any t. However, the evolution equations for $\rho(x,t)$ and $j^{(n)}(x,t)$ derived in Sec. II B imply that if the initial-state probability density of the system $s(x,u,t_0)$ is semipure, $s(x,u,t)$ will be semipure for any $t>t_0$. Namely, if Eq. (18) is assumed to be satisfied for any n at the time t_0, we have

$$\partial_t^k[j^{(n)}(x,t) - v^n(x,t)\rho(x,t)]_{t=t_0} = 0 \tag{25}$$

for any k and n, which means that Eq. (18) will also be satisfied for any n at any time $t>t_0$ [23]. Thus by assuming the satisfaction of Eq. (18) for any n we actually require the system's initial state to have specific properties that are preserved during the evolution of the system. Such an assumption can be made for any system and it is absolutely justified as long as we are not interested in the cases in which the system's initial state does not have the properties implied by the assumption.

D. Probability liquid in quantum mechanics

In quantum mechanics, the evolution equations for the probability liquid can be derived directly from the Schrödinger equation [1,22,23].

To illustrate this we consider the quantum counterpart of the system discussed in Sec. II B and observe that, as long as the laws of quantum mechanics are assumed to hold, its wave function $\psi(x,t)$ must satisfy the following Schrödinger equation:

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2\psi(x,t) + V(x)\psi(x,t), \tag{26}$$

where $V(x)$ is the potential of the force $F(x)$.

Now, the configurational probability density and the probability velocity field of the system can be defined in terms of $\psi(x,t)$, viz. [1,23]

$$\rho(x,t) = \psi^*(x,t)\psi(x,t), \tag{27}$$

$$\rho(x,t)v(x,t) = \frac{i\hbar}{2m}[\psi(x,t)\partial_x\psi^*(x,t) - \psi^*(x,t)\partial_x\psi(x,t)]. \tag{28}$$

Conversely, the wave function $\psi(x,t)$ can be expressed in terms of $\rho(x,t)$ and $v(x,t)$ in the following way:

$$\psi(x,t) = \sqrt{\rho(x,t)}\exp\left[i\frac{m}{\hbar}\int_{x_0}^{x} dqv(q,t)\right], \tag{29}$$

and the right-hand side of this equation can be substituted for $\psi(x,t)$ in Eq. (26). Then, the resulting equation can be divided by the exponential factor that appears on the right-hand side of Eq. (29) and we get a system of two equations for the amplitude and phase of the wave function being real-valued functions of x and t. Next, we use the observation that the amplitude of the wave function can be expressed in terms of $\rho(x,t)$ and that the derivative with respect to x of its phase is proportional to the velocity field $v(x,t)$ [1], which allows us to state that $\rho(x,t)$ satisfies the continuity equation, Eq. (11), and to obtain the evolution equation for $v(x,t)$:

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = \frac{F(x)}{m} + \frac{\hbar^2}{2m^2}\partial_x T_x[\rho(x,t)], \tag{30}$$

where $T_x[f]$ is defined by

$$T_x[f] = \frac{\partial_x^2\sqrt{f}}{\sqrt{f}}, \tag{31}$$

for any non-negative function f depending on the system's configurational coordinate x.

Obviously the only difference between Eq. (30) and its classical-mechanical counterpart, Eq. (19), is the presence of the term proportional to $(\hbar/m)^2$ on the right-hand side of the former. If $(\hbar/m)$ is very small this term can be neglected and the quantum-mechanical evolution equations become their classical-mechanical counterparts. Then, the quantum-mechanical description of the system can be replaced with the classical one without the risk of the appearance of any substantial inaccuracy. This is the so-called *classical limit* of quantum mechanics, and we can say, for example, that Eq. (19) follows from Eq. (30) in the classical limit. Still, this does not mean that for $(\hbar/m)$ being negligibly small the behavior of the system must be predictable exactly. On the contrary, it generally is necessary to represent physical quantities by random variables in the classical limit too [25].

It may seem to be a problem that in classical mechanics Eqs. (11) and (19) are only satisfied for systems in special initial states, but in fact the same observation applies to the Schrödinger equation in quantum mechanics. Namely, in quantum mechanics the solutions of the Schrödinger equation correspond only to the so-called pure states of the system that are special. In quantum mechanics states of the system are generally represented by density matrices that satisfy the von Neumann–Liouville equation [1,2,26]. Some density matrices correspond to individual wave functions satisfying the Schrödinger equation and they are regarded as the representations of the pure states of the system, but there are also density matrices that do not correspond to any individual wave functions and the states represented by them are referred to as mixed states of the system [1]. In general, the quantum-mechanical pure states of the system correspond to the states of its classical counterpart that are represented by semipure state probability densities $s(x,u,t)$ introduced in Sec. II C [23].

It is instructive to notice that for the considered quantum-mechanical system there also is a semipure probability density $s(x,u,t)$ given by Eq. (24), where $\rho(x,t)$ and $v(x,t)$ satisfy Eqs. (11) and (30). Obviously it is a joint probability density of two random variables $\tilde{x}(t)$ and $\tilde{u}(t)$, the first of which represents the system's position at the time t. Unfortunately the random variable $\tilde{u}(t)$ cannot be regarded as a representation of the system's velocity, because doing so would contradict the widely accepted interpretation of quantum mechanics in which the momentum of the system is represented by a differential operator and the moments of the joint probability distribution of the system's position and velocity are not well defined [1,2,26]. Still, in the classical limit $\tilde{u}(t)$ becomes the random variable representing the system's velocity considered in Sec. II C.

Of course, in general the quantum-mechanical probability density $s(x,u,t)$ does not satisfy the Liouville equation. In the classical limit, however, it becomes the state probability density of the system discussed in Sec. II C, which satisfies this equation. Thus the probability density $s(x,u,t)$ can be used in studying the correspondence between quantum and classical mechanics. Furthermore, since the probability density $s(x,u,t)$ can be obtained with the help of the system's wave function $\psi(x,t)$ and vice versa, as they both can be expressed in terms of the system's configurational probability density and probability velocity field, it may be regarded as a representation of the system's state at the time t. Accordingly, it may be referred to as the state probability density of the system in analogy with its classical-mechanical counterpart, although in practice in quantum mechanics it usually is more convenient to represent states by wave functions.

E. Energy of concentration

Having obtained two systems of evolution equations for the probability liquid, one of which, consisting of Eqs. (11) and (30), is quantum mechanical, and the other, consisting of Eqs. (11) and (19), is consistent with the laws of classical mechanics, we are now going to argue that the difference between them is strictly related to the existence of an additional contribution to the system's average energy in quantum mechanics.

First, we note that in classical mechanics the energy of the system discussed in Sec. II B is defined by

$$E = \frac{m}{2}\left(\frac{dq}{dt}\right)^2 + V(q), \tag{32}$$

where $q(t)$ is a function satisfying Eq. (3) that can be determined if the system's velocity and position at the initial instant of time are known exactly. Otherwise we have to treat the energy as a random variable and we can only compute the moments of its probability distribution. In particular, we can compute its mean value that is given by

$$\langle E\rangle = \frac{m}{2}\langle v^2\rangle + \langle V\rangle, \tag{33}$$

where $\langle v^2\rangle$ denotes the average square of the system's velocity and $\langle V\rangle$ is its average potential energy. According to Eq. (20) the right-hand side of Eq. (33) can be rewritten in the following form:

$$\langle E\rangle = \frac{m}{2}\int dx\ j^{(2)}(x,t) + \int dx\ V(x)\rho(x,t), \tag{34}$$

so if the system's state probability density is semipure, in which case Eq. (18) is satisfied for any n, we have

$$\langle E\rangle = \frac{m}{2}\int dx\ v^2(x,t)\rho(x,t) + \int dx\ V(x)\rho(x,t). \tag{35}$$

By differentiating Eq. (35) with respect to time and using the corresponding evolution equations, Eqs. (11) and (19), to eliminate the time derivatives of $\rho(x,t)$ and $v(x,t)$, respectively, from the resulting equation, we can show that $\langle E\rangle$ is independent of time [23]. Therefore it may be treated as a constant of motion.

In quantum mechanics the average energy of the system considered in Sec. II D is defined by

$$\langle E_q\rangle = -\frac{\hbar^2}{2m}\int dx\ \psi^*(x,t)\frac{\partial^2}{\partial x^2}\psi(x,t) + \int dx\ \psi^*(x,t)V(x)\psi(x,t), \tag{36}$$

where $\psi(x,t)$ is the system's wave function satisfying Eq. (26). Hence if the right-hand side of Eq. (29) is substituted for $\psi(x,t)$ in Eq. (36), we get

$$\langle E_q\rangle = \frac{m}{2}\int dx\ v^2(x,t)\rho(x,t) + \int dx\ V(x)\rho(x,t) + \frac{\hbar^2}{2m}U[\rho(x,t)], \tag{37}$$

where the functional $U[f]$ is defined by

$$U[f] = \int dx(\partial_x\sqrt{f})^2, \tag{38}$$

for any non-negative function f depending on the system's configurational coordinate x.

We now see that the only difference between the quantum-mechanical formula for the system's average energy, Eq. (37), and its classical counterpart, Eq. (35), is the term proportional to $(\hbar/m)$ appearing on the right-hand side of the former, which is not present in the latter. It clearly follows from this observation that in quantum mechanics there is an additional contribution to the average energy of the system. Moreover, this contribution is always positive and independent of the potential $V(x)$ as well as of the flow of the probability liquid represented by the velocity field. It depends only on the configurational probability density of the system and it is greater for more concentrated $\rho(x,t)$ and lesser for $\rho(x,t)$ that are relatively flat, which is shown in Appendix B. Therefore it may be referred to as the *energy of concentration*. Traditionally, however, it is regarded as a component of the average kinetic energy of the system, which allows one to prove the satisfaction of the uncertainty principle, as demonstrated in Appendix C.

Having identified the energy of concentration we are now going to show that there is a strict relationship between this energy and the last term on the right-hand side of Eq. (30). Namely, if $U[\rho(x,t)]$ is differentiated with respect to time, we get

$$\frac{d}{dt}U[\rho(x,t)] = -\int dxv(x,t)\rho(x,t)\partial_x T_x[\rho(x,t)], \tag{39}$$

where $T_x[f]$ is defined by Eq. (31), which follows from Eq. (38) and from the continuity equation, Eq. (11). The appearance of $T_x[\rho(x,t)]$ in this equation is hardly a coincidence. Indeed, if Eq. (37) is differentiated with respect to time and Eqs. (11) and (30) are used to eliminate the time derivatives of $\rho(x,t)$ and $v(x,t)$, respectively, from the resulting equation, we get

$$\frac{d}{dt}\langle E_q\rangle = \frac{\hbar^2}{2m}\left\{\int dxv(x,t)\rho(x,t)\partial_x T_x[\rho(x,t)] + \frac{d}{dt}U[\rho(x,t)]\right\}, \tag{40}$$

where the source of the first term on the right-hand side is the time derivative of the velocity field. Thus in quantum mechanics, if the force $F(x)$ is conservative, the average energy $\langle E_q\rangle$ is a constant of motion. However, this is only possible because the last term on the right-hand side of Eq. (30) has the correct form [23]. Consequently, we can assume that the role of this term in Eq. (30) is exactly to cancel the contribution of the energy of concentration to the time derivative of the average energy $\langle E_q\rangle$.

III. HYDRODYNAMIC QUANTIZATION

A. Quantization of one-dimensional dissipative system

In Sec. II we have made some important observations regarding the correspondence between classical and quantum mechanics. Now, we are going to use them to obtain quantum-mechanical evolution equations for a dissipative system.

For simplicity we consider a system for which the following Newton equation is satisfied in classical mechanics:

$$\frac{d^2q}{dt^2} = -\frac{\gamma}{m}\frac{dq}{dt} + \frac{F(q)}{m}, \tag{41}$$

where γ is a non-negative constant and the force $F(x)$ is potential.

We first use the observation following from Secs. II B and II D that on the basis of Eq. (41) we can derive the classical-mechanical equations for the probability liquid which ought to result from the quantum-mechanical evolution equations for the system in the classical limit. To obtain these equations we use the same approach that we have used in Sec. II B for the analogous nondissipative system. Doing so we find that

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{\gamma}{m}v(x,t) + \frac{F(x)}{m} - \frac{1}{\rho(x,t)}\partial_x[j^{(2)}(x,t) - \rho(x,t)v^2(x,t)], \tag{42}$$

$$\partial_t j^{(n)}(x,t) = -\frac{\gamma}{m}j^{(n)}(x,t) + \frac{F(x)}{m}j^{(n-1)}(x,t) - \partial_x j^{(n+1)}(x,t), \quad n > 1, \tag{43}$$

where the functions $j^{(n)}(x,t)$ are defined in exactly the same way as in Sec. II B, and the continuity equation, Eq. (11), is satisfied [23]. Hence if the satisfaction of Eq. (18) for any n is assumed, which means that the system's state probability density is semipure, we get

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{\gamma}{m}v(x,t) + \frac{F(x)}{m}. \tag{44}$$

Thus we can state that in the classical limit the configurational probability density and probability velocity field of the system, $\rho(x,t)$ and $v(x,t)$, satisfy Eqs. (11) and (44). Moreover, with the help of these equations we can derive the counterpart of the Liouville equation for the system's state probability density using the approach presented in Appendix A.

Next, we notice that for $\gamma=0$ the considered system becomes the system discussed in Sec. II D. Consequently, if $\gamma=0$, the functions $\rho(x,t)$ and $v(x,t)$ have to satisfy Eqs. (11) and (30). Moreover, the average energy of the system is then given by Eq. (37) and there is no reason for which it should be given by a different formula if $\gamma\neq 0$. Thus we state that the quantum-mechanical average energy of the considered dissipative system is given by Eq. (37).

Now we can assume that the system of quantum-mechanical evolution equations for $\rho(x,t)$ and $v(x,t)$ should be closed, because it is closed in the classical limit as well as for $\gamma=0$. Therefore as it is obvious that the continuity equation, Eq. (11), must be one of these equations, we only need to find the second equation allowing us to determine the time derivative of the probability velocity field $v(x,t)$. Of course we know that for $(\hbar/m)$ being negligibly small this equation should become Eq. (44) and for $\gamma=0$ it should be Eq. (30), so we can assert that in general

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{\gamma}{m}v(x,t) + \frac{F(x)}{m} + \frac{\hbar^2}{2m^2}\partial_x T_x[\rho(x,t)] - \frac{\gamma\hbar}{m^2}\Xi[\rho(x,t),v(x,t)], \tag{45}$$

where $T_x[f]$ is given by Eq. (31), $\Xi[\rho(x,t),v(x,t)]$ is unknown, and the last term on the right-hand side of the equation vanishes in the classical limit as well as for $\gamma=0$.

The observation that the role of the next-to-last term on the right-hand side of Eq. (45) is to cancel the contribution of the energy of concentration to the time derivative of the total average energy of the system, consistent with the discussion in Sec. II E, suggests strongly that $\Xi[\rho(x,t),v(x,t)]$ should be equal to zero, as there is no particular reason for its existence. Furthermore, there are less heuristic arguments indicating that $\Xi[\rho(x,t),v(x,t)]$ ought to vanish and they are presented in Appendix D. Accordingly, we postulate that $\Xi[\rho(x,t),v(x,t)]$ is identically equal to zero and get

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{\gamma}{m}v(x,t) + \frac{F(x)}{m} + \frac{\hbar^2}{2m^2}\partial_x T_x[\rho(x,t)], \tag{46}$$

which is the required second quantum-mechanical evolution equation for the considered system [23].

We note that the assumption allowing us to obtain Eq. (46) may be referred to as the *principle of classical dissipation*, because it leads to the conclusion that in quantum mechanics the formula for the time derivative of the system's average energy is always the same as in the classical limit [23]. Of course, the discussion in Appendix D nonwithstanding, it is only a postulate and requires empirical verification.

B. Energy of concentration in many dimensions

Having found the quantum-mechanical evolution equations for a simple one-dimensional dissipative system, Eqs. (11) and (46), we notice that the approach used to obtain them can be extended to multidimensional systems in a straightforward manner, since every multidimensional system may be regarded as an ensemble of one-dimensional systems, possibly not independent of each other. Namely, each Cartesian coordinate of the system's position can be treated as an individual entity subject to the most of the observations made in Secs. II and III A.

To illustrate this we consider a two-dimensional system consisting of two noninteracting bodies of masses m_1 and m_2 for which the following Newton equations are satisfied in classical mechanics:

$$\frac{d^2q_1}{dt^2} = -\frac{\gamma_1}{m_1}q_1 + \frac{F_1(q_1)}{m_1}, \tag{47}$$

$$\frac{d^2q_2}{dt^2} = -\frac{\gamma_2}{m_2}q_2 + \frac{F_2(q_2)}{m_2}, \tag{48}$$

where $F_1(x_1)$, $F_2(x_2)$ denote potential forces, and the constants γ_1, γ_2 are non-negative. Clearly, Eqs. (47) and (48) are independent of each other, so we can apply the approach used in Sec. III A to each of them individually. Hence we obtain two independent systems of evolution equations for two pairs of functions $\rho_1(x_1,t)$, $v_1(x_1,t)$ and $\rho_2(x_2,t)$, $v_2(x_2,t)$ that each consist of configurational probability densities and probability velocity fields. However, it is often necessary to use the joint probability density $\rho(x_1,x_2,t)$ of the positions of the first and second body.

For the considered system $\rho(x_1,x_2,t)$ can be defined as the product of $\rho_1(x_1,t)$ and $\rho_2(x_2,t)$, because the random variables that represent the positions of the two bodies can be regarded as statistically independent. Using this observation we are able to express the moments of the probability distributions of many physical quantities by means of $\rho(x_1,x_2,t)$. For instance, if $\rho(x_1,x_2,t)=\rho_1(x_1,t)\rho_2(x_2,t)$, the quantum-mechanical total average energy of the system, equal to the sum of the average energies of the two bodies it consists of, i.e.,

$$\langle E_q\rangle = \sum_{\alpha=1,2}\left\{\int dx_\alpha\left[\frac{m_\alpha}{2}v_\alpha^2(x_\alpha,t)+V_\alpha(x_\alpha)\right]\rho_\alpha(x_\alpha,t) + \frac{\hbar^2}{2m_\alpha}U_\alpha[\rho_\alpha(x_\alpha,t)]\right\}, \tag{49}$$

where $V_1(x_1)$ and $V_2(x_2)$ are the potentials of the forces $F_1(x_1)$ and $F_2(x_2)$, respectively, and

$$U_\alpha[f]=\int dx_\alpha[\partial_\alpha\sqrt{f}]^2, \quad \alpha=1,2, \tag{50}$$

can be represented as

$$\langle E_q\rangle = \sum_{\alpha=1,2}\left\{\int d^2x\left[\frac{m_\alpha}{2}v_\alpha^2(x_\alpha,t)+V_\alpha(x_\alpha)\right]\rho(x_1,x_2,t) + \langle\delta E\rangle_\alpha\right\}, \tag{51}$$

where

$$\langle\delta E\rangle_\alpha = \frac{\hbar^2}{2m_\alpha}\int d^2x[\partial_\alpha\sqrt{\rho(x_1,x_2,t)}]^2, \quad \alpha=1,2, \tag{52}$$

because we have

$$U_\alpha[\rho_\alpha(x_\alpha,t)] = \int d^2x[\partial_\alpha\sqrt{\rho(x_1,x_2,t)}]^2, \quad \alpha=1,2. \tag{53}$$

Hence the total energy of concentration of the system is the sum of $\langle\delta E\rangle_1$ and $\langle\delta E\rangle_2$.

The evolution equations for the considered system can be expressed in terms of $\rho(x_1,x_2,t)$ too. Namely, it follows from the considerations in Appendix E that the approach presented in Sec. II B is applicable to $\rho(x_1,x_2,t)$ and the associated two-dimensional velocity field. In particular, for the considered system it leads to the evolution equations

$$\partial_t\rho(x_1,x_2,t) = -\sum_{\alpha=1,2}\partial_\alpha[v_\alpha(x_1,x_2,t)\rho(x_1,x_2,t)], \tag{54}$$

$$\partial_t v_1(x_1,x_2,t) + \sum_{\alpha=1,2} v_\alpha(x_1,x_2,t)\partial_\alpha v_1(x_1,x_2,t) = -\frac{\gamma_1}{m_1}v_1(x_1,x_2,t) + \frac{F_1(x_1)}{m_1}, \tag{55}$$

$$\partial_t v_2(x_1,x_2,t) + \sum_{\alpha=1,2} v_\alpha(x_1,x_2,t)\partial_\alpha v_2(x_1,x_2,t) = -\frac{\gamma_2}{m_2}v_2(x_1,x_2,t) + \frac{F_2(x_2)}{m_2}, \tag{56}$$

where $v_1(x_1,x_2,t)$ and $v_2(x_1,x_2,t)$ are the components of the velocity field. Now, on the basis of the discussion in Sec. III A and Appendix D we can expect that in order to obtain the quantum-mechanical counterparts of Eqs. (55) and (56) it is only necessary to complement these equations with the terms that will cancel the contribution of the energy of concentration to the time derivative of the system's total average energy.

To determine these terms we differentiate the sum of $\langle\delta E\rangle_1$ and $\langle\delta E\rangle_2$ with respect to time and use the continuity equation, Eq. (54), to eliminate the time derivatives of $\rho(x_1,x_2,t)$ from the resulting equation. Hence we get

$$\frac{d}{dt}(\langle\delta E\rangle_1+\langle\delta E\rangle_2) = -\frac{\hbar^2}{2}\int d^2x\rho(x_1,x_2,t) \times\left\{\sum_{\alpha=1,2}v_\alpha(x_1,x_2,t)\partial_\alpha S[\rho(x_1,x_2,t)]\right\}, \tag{57}$$

where

$$S[f]=\frac{T_1[f]}{m_1}+\frac{T_2[f]}{m_2}, \tag{58}$$

and the operators $T_1[f]$ and $T_2[f]$ are defined in analogy with Eq. (31). Thus we see that the quantum-mechanical counterparts of Eqs. (55) and (56) should be the following equations:

$$\partial_t v_1 + \sum_{\alpha=1,2} v_\alpha\partial_\alpha v_1 = -\frac{\gamma_1}{m_1}v_1 + \frac{F_1}{m_1} + \frac{\hbar^2}{2m_1}\partial_1 S[\rho], \tag{59}$$

$$\partial_t v_2 + \sum_{\alpha=1,2} v_\alpha\partial_\alpha v_2 = -\frac{\gamma_2}{m_2}v_2 + \frac{F_2}{m_2} + \frac{\hbar^2}{2m_2}\partial_2 S[\rho], \tag{60}$$

respectively. It can be verified by direct computation that if $\rho(x_1,x_2,t)$ is the product of $\rho_1(x_1,t)$ and $\rho_2(x_2,t)$, Eqs. (54), (59), and (60) are exactly equivalent to the two systems of evolution equations for $\rho_1(x_1,t)$, $v_1(x_1,t)$ and $\rho_2(x_2,t)$, $v_2(x_2,t)$ obtainable by applying the approach used in Sec. III A to each of Eqs. (47) and (48) individually. However, if $\rho(x_1,x_2,t)$ cannot be factorized into $\rho_1(x_1,t)$ and $\rho_2(x_2,t)$, this equivalence is no longer retained, since the above considerations are based on Eq. (53) which generally is not valid if $\rho(x_1,x_2,t)$ cannot be factorized. Therefore the question arises whether the contribution corresponding to the system's Cartesian configurational coordinate x_α to its total energy of concentration is generally given by Eq. (52), or it should rather be given by $U_\alpha[\rho_\alpha(x_\alpha,t)]$.

To answer this question we consider a system consisting of the same two bodies in which all forces depend on both their positions and are potential. Of course in classical mechanics the following Newton equations are satisfied for it:

$$\frac{d^2q_1}{dt^2}=\frac{F_1(q_1,q_2)}{m_1}, \tag{61}$$

$$\frac{d^2q_2}{dt^2}=\frac{F_2(q_1,q_2)}{m_2}, \tag{62}$$

where $F_1(x_1,x_2)=-\partial_1 V(x_1,x_2)$ and $F_2(x_1,x_2)=-\partial_2 V(x_1,x_2)$. Thus if the approach used in Sec. II B and Appendix E is applied to this system, we find that Eq. (54), the continuity equation, is satisfied along with the following pair of equations:

$$\partial_t v_1(x_1,x_2,t)+\sum_{\alpha=1,2} v_\alpha(x_1,x_2,t)\partial_\alpha v_1(x_1,x_2,t)=\frac{F_1(x_1,x_2)}{m_1}, \tag{63}$$

$$\partial_t v_2(x_1,x_2,t)+\sum_{\alpha=1,2} v_\alpha(x_1,x_2,t)\partial_\alpha v_2(x_1,x_2,t)=\frac{F_2(x_1,x_2)}{m_2}. \tag{64}$$

Next, if the energy of concentration is the sum of $\langle\delta E\rangle_1$ and $\langle\delta E\rangle_2$, its time derivative is given by Eq. (57), which follows directly from the continuity equation. Accordingly, on the basis of our discussion in Sec. III A and Appendix D, we expect the quantum-mechanical counterparts of Eqs. (63) and (64) to be given by

$$\partial_t v_1(x_1,x_2,t)+\sum_{\alpha=1,2} v_\alpha(x_1,x_2,t)\partial_\alpha v_1(x_1,x_2,t)=\frac{F_1(x_1,x_2)}{m_1}+\frac{\hbar^2}{2m_1}\partial_1 S[\rho], \tag{65}$$

$$\partial_t v_2(x_1,x_2,t)+\sum_{\alpha=1,2} v_\alpha(x_1,x_2,t)\partial_\alpha v_2(x_1,x_2,t)=\frac{F_2(x_1,x_2)}{m_2}+\frac{\hbar^2}{2m_2}\partial_2 S[\rho]. \tag{66}$$

Indeed, it is straightforward to verify that the following Schrödinger equation:

$$i\hbar\dot{\psi}(x_1,x_2,t)=\left[-\sum_{\alpha=1,2}\frac{\hbar^2}{2m_\alpha}\partial_\alpha^2+V(x_1,x_2)\right]\psi(x_1,x_2,t), \tag{67}$$

leads to Eqs. (54), (65), and (66). Moreover, if $F_1(x_1,x_2)$ is actually independent of x_2, and $F_2(x_1,x_2)$ is independent of x_1, it leads to Eqs. (54), (59), and (60) for $\gamma_1=\gamma_2=0$, independently of whether $\rho(x_1,x_2,t)$ can be factorized or not. Therefore the assumption that the total energy of concentration of the system is the sum of $\langle\delta E\rangle_1$ and $\langle\delta E\rangle_2$ leads to the results that are consistent with the standard equations of quantum mechanics. Consequently, we can conclude that the contribution of the system's degree of freedom represented by x_α to its total energy of concentration should always be given by the right-hand side of Eq. (52), which is only equal to $U_\alpha[\rho_\alpha(x_\alpha,t)]$ in special cases.

It is not difficult to verify that the above discussion can be repeated for a system of any number of dimensions, so we can assume that in general the total energy of concentration of an n-dimensional system is given by

$$\langle\delta E\rangle=\sum_{\alpha=1}^{n}\frac{\hbar^2}{2m_\alpha}\int d^n x[\partial_\alpha\sqrt{\rho(\mathbf{x},t)}]^2, \tag{68}$$

which is a trivial generalization of the sum of $\langle\delta E\rangle_1$ and $\langle\delta E\rangle_2$ defined by Eq. (52) for $\alpha=1$, 2, respectively. Therefore we have

$$\frac{d}{dt}\langle\delta E\rangle=-\frac{\hbar^2}{2}\int d^n x\rho(\mathbf{x},t)\left\{\sum_{\alpha=1}^{n} v_\alpha(\mathbf{x},t)\partial_\alpha S[\rho(\mathbf{x},t)]\right\}, \tag{69}$$

where

$$S[f]=\sum_{\alpha=1}^{n}\frac{1}{m_\alpha}\frac{\partial_\alpha^2\sqrt{f}}{\sqrt{f}}, \tag{70}$$

because the continuity equation, Eq. (2), must be satisfied for any system, as stated in Sec. II A. Consequently, for each component of the probability velocity field $v_\alpha(\mathbf{x},t)$ the evolution equation allowing us to determine its time derivative ought to contain the term

$$Q_\alpha[\rho(\mathbf{x},t)]=\frac{\hbar^2}{2m_\alpha}\partial_\alpha S[\rho(\mathbf{x},t)], \tag{71}$$

that cancels one component of the sum on the right-hand side of Eq. (69) in the time derivative of the total average energy of the system.

C. Hydrodynamic quantization rules

On the basis of the observations made in Sec. III B we can establish a simple set of quantization rules applicable to virtually any mechanical system. Yet, as follows from the considerations in Secs. III A and III B, in order to use these observations to obtain quantum-mechanical evolution equations we first need to derive the corresponding classical-mechanical equations for the probability liquid. Of course, they may be derived directly from the Newton equations, which is shown, for example, in Appendix E, but this approach is rather tedious. Fortunately, instead of using it we can make some more observations that allow us to simplify this step substantially.

First of all we notice that the classical-mechanical evolution equations for the probability liquid have always the same structure, independently of the number of dimensions of the system. Namely, there always is the continuity equation which is mandatory, as follows from Sec. II A. There also are the equations for the time derivatives of n components of the velocity field, which are quite similar to each other. Specifically, for each component of the velocity field $v_\alpha(\mathbf{x},t)$, $\alpha=1,2,\ldots,n$, the corresponding evolution equation contains not only the time derivative of $v_\alpha(\mathbf{x},t)$, but the whole total derivative of it, which for an arbitrary function $\phi(\mathbf{x},t)$ is defined by [28]

$$\frac{d}{dt}\phi(\mathbf{x},t)=\partial_t\phi(\mathbf{x},t)+\sum_{\alpha=1}^{n}v_\alpha(\mathbf{x},t)\partial_\alpha\phi(\mathbf{x},t). \tag{72}$$

Moreover, if the total derivatives of $v_\alpha(\mathbf{x},t)$ are placed on the right-hand sides of the respective evolution equations, on the left-hand sides of these equations there only are the respective forces that affect the evolution of $v_\alpha(\mathbf{x},t)$, possibly expressed by $v_\alpha(\mathbf{x},t)$ themselves and by the other components of the velocity field, which follows from the discussion in Appendix E.

On the other hand, in classical mechanics we can always introduce the function $\mathbf{v}(t)$ defined as the time derivative of the system's position at the time t which can be referred to as the velocity function of the system. Then, as demonstrated in Appendix E, the Newton equations of motion can be represented as a system of $2n$ first-order differential equations with respect to time, where the first n equations define n components of the velocity function $v_\alpha(t)$, $\alpha=1,2,\ldots,n$, and the remaining n equations contain each the time derivatives of $v_\alpha(t)$ on the left-hand sides and the respective forces that affect the evolution of $v_\alpha(t)$, possibly expressed by $v_\alpha(t)$ themselves and by the other components of the velocity function, on the right-hand sides. Thus if (i) the first n equations are replaced with the continuity equation, (ii) in the remaining n equations the time derivative of each $v_\alpha(t)$ is replaced with the total derivative of the corresponding velocity field's component $v_\alpha(\mathbf{x},t)$, (iii) each $v_\alpha(t)$ is replaced with the corresponding $v_\alpha(\mathbf{x},t)$, and (iv) each component of the system's position at the time t is replaced with the corresponding configurational coordinate of the system x_α, we get the required classical-mechanical equations for the probability liquid.

Having shown that the necessary classical-mechanical equations for the probability liquid can be obtained relatively easily, we are now able to propose the following rules of quantization of mechanical systems:

(i) Given the Newton equations for the considered n-dimensional mechanical system one should represent them as $2n$ first-order differential equations with respect to time for the system's position at the time t, $\mathbf{q}(t)$, and its derivative, $\mathbf{v}(t)$, i.e.,

$$\frac{d}{dt}q_\alpha=v_\alpha,\quad \alpha=1,2,\ldots,n, \tag{73}$$

$$\frac{d}{dt}v_\alpha=\frac{F_\alpha(\mathbf{q},\mathbf{v},t)}{m_\alpha},\quad \alpha=1,2,\ldots,n. \tag{74}$$

(ii) The first n equations defining the components of $\mathbf{v}(t)$, Eqs. (73), ought to be replaced with the continuity equation, Eq. (2).

(iii) In each of the remaining equations, Eqs. (74), (i) the time derivative of $v_\alpha(t)$ should be replaced with the total derivative of the corresponding velocity field's component $v_\alpha(\mathbf{x},t)$, (ii) the system's position $\mathbf{q}(t)$ should be replaced with $\mathbf{x}$, and (iii) $\mathbf{v}(t)$ should be replaced with the probability velocity field $\mathbf{v}(\mathbf{x},t)$. This leads to the following result:

$$\frac{d}{dt}v_\alpha(\mathbf{x},t)=\frac{F_\alpha(\mathbf{x},\mathbf{v}(\mathbf{x},t),t)}{m_\alpha},\quad \alpha=1,2,\ldots,n. \tag{75}$$

(iv) One ought to modify the equations for the total derivatives of the velocity field's components $v_\alpha(\mathbf{x},t)$, Eqs. (75), by adding the respective term $Q_\alpha[\rho(\mathbf{x},t)]$ defined by Eq. (71) to the right-hand side of each of them.

Simultaneously, the rules leading to the formula for the average energy of the system are the following:

(i) In the classical-mechanical formula for the energy of the system one should replace the system's position at the time t, $\mathbf{q}(t)$, with $\mathbf{x}$, and its derivative, $\mathbf{v}(t)$, with the probability velocity field $\mathbf{v}(\mathbf{x},t)$.

(ii) The formula obtained in the previous step ought to be multiplied by the system's configurational probability density $\rho(\mathbf{x},t)$ and integrated over the entire configurational space of the system.

(iii) The total energy of concentration of the system given by Eq. (68) should be added to the formula obtained in the previous step.

We note that the evolution equations resulting from the application of the above quantization rules are similar to well-known equations of hydrodynamics [1,22,23], so this approach may be referred to as the *hydrodynamic quantization* of mechanical systems.

IV. EXAMPLES

A. One-dimensional dissipative system

As the first example we consider the application of the hydrodynamic quantization procedure described in Sec. III C to a one-dimensional system consisting of one body of mass m for which the Newton equation of motion is given by

$$\frac{d^2q}{dt^2}=-\frac{\gamma^{1+\kappa}}{m}\frac{dq}{dt}\left|\frac{dq}{dt}\right|^{\kappa}+\frac{F(q)}{m}, \tag{76}$$

where γ and κ are positive constants, and the force $F(x)$ is potential.

The first two steps of the hydrodynamic quantization lead to the continuity equation, Eq. (11), and by completing the third step of the procedure we get

$$\partial_t v(x,t)+v(x,t)\partial_x v(x,t)=-\frac{\gamma^{1+\kappa}}{m}v(x,t)|v(x,t)|^{\kappa}+\frac{F(x)}{m}. \tag{77}$$

Next, if Eq. (77) is complemented with the appropriate term $Q_\alpha[\rho(\mathbf{x},t)]$, which in the present case is equal to $(\hbar^2/2m^2)\partial_x T_x[\rho(x,t)]$, where the operator $T_x[f]$ is defined by Eq. (31), we obtain

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{\gamma^{1+\kappa}}{m} v(x,t)|v(x,t)|^{\kappa} + \frac{F(x)}{m} + \frac{\hbar^2}{2m^2}\partial_x\left[\frac{\partial_x^2\sqrt{\rho(x,t)}}{\sqrt{\rho(x,t)}}\right]. \quad (78)$$

Of course, according to the quantization procedure, the average energy of the system is then given by Eq. (37), which is consistent with our considerations in Sec. II.

We note that if γ vanishes, Eq. (78) becomes Eq. (30) and we get the system of equations obtained in Sec. II D. Moreover, if γ does not vanish, but κ is equal to zero, we get the equations obtained in Sec. III A that allow us to derive the Schrödinger-Langevin equation for the system's wave function defined by Eq. (28) [6,10,23,30]. If both γ and κ are different to zero, we are still able to obtain an evolution equation for the system's wave function, but it is a complicated differential-integral equation that cannot be easily solved.

B. Two-dimensional dissipative system

Having presented the application of the hydrodynamic quantization to a one-dimensional system, we now apply it to a two-dimensional dissipative system. Namely, we consider a system for which the following Newton equations are satisfied in classical mechanics:

$$\frac{d^2q_1}{dt^2} = \frac{1}{m_1}\left[-\gamma\frac{dq_1}{dt} + B\frac{dq_2}{dt} + F_1(q_1,q_2)\right], \quad (79)$$

$$\frac{d^2q_2}{dt^2} = \frac{1}{m_2}\left[-\gamma\frac{dq_2}{dt} - B\frac{dq_1}{dt} + F_2(q_1,q_2)\right], \quad (80)$$

where γ and B are non-negative constants, and $\mathbf{F}(q_1,q_2)$ is a potential force.

If the hydrodynamic quantization procedure is applied to this system, we find that the two-dimensional continuity equation, Eq. (54), is satisfied along with the following pair of equations:

$$\partial_t v_1 + \sum_{\alpha=1,2} v_\alpha\partial_\alpha v_1 = \frac{1}{m_1}\left\{-\gamma v_1 + Bv_2 + F_1 + \frac{\hbar^2}{2}\partial_1 S[\rho]\right\}, \quad (81)$$

$$\partial_t v_2 + \sum_{\alpha=1,2} v_\alpha\partial_\alpha v_2 = \frac{1}{m_2}\left\{-\gamma v_2 - Bv_1 + F_2 + \frac{\hbar^2}{2}\partial_2 S[\rho]\right\}, \quad (82)$$

where $S[f]$ is defined by Eq. (58). At the same time the average energy of the system is given by

$$\langle E\rangle = \int d^2x\left[\sum_{\alpha=1,2}\frac{m_\alpha}{2}v_\alpha^2(x_1,x_2,t) + V(x_1,x_2)\right]\rho(x_1,x_2,t) + \sum_{\alpha=1,2}\frac{\hbar^2}{2m_\alpha}\int d^2x[\partial_\alpha\sqrt{\rho(x_1,x_2,t)}]^2, \quad (83)$$

where $V(x_1,x_2)$ denotes the potential of the force $\mathbf{F}(x_1,x_2)$. Of course, if $V(x_1,x_2)=V_1(x_1)+V_2(x_2)$ this equation is practically equivalent to Eq. (51), which is consistent with our considerations in Sec. III B.

Now, it follows from the discussion in Appendix F that if $B=0$, the time derivative of the quantity Γ_C defined by

$$\Gamma_C = \oint_C d\mathbf{s}\cdot\mathbf{p}(s_1,s_2,t), \quad (84)$$

where C is a closed contour in the system's configurational space, and $p_\alpha(x_1,x_2,t)=m_\alpha v_\alpha(x_1,x_2,t)$, $\alpha=1$, 2, is proportional to the circulation of the velocity field. Thus if the initial circulation of the velocity field is equal to zero and Γ_C vanishes initially for any closed contour C, we can define the system's wave function in the following way:

$$\psi(x_1,x_2,t) = \sqrt{\rho(x_1,x_2,t)}\exp\left[\frac{i}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{s}\cdot\mathbf{p}(s_1,s_2,t)\right], \quad (85)$$

where $\mathbf{x}_0$ is a fixed point in the system's configurational space and $\mathbf{x}=(x_1,x_2)$. Then, we can use Eqs. (54), (81), and (82) to derive the evolution equation for $\psi(x_1,x_2,t)$, which is the known Schrödinger-Langevin equation [6,10].

If $B\neq 0$, but $\gamma=0$, the quantity $\Gamma_C(B)$ defined by

$$\Gamma_C(B) = \oint_C d\mathbf{s}\cdot[\mathbf{p}(s_1,s_2,t) + \mathbf{A}(s_1,s_2)], \quad (86)$$

where $\mathbf{A}(x_1,x_2)=(B/2)(-x_2,x_1)$, is a constant of motion for any closed contour C in the system's configurational space, which is shown in Appendix F. Then, if $\Gamma_C(B)$ is initially equal to $2\pi n\hbar$, where n is an integer, for any closed contour C, we can define $\psi(x_1,x_2,t)$ in analogy with Eq. (85), i.e.,

$$\psi(x_1,x_2,t) = \sqrt{\rho(x_1,x_2,t)}\exp\left\{\frac{i}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{s}\cdot[\mathbf{p}(s_1,s_2,t) + \mathbf{A}(s_1,s_2)]\right\}. \quad (87)$$

Next, we can use Eqs. (54), (81), and (82) to obtain the following Schrödinger equation for $\psi(x_1,x_2,t)$:

$$i\hbar\partial_t\psi = \frac{1}{2m_1}\left(\frac{\hbar}{i}\partial_1 + \frac{B}{2}x_2\right)^2\psi + \frac{1}{2m_2}\left(\frac{\hbar}{i}\partial_2 - \frac{B}{2}x_1\right)^2\psi + V\psi, \quad (88)$$

which may be verified by deriving the evolution equations for the amplitude and phase of the wave function from this equation and using Eq. (87) to relate these quantities to the configurational probability density and probability velocity field of the system [1].

Of course, Eq. (88) is very similar to the Schrödinger equation for a particle in magnetic field, since the force represented by the terms proportional to B in Eqs. (79) and (80) has been defined in analogy with the Lorentz force. Still, in principle this force need not be a magnetic one. In fact, it may be an arbitrary force that is always perpendicular to the system's velocity field.

If both γ and B are different than zero, neither Γ_C nor $\Gamma_C(B)$ can be independent of time unless the velocity field is constant and equal to zero, which is demonstrated in Appendix F. For this reason, if both γ and B are different to zero, it is impossible to define the system's wave function and, consequently, to use the Schrödinger equation or another wave equation for describing its evolution.

At first sight this may seem to be a serious issue, but fortunately it can be easily understood. Namely, as far as dissipative systems are concerned, we can always assume that they are in contact with another, possibly large, system called the *environment* and they lose energy as a result of some, possibly complicated, interactions with it [6,7]. Moreover, in principle it is possible to treat the system and the environment together as yet another, larger system called the *universe*, which can be regarded as nondissipative [6,7,9]. Then, it is reasonable to expect that the evolution equations for the dissipative system under consideration can be obtained by averaging the evolution equations for the universe over the environmental degrees of freedom. It is even possible to prove this for some simple models of the interactions between the system and the environment [6,7,9]. Therefore in general we should rather not expect any dissipative system to have a well-defined wave function, as this would mean that the system's evolution is in fact independent of the evolution of the environment. Accordingly, it is plausible that no wave function can be defined for a dissipative system. On the other hand, if the considered system is not dissipative, it is possible to obtain the Schrödinger equation for it with the help of Eqs. (54), (81), and (82), provided that the system's probability velocity field satisfies specific condition at the initial instant of time.

C. Dissipative system in magnetic field

To provide yet another example application of the quantization procedure described in Sec. III we consider a system consiting of one body of mass m and electric charge e that moves in a three-dimensional space and is acted on by electromagnetic forces as well as by a friction force proportional to its velocity.

In classical mechanics we have the following Newton equation for the system:

$$\frac{d^2\mathbf{q}}{dt^2} = \frac{e}{m}\left[\mathbf{E}(\mathbf{q}) + \frac{d\mathbf{q}}{dt} \times \mathbf{B}(\mathbf{q})\right] - \frac{\gamma}{m}\frac{d\mathbf{q}}{dt}, \tag{89}$$

where γ is a non-negative constant, and $\mathbf{E}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$ denote the electric and magnetic fields, respectively. Therefore by applying the hydrodynamic quantization procedure to the system we get the following equations:

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) = -\boldsymbol{\nabla}\cdot[\rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)], \tag{90}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{v}(\mathbf{x},t) &+ [\mathbf{v}(\mathbf{x},t)\cdot\boldsymbol{\nabla}]\mathbf{v}(\mathbf{x},t)\\ &= \frac{e}{m}[\mathbf{E}(\mathbf{x}) + \mathbf{v}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x})] - \frac{\gamma}{m}\mathbf{v}(\mathbf{x},t)\\ &+ \frac{\hbar^2}{2m^2}\nabla\left[\frac{1}{\sqrt{\rho(\mathbf{x},t)}}\triangle\sqrt{\rho(\mathbf{x},t)}\right],\end{aligned} \tag{91}$$

which is a known result [21]. Moreover, we find that the average energy of the system is given by

$$\langle E\rangle = \int d^3x\left\{\left[\frac{m}{2}\mathbf{v}^2(\mathbf{x},t) + e\ \Phi(\mathbf{x})\right]\rho(\mathbf{x},t) + \frac{\hbar^2}{2m}[\nabla\sqrt{\rho(\mathbf{x},t)}]^2\right\}, \tag{92}$$

where $\Phi(\mathbf{x})$ is the potential of the electric field $\mathbf{E}(\mathbf{x})$.

If $\gamma=0$, Eqs. (90) and (91) become the well-known hydrodynamic equations of quantum mechanics for a system in magnetic field [1,27,29]. Then, it is possible to obtain the Schrödinger-Langevin equation from these equations provided that the quantity $\Gamma_C(B)$, defined in analogy with Eq. (86), is equal to $2\pi n\hbar$ for any C at the initial instant of time [1].

If the magnetic field $\mathbf{B}(\mathbf{x})$ vanishes, it is possible to obtain the Schrödinger equation with the help of Eqs. (90) and (91) if the quantity Γ_C, defined in analogy with Eq. (84), is equal to zero for any C at the initial instant of time.

If $\gamma\neq 0$ and $\mathbf{B}(\mathbf{x})$ does not vanish, the system's wave function is generally not well defined, because neither $\Gamma_C(B)$, nor Γ_C is independent of time, unless the velocity field is constant and equal to zero. Consequently, in that case it is generally impossible to use any wave equation for describing the system [21]. Still, to explain this we can use the same arguments that have been used in Sec. IV B.

V. STATIONARY STATES

A. Stationary states of dissipative systems

In Sec. IV we have applied the hydrodynamic quantization proceure to some dissipative systems and we have obtained evolution equations that become well-known equations of quantum mechanics when the dissipation of energy in these systems is negligible. In principle these equations can be used to analyze the behavior of the corresponding systems from the quantum-mechanical point of view, but unfortunately they are heavily nonlinear and therefore generally difficult to solve. Yet, their solutions that correspond to stationary states of the systems can be found relatively easily.

To show this we first observe that for any dissipative system there is a state of minimal energy, such that if the system is in this state, its energy cannot decrease any more. In classical mechanics this state corresponds to the global minimum of the system's potential energy, if one exists [31]. In quantum mechanics it should correspond to the minimum of the system's average energy regarded as a functional of its configurational probability density $\rho(\mathbf{x},t)$ and probability velocity field $\mathbf{v}(\mathbf{x},t)$.

It follows from the discussion in Sec. III C that in quantum mechanics the average energy of a typical n-dimensional system is given by

$$\begin{aligned}\langle E\rangle = \int d^nx\Bigg\{&\left[\sum_{\alpha=1}^{n}\frac{m_\alpha}{2}v_\alpha^2(\mathbf{x},t) + V(\mathbf{x})\right]\rho(\mathbf{x},t)\\ &+ \sum_{\alpha=1}^{n}\frac{\hbar^2}{2m_\alpha}[\partial_\alpha\sqrt{\rho(\mathbf{x},t)}]^2\Bigg\},\end{aligned} \tag{93}$$

where $V(\mathbf{x})$ is the combined potential of all potential forces in the system, including the forces of interactions between its components. Hence the contribution of the velocity field to system's average energy is always non-negative, so if we find the minimum of $\langle E\rangle$ for $\mathbf{v}(\mathbf{x},t)$ equal to zero, it will be lesser than any value of $\langle E\rangle$ for a nonzero velocity field. Thus we only need to find the minimum of the average energy for $\mathbf{v}(\mathbf{x},t)$ equal to zero, in which case it can be treated as a functional of $\rho(\mathbf{x},t)$.

For this purpose it is convenient to denote the square root of $\rho(\mathbf{x},t)$ by $\varphi(\mathbf{x})$, omitting the irrelevant time variable, and regard $\langle E\rangle$ as a functional of it. Then, to find the minimum of the average energy for $\mathbf{v}(\mathbf{x},t)$ equal to zero, we need to find $\varphi(\mathbf{x})$ for which the functional

$$E[\varphi] = \int d^n x \left\{ V(\mathbf{x})\varphi^2(\mathbf{x}) + \sum_{\alpha=1}^{n} \frac{\hbar^2}{2m_\alpha}[\partial_\alpha \varphi(\mathbf{x})]^2 \right\} \quad (94)$$

attains its minimal value under the additional condition that the integral of $\varphi^2(\mathbf{x})$ over the whole available space must be equal to 1. The necessary condition of the existence of such a minimum is the vanishing of the functional derivative with respect to $\varphi(\mathbf{x})$ of the functional

$$I[\varphi] = \int d^n x \left\{ [V(\mathbf{x}) - E_\lambda]\varphi^2(\mathbf{x}) + \sum_{\alpha=1}^{n} \frac{\hbar^2}{2m_\alpha}[\partial_\alpha \varphi(\mathbf{x})]^2 \right\}, \quad (95)$$

where E_λ is an unknown constant that has to be determined along with $\varphi(\mathbf{x})$.

The functional derivative of $I[\varphi]$ with respect to $\varphi(\mathbf{x})$ is given by

$$\frac{\delta I[\varphi]}{\delta \varphi(\mathbf{x})} = 2\left[V(\mathbf{x}) - E_\lambda - \sum_{\alpha=1}^{n} \frac{\hbar^2}{2m_\alpha}\partial_\alpha^2 \right]\varphi(\mathbf{x}), \quad (96)$$

so $E[\varphi]$ can only attain its minimum for $\varphi_\lambda(\mathbf{x})$ that satisfies the following Schrödinger equation without time:

$$\left[-\sum_{\alpha=1}^{n} \frac{\hbar^2}{2m_\alpha}\partial_\alpha^2 + V(\mathbf{x}) \right]\varphi(\mathbf{x}) = E_\lambda \varphi(\mathbf{x}). \quad (97)$$

For a typical boundary conditions this equation has a countable set of real-valued solutions $\varphi_\lambda(\mathbf{x})$ that generally correspond to different constants E_λ, and the set of all E_λ is lower bounded [2]. Moreover, we have

$$E[\varphi_\lambda] = E_\lambda, \quad (98)$$

which follows directly from Eqs. (94) and (97), so we can state that the minimal average energy of the system is equal to the least of E_λ. Thus if the least of E_λ is denoted by E_0, the configurational probability density for which the system's average energy attains its minimal value is equal to $\varphi_0^2(\mathbf{x})$. Moreover, since the coefficient that multiplies $[\partial_\alpha \varphi(\mathbf{x})]^2$ in Eq. (95) is positive and independent of $\mathbf{x}$, each of $\varphi_\lambda(\mathbf{x})$ corresponds to a local minimum of the system's average energy.

Now, we assume that the initial velocity field $\mathbf{v}(\mathbf{x},t_0)$ is equal to zero and the initial configurational probability density of the system $\rho(\mathbf{x},t_0)$ is equal to the square of an arbitrary real-valued solution of Eq. (97), $\varphi_\lambda(\mathbf{x})$. Then, it follows from the continuity equation, Eq. (2), that the time derivative of $\rho(\mathbf{x},t)$ at the time t_0 vanishes, so the configurational probability density does not change at this instant of time. Moreover, if zero and $\varphi_\lambda^2(\mathbf{x})$ are substituted for $\mathbf{v}(\mathbf{x},t_0)$ and $\rho(\mathbf{x},t_0)$, respectively, in the evolution equations for $v_\alpha(\mathbf{x},t)$, we get

$$\partial_t v_\alpha(\mathbf{x},t)_{t=t_0} = \frac{F_\alpha(\mathbf{x},0)}{m_\alpha} + \frac{\hbar^2}{2m_\alpha}\partial_\alpha\left[\sum_{\beta=1}^{n} \frac{1}{m_\beta}\frac{\partial_\beta^2 \varphi_\lambda(\mathbf{x})}{\varphi_\lambda(\mathbf{x})} \right], \quad \alpha = 1,2,\ldots,n, \quad (99)$$

where we have assumed that the force does not explicitly depend on time. Hence, since in a typical situation we can assume that $F_\alpha(\mathbf{x},0) = -\partial_\alpha V(\mathbf{x})$, we have

$$\partial_t v_\alpha(\mathbf{x},t)_{t=t_0} = -\frac{1}{m_\alpha}\partial_\alpha\left[V(\mathbf{x}) - \sum_{\beta=1}^{n} \frac{\hbar^2}{2m_\beta}\frac{\partial_\beta^2 \varphi_\lambda(\mathbf{x})}{\varphi_\lambda(\mathbf{x})} \right], \quad \alpha = 1,2,\ldots,n, \quad (100)$$

and it follows from Eq. (97) that the expression in the square bracket is a constant. Thus we find that the time derivative of $\mathbf{v}(\mathbf{x},t)$ at the time t_0 vanishes, so the velocity field does not change at this instant of time too. Consequently, since neither $\rho(\mathbf{x},t)$ nor $\mathbf{v}(\mathbf{x},t)$ changes at the time t_0, they remain constant with respect to time and the system does not evolve. It is therefore in a stationary state represented by the time-independent state probability density $s(\mathbf{x},\mathbf{u}) = \varphi_\lambda^2(\mathbf{x})\delta(\mathbf{u})$.

We can conclude that for a quantum-mechanical dissipative system every real-valued solution of Eq. (97) corresponds to a stationary state of it and, simultaneously, to a local minimum of its average energy. Moreover, in a typical situation there are no other stationary states of the system, because it starts to dissipate energy once its velocity field is different than zero. Thus contrary to dissipative systems in classical mechanics quantum-mechanical dissipative systems have many stationary states in which they can stay forever. Still, this is consistent with the observation made in the early days of the quantum theory that there were stable "orbits" in which the quantum system could stay forever, although in the same physical situation its classical counterpart would continuously dissipate energy and would end up in the minimum of its potential energy [1].

It is important to stress that we do not need to use the evolution equations to determine the system's stationary states, because they correspond to local minima of its average energy. We have only used the evolution equations to show that these states are really stationary, but in fact even this is not necessary. Namely, if the system is initially in a state that corresponds to a local minimum of its average energy, its state cannot change until it is provided with some additional energy, so for a dissipative system this state must be stationary. Thus the existence of the system's stationary states as well as their representation are consequences of the fact that the system's average energy depends on the configurational probability density in a specific way. Therefore the existence of these states is strictly related to the existence of the energy of concentration.

B. Dissipative vs nondissipative systems

As far as stationary states are concerned, quantum-mechanical nondissipative systems can be divided into two distinct categories comprising, respectively, the systems in which there are forces that are always perpendicular to the system's probability velocity field, and the systems in which there are no such forces.

It turns out that the stationary states of the systems belonging the second category can generally be determined with the help of Eq. (97), although for a nondissipative system they correspond to complex-valued as well as to real-valued solutions of this equation [1].

The real valued solutions of Eq. (97) that we denote by $\varphi_\lambda(\mathbf{x})$ correspond to the stationary states of the system in which the probability velocity field is equal to zero, represented by the state probability densities $s(\mathbf{x},\mathbf{u})=\varphi_\lambda^2(\mathbf{x})\delta(\mathbf{u})$. Thus they also correspond to the stationary states of many different quantum-mechanical dissipative systems that are acted on by the same set of potential forces. Hence a quantum-mechanical nondissipative system being in one of these states is practically indistinguishable from a dissipative system subject to the same set of potential forces. Generally, as long as the probability velocity field is identically equal to zero, all quantum-mechanical systems acted on by the same set of potential forces are indistinguishable.

The complex-valued solutions of Eq. (97) that we denote by $\phi_\lambda(\mathbf{x})$ correspond to the stationary states of the system in which the probability velocity field is nonzero. Of course in those states the divergence of the product of the configurational probability density $\rho(\mathbf{x})$ and velocity field $\mathbf{v}(\mathbf{x})$ has to vanish so that $\rho(\mathbf{x})$ can be independent of time, which is only possible in two or more dimensions. Moreover, as we show in Appendix G, $\phi_\lambda(\mathbf{x})$ correspond to the local minima of the system's average energy determined under the additional condition that for any closed contour C in the system's configurational space

$$\oint_C d\mathbf{s}\cdot\mathbf{p}(\mathbf{s}) = 2\pi k\hbar, \quad k=\pm 1,\pm 2,\ldots, \tag{101}$$

where $p_\alpha(\mathbf{x})=m_\alpha v_\alpha(\mathbf{x})$, and the scalar product is defined in a standard way. Accordingly, the relationship between $\phi_\lambda(\mathbf{x})$ and the pair of quantities $\rho(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ is such that the components of the velocity field can be obtained with the help of the formula

$$\rho(\mathbf{x})v_\alpha(\mathbf{x}) = \frac{i\hbar}{2m_\alpha}[\phi_\lambda(\mathbf{x})\partial_\alpha\phi_\lambda^*(\mathbf{x}) - \phi_\lambda^*(\mathbf{x})\partial_\alpha\phi_\lambda(\mathbf{x})], \tag{102}$$

which is analogous to Eq. (28). Therefore the states that correspond to $\phi_\lambda(\mathbf{x})$ cannot be stationary for a dissipative system, because the energy of it is decreasing whenever its probability velocity field is different than zero. As a result of this, any quantum-mechanical nondissipative multidimensional system that is only subject to potential forces has substantially more stationary states than dissipative systems acted on by the same set of potential forces, but one group of its stationary states overlaps with the stationary states of the dissipative systems. Consequently, if a once nondissipative quantum-mechanical system of this kind starts to be dissipative, it loses the majority of its stationary states, but some of them remain stationary in the new physical situation.

A quantum-mechanical nondissipative system acted on by forces that are always perpendicular to its probability velocity field has a completely different pattern of stationary states, which is a direct consequence of a specific assumption regarding the circulation of the velocity field of the system. Namely, for an n-dimensional system the component $F_\alpha^\perp(\mathbf{x},t)$ of the sum of forces that are always perpendicular to the system's probability velocity field can be represented by

$$F_\alpha^\perp(\mathbf{x},t) = \sum_{\beta=1}^{n} v_\beta(\mathbf{x},t)[\partial_\alpha A_\beta(\mathbf{x}) - \partial_\beta A_\alpha(\mathbf{x})], \tag{103}$$

where the quantity $\mathbf{A}(\mathbf{x})$ plays the role of a vector potential, and it is required that for any closed contour C in the system's configurational space

$$\oint_C d\mathbf{s}\cdot[\mathbf{p}(\mathbf{s},t) + \mathbf{A}(\mathbf{s})] = 2\pi k\hbar, \tag{104}$$

where k is an integer, so that the system's wave function is well defined and the Schrödinger equation can be used to describe its evolution [1].

Now, it is shown in Appendix G that under the above condition the stationary states of the system correspond to the solutions of the equation

$$\left\{\sum_{\alpha=1}^{n}\frac{1}{2m_\alpha}\left[\frac{\hbar}{i}\partial_\alpha - A_\alpha(\mathbf{x})\right]^2 + V(\mathbf{x})\right\}\phi_\lambda(\mathbf{x}) = E_\lambda\phi_\lambda(\mathbf{x}), \tag{105}$$

where $V(\mathbf{x})$ is the combined potential of all potential forces in the system, and they also correspond to local minima of the system's average energy. Thus all of the stationary states of the system correspond to nonzero velocity fields and none of them can be stationary if the system is dissipative. Consequently, if a once nondissipative quantum-mechanical system of this kind starts to be dissipative, it loses all of its "original" stationary states, but it acquires "new" stationary states that correspond to the vanishing velocity field.

VI. CONCLUSIONS

The proposed hydrodynamic method of the quantization of mechanical systems is based on two main assumptions. The first of them is related to the existence of the energy of concentration, which is a special type of energy that is not taken into consideration in classical mechanics. The second one, the principle of classical dissipation, means that the only new forces appearing in quantum mechanics are those necessary to make the principle of the conservation of energy hold in spite of the existence of the energy of concentration. Consequently, all of the quantum-mechanical corrections to the system's average energy as well as to the evolution equations are of the orders of $\hbar^2$, since all of the contributions to the energy of concentration are proportional to the square of $\hbar$.

The structure of the energy of concentration is analogous to the structure of the classical-mechanical kinetic energy, since for any system it is symmetrically distributed among its Cartesian configurational coordinates. Namely, for each Cartesian configurational coordinate of the system there is a separate contribution to the energy of concentration, the contributions corresponding to different coordinates are mutually symmetrical and additive, and none of the coordinates is favored in any way. Therefore the energy of concentration can be and usually is regarded as a part of the system's kinetic energy. Still, the energy of concentration depends only on the system's configurational probability density and is always exactly positive, even for the system being in a stationary state.

In classical mechanics the system can only be in a stationary state if its velocity field is equal to zero. Moreover, if there are any minima of the system's potential energy, its configurational probability density corresponding to a stationary state must be singular and equal to $\delta(\mathbf{x}_0)$, where $\mathbf{x}_0$ is a point at which the system's potential energy attains its locally minimal value. In quantum mechanics, thanks to the energy of concentration, typically there are infinitely many stationary states of the system that are not necessarily directly related to local minima of the system's potential energy, since they all correspond to local minima of the system's total quantum-mechanical average energy including the energy of concentration. Therefore they exist even if the system is in principle dissipative, in which case they represent the stable "orbits" that can be occupied by the quantum-mechanical system, even if its classical counterpart in the analogous physical situation would continuously dissipate energy.

For nondissipative systems the proposed method of quantization reproduces the equations that are well known and can be obtained with the help of some other quantization procedures. For dissipative systems it also allows us to reproduce some known results, such as the Schrödinger-Langevin equation, or the hydrodynamiclike equations for a three-dimensional dissipative system in magnetic field. However, it can also be applied to dissipative systems where the Hamiltonian and the Lagrangian do not exist, and for some dissipative systems it leads to completely different evolution equations that cannot be obtained using any other systematic approach like, for example, Eq. (78).

Importantly enough, the presented approach allows us to treat noninteracting parts of a quantum-mechanical dissipative system as separate systems satisfying independent evolution equations. Namely, if the system consists of two or more noninteracting parts, its configurational probability density factorizes and this leads to the separation of its noninteracting subsystems, although the nonconservative forces responsible for the dissipation of energy acting on each of these subsystems individually can be arbitrary. Moreover, there is no reason for which this approach should not be applicable to systems acted on by forces that cause their energy to increase. It is generally applicable to virtually any system acted on by forces that in classical mechanics can be expressed in terms of the system's position and velocity. Therefore it is particularly well suited for the treatment of systems in which the forces are only known up to some phenomenological coefficients determined by fitting theoretical results to experimental data.

As far as dissipative systems are concerned, it is reasonable to assume that their energy decreases because of some interactions between them and other larger systems of many more degrees of freedom in which the energy can be distributed in many more different ways. Thus if the interactions between the system in question and the other systems are known exactly, it is formally more appropriate to consider all of the interacting systems together as one huge nondissipative system and derive the necessary evolution equations for one of its parts from its own evolution equations. Yet, in many situations this is impractical or even impossible due to overwhelming complications. In that cases it sometimes is possible to guess the forces acting on the system by comparing experimental data with a simple theoretical model, and the proposed quantization method can be used in developing such models. Namely, it allows us to devise some formulae for the nonconservative forces acting on the system, put them into the system's evolution equations and look if they lead to results consistent with the available empirical data. If they do, we can use the model to make some predictions without even knowing what the interactions between the system and its neighborhood may be, and we do not need to take the structure of the system's neighborhood into consideration. This may be a great advantage in the situations in which the system's neighborhood is highly complicated and its interactions with the system are convoluted.

Arguably, quantum mechanics is only necessary to make predictions regarding systems of the subatomic or atomic scale which interact via well-known forces, so the need for the phenomenological approach described above seems to be very limited. Still, some effects of apparently quantum-mechanical origin have been observed in the systems that are in fact macroscopic and interact with their neighborhood in a complicated way. Besides, the scale on which the existence of the energy of concentration starts to affect the system's behavior is actually defined by the ratios of the Planck's constant to the masses of the system's components, so it is different for systems in which these masses are different. Accordingly, even in a macroscopic system there can be a parameter playing the role of effective mass, so small that in predicting the system's evolution we ought to take the energy of concentration into account. In such a case the presented hydrodynamic quantization may be quite useful.

It is also important to remember that evolution equations are used not only in physics. In the other sciences, like economics or biology, there are dynamical systems that can be modeled with the help of various evolution equations. The structure of some of these equations is similar to the structure of the equations of classical mechanics, although usually they are nonlinear and correspond to the classical-mechanical equations for systems acted on by nonconservative forces. It is therefore interesting whether these nonmechanical systems may behave like quantum systems in physics and, if so, in what conditions. However, we can only verify this by using evolution equations that correspond to the quantum-mechanical equations of motion for analogous physical systems, and the proposed hydrodynamic quantization method seems to be suitable for this purpose.

APPENDIX A: SEMIPURE JOINT P ROBABILITY DENSITIES

Let us assume we have two random variables $\tilde{x}$ and $\tilde{u}$ that represent the position of certain physical system and another quantity which may be related to the system's velocity, respectively.

For these random variables there is the set Ω of all functions that can be their joint probability densities, some elements of which are special. Consider, for instance, the function

$$d(x,u) = \delta(x - x_0)\,\delta(u - u_0), \tag{A1}$$

where x_0 is a fixed point in the system's configurational space, and u_0 is a fixed real number. If $d(x,u)$ is the joint probability density of $\tilde{x}$ and $\tilde{u}$, they are not really random variables, since their only possible values are x_0 and u_0, respectively. In that case, if $\tilde{x}$ represents the system's position and $\tilde{u}$ represents its velocity at certain time t, the system's position and velocity at the time t are known exactly. Thus since the states in which the system's position and velocity are known exactly can be referred to as pure states of the system [1], $d(x,u)$ may be referred to as a *pure* joint probability density of $\tilde{x}$ and $\tilde{u}$.

Further, it is instructive to observe that in classical mechanics the state probability density of the system may be pure at any instant of time. Namely, if the function $q(t)$ satisfies the system's Newton equation of motion, its state probability density can be defined by

$$s_q(x,u,t) = \delta(x - q(t))\,\delta(u - \dot{q}(t)), \tag{A2}$$

where $\dot{q}(t)$ denotes the time derivative of $q(t)$. Of course $q(t)$ can only be determined if the initial position and velocity of the system are known exactly. However, if the system's position at certain instant of time t is not known exactly, we need to use a random variable $\tilde{x}$ with the probability density $\rho(x)$ to represent its position at that time. In that case, if the system's velocity at the time t is known independently of its position and equal to u_0, the system's state at the time t can be represented by the probability density

$$p_s(x,u) = \rho(x)\,\delta(u - u_0), \tag{A3}$$

which is pure only with respect to the system's velocity. Accordingly, $p_s(x,u)$ may be called a *strictly semipure* joint probability density of $\tilde{x}$ and $\tilde{u}$.

In classical mechanics the strictly semipure probability density $p_s(x,u)$ represents the state in which the system's velocity is exactly equal to u_0 independently of the actual value of its position. There are some cases in which such an assumption can be made. For example, if the force $F(x)$ in Eq. (19) is in fact independent of x, there are solutions of Eqs. (11) and (19) that correspond to strictly semipure state probability densities. Usually, however, it is unrealistic, so the usefulness of strictly semipure probability densities is quite limited. Yet, often the system's velocity is not really random, but it depends on the system's position, which is random. In other words, we can often say exactly what the system's velocity is, provided that someone tells us the actual value of its position, but as long as the system's position is not known, its velocity is not known either. Then, the information on the system's velocity that we have can be represented as a function $v_p(x)$, and we can generalize Eq. (A3) by using $v_p(x)$ instead of u_0. Hence we get

$$p(x,u) = \rho(x)\,\delta(u - v_p(x)), \tag{A4}$$

and $p(x,u)$ may be referred to as a *semipure* joint probability density of $\tilde{x}$ and $\tilde{u}$, since it represents the state in which the system's velocity is only random because it depends on the system's position that is really random [23]. Moreover, since Eq. (A4) is analogous to Eq. (24), $v_p(x)$ may be referred to as the *velocity field* of $p(x,u)$.

The applications of semipure joint probability densities are not as limited as the applications of strictly semipure probability densities. For example, we have shown in Sec. II C that Eqs. (11) and (19) can only be satisfied if the system's initial state is represented by a semipure probability density, in which case its state probability density is semipure at any instant of time. Furthermore, it turns out that this probability density satisfies the Liouville equation.

To show this we differentiate the probability density $s(x,u,t)$ given by Eq. (24), where $\rho(x,t)$ and $v(x,t)$ satisfy Eqs. (11) and (19), with respect to time. Hence we get

$$\partial_t s(x,u,t) = -\partial_x[\rho(x,t)v(x,t)]\,\delta(u - v(x,t)) + \left[v(x,t)\partial_x v(x,t) - \frac{F(x)}{m}\right]\rho(x,t)\,\partial_u \delta(u - v(x,t)), \tag{A5}$$

which can be rewritten in the following way:

$$\begin{aligned}\partial_t s(x,u,t) = {}& -\frac{F(x)}{m}\partial_u s(x,u,t) - u\,\delta(u - v(x,t))\,\partial_x \rho(x,t) \\ & + \rho(x,t)v(x,t)[\partial_x v(x,t)]\partial_u \delta(u - v(x,t)) \\ & - \rho(x,t)\,\delta(u - v(x,t))\,\partial_x v(x,t).\end{aligned} \tag{A6}$$

Next, we notice that

$$\begin{aligned}& u\rho(x,t)[\partial_x v(x,t)]\partial_u \delta(u - v(x,t)) \\ &\quad = \rho(x,t)v(x,t)[\partial_x v(x,t)]\partial_u \delta(u - v(x,t)) \\ &\qquad - \rho(x,t)\,\delta(u - v(x,t))\,\partial_x v(x,t),\end{aligned} \tag{A7}$$

and

$$\begin{aligned}\partial_x s(x,u,t) = {}& \delta(u - v(x,t))\,\partial_x \rho(x,t) \\ & - \rho(x,t)[\partial_x v(x,t)]\partial_u \delta(u - v(x,t)).\end{aligned} \tag{A8}$$

This allows us to transform Eq. (A6) so that

$$\partial_t s(x,u,t) = -\frac{F(x)}{m}\partial_u s(x,u,t) - u\,\partial_x s(x,u,t), \tag{A9}$$

which is the Liouville equation for $s(x,u,t)$.

Interestingly enough, semipure joint probability densities can also be used for representing states of quantum-mechanical systems, which is discussed in Sec. II D, although in these cases the interpretation of the second random variable $\tilde{u}$ is not clear.

APPENDIX B: PROPERTIES OF THE ENERGY OF CONCENTRATION

To understand the influence of the term proportional to $(\hbar/m)$ in Eq. (37), it is advisable to examine the behavior of the functional $U[f]$ given by Eq. (38). For this purpose we note that the square root is monotonic, so the sign of the derivative of the square root of a function always reflects the sign of the derivative of the function itself. Therefore the values of $U[f]$ are greater for wildly varying functions and smaller for functions that do not vary a lot. In particular, for constant functions $U[f]$ is equal to zero.

Of course, the system's configurational probability density cannot be a constant function as long as it is assumed to be smooth. Namely, it must vanish on the boundaries of the system's configurational space, as the probability of finding the system outside of its configurational space is zero, and at the same time the integral of it over the whole system's configurational space must be equal to 1. For this reason $U[\rho(x,t)]$ is always positive. Furthermore, the normalization of $\rho(x,t)$ implies that in can only be significantly varying when it has steep and narrow maxima.

To illustrate this we consider the well-known Gaussian probability density defined by

$$\rho_G(x) = \frac{1}{\sqrt{2\pi}\Delta x}\exp\left[-\frac{(x-x_0)^2}{2(\Delta x)^2}\right], \tag{B1}$$

where x_0 is the average x and Δx is the standard deviation of x from x_0. Obviously the maximal value of $\rho_G(x)$ which occurs for $x{=}x_0$ is inversely proportional to Δx. It is also known that the probability of x being less than $(x_0-3\Delta x)$ or greater than $(x_0+3\Delta x)$ is negligible. In other words $\rho_G(x)$ is *concentrated* between $(x_0-3\Delta x)$ and $(x_0+3\Delta x)$. Accordingly, it is *more concentrated* if Δx is small and it is *less concentrated* if Δx is large. Hence the more concentrated is the function, the steeper and narrower is its maximum at x_0.

Now, it is straightforwad to verify that

$$U[\rho_G(x)] = (\Delta x)^{-2}, \tag{B2}$$

so $U[\rho_G(x)]$ is a measure of the concentration of $\rho_G(x)$. Moreover, it is not difficult to convince oneself that for any probability density $\rho(x)$ the value $U[\rho(x)]$ is a measure of its concentration. Indeed, if $\rho(x)$ is highly concentrated, it has at least one peak that corresponds to a relatively high maximal value, which follows from the fact that $\rho(x)$ must be normalized. Then, this peak is steep and narrow, so in the vicinity of it the derivative of $\sqrt{\rho(x)}$ attains considerably large absolute values and the contribution of it to the integral on the right-hand side of Eq. (38) is substantial. On the other hand, if $\rho(x)$ is not concentrated, its maxima are broad and smooth and the derivative of $\sqrt{\rho(x)}$ is close to zero for any x, so $U[\rho(x)]$ is small. Accordingly, the quantum-mechanical contribution to the average energy of the system proportional to $U[\rho(x,t)]$ may be referred to as *the energy of concentration*, since it is a measure of the concentration of the system's configurational probability density. Apparently, it represents the amount of energy available to the system just because its configurational probability density is more or less concentrated.

To provide a more in-depth interpretation of the energy of concentration, it is convenient to consider the problem of forcing a particle to stay in certain area of space. Namely, to make a particle stay in a specific area we have to use some forces that will prevent it from leaving that area and the question arises how much energy we will have to spend to do this.

For example, if the particle is confined to only one dimension we can formally use two infinitely high potential barriers which can be moved along the track of the particle to make it stay in a specific section of its configurational space. We can also assume for simplicity that the barriers are initially infinitely far away from each other and that if there is no particle between them, the energetic cost of moving them is zero. Now, the question is how much energy it will cost to set the distance between the barriers to D if we put the particle between them.

It should be clear that if we move the barriers carelessly, we will probably waste a considerable amount of energy for shorting the distance between them, because each time we move a barrier we can potentially accelerate the particle. Still, according to the laws of classical mechanics, if we take maximal care when moving the barriers, we ought to be able to avoid any transfers of energy from the barriers to the particle. Thus it follows from classical mechanics that we do not need to spend any amount of energy to move the barriers to the distance D from each other, whether there is a particle between them or not. In quantum mechanics, however, there is the energy of concentration which represents the amount of energy available to the particle just because the probability of finding it outside of the area between the barriers is equal to zero. This energy may not come from nowhere, so it has to be provided by us in the process of moving the barriers. Hence in quantum mechanics the minimal energy required to move the barriers to the distance D from each other is equal to the minimal energy of concentration of the particle trapped between them. We are now going to compute this energy.

For this purpose we treat the energy of concentration as a functional of the function $\varphi(x)$ equal to the square root of the particle's configurational probability density $\rho(x)$, i.e.,

$$E_c[\varphi] = \frac{\hbar^2}{2m}\int dx[\partial_x\varphi(x)]^2, \tag{B3}$$

where m denotes the mass of the particle. To find the minimal particle's energy of concentration we need to determine the global minimum of $E_c[\varphi]$ with the additional requirement that the integral of the square of $\varphi(x)$ over the whole configurational space of the particle be equal to 1. The necessary condition of the existence of such a minimum is the vanishing of the functional derivative with respect to $\varphi(x)$ of the functional

$$I[\varphi] = \frac{\hbar^2}{2m}\int dx\{[\partial_x\varphi(x)]^2 - E_\lambda\varphi^2(x)\}, \tag{B4}$$

where the unknown constant E_λ has to be determined along with $\varphi(x)$.

The functional derivative of $I[\varphi]$ with respect to $\varphi(x)$ vanishes for $\varphi_\lambda(x)$ that satisfy the following equation:

$$-\frac{\hbar^2}{2m}\partial_x^2\varphi_\lambda(x) = E_\lambda\varphi_\lambda(x), \tag{B5}$$

which of course is the well-known Schrödinger equation without time for a free particle and the solutions of it are also well known [1,2]. Namely, if the infinite potential barriers are located in the positions $-D/2$ and $D/2$, the solution of Eq. (B5) should vanish outside of the area between the barriers and it also ought to be continuous. For this boundary condition the solutions of Eq. (B5) are given by

$$\varphi_\lambda(x) = \begin{cases} \sqrt{\frac{2}{D}}\cos\left[\frac{\lambda\pi}{D}x\right], & \lambda = 1,3,5,\ldots \\ \sqrt{\frac{2}{D}}\sin\left(\frac{\lambda\pi}{D}x\right), & \lambda = 2,4,6,\ldots \end{cases}, \tag{B6}$$

$$E_\lambda = \frac{1}{2m}\left(\lambda\frac{\hbar\pi}{D}\right)^2, \quad \lambda = 1,2,\ldots. \tag{B7}$$

Moreover, it follows from Eqs. (B3) and (B5) that $E_c[\varphi_\lambda]$ is equal to E_λ, for each λ, so the constants E_λ represent different values of the energy of concentration corresponding to different local minima of $E_c[\varphi_\lambda]$.

We only want to know the minimal energy required to place the barriers at the distance D from each other, so we only consider the solution of Eq. (B5) that corresponds to the least of E_λ. In other words we assume that

$$\rho(x) = \varphi_1^2(x) = \frac{2}{D}\cos^2\left(\frac{\pi}{D}x\right), \tag{B8}$$

and

$$E_c = E_1 = \frac{1}{2m}\left(\frac{\hbar\pi}{D}\right)^2. \tag{B9}$$

Hence we see that to make the particle stay in the area of the size D between the barriers we need to spend at least the amount of energy equal to E_c, which is inversely proportional to the mass of the particle and to the square of the distance D.

We can conclude that the existence of the energy of concentration makes it impossible to localize a particle in an area that is too small, since the amount of energy required to achieve this goal would be insanely large. Moreover, since the energy of concentration is inversely proportional to the mass of the particle, massive particles can generally be localized in smaller areas than less massive ones.

APPENDIX C: HEISENBERG UNCERTAINTY PRINCIPLE

We have asserted in Sec. II E that the inclusion of the energy of concentration in the kinetic energy of the system allows one to prove the Heisenberg uncertainty principle. Indeed, if the energy of concentration is included in the kinetic energy, we have

$$\langle v^2\rangle = \frac{2}{m}[\langle E_q\rangle - \langle V\rangle] = \int dx\, v^2(x,t)\rho(x,t) + \frac{\hbar^2}{m^2}U[\rho(x,t)], \tag{C1}$$

where the functional $U[f]$ is given by Eq. (38), which follows from Eq. (37). Then, the variance of the system's velocity is given by

$$\sigma^2(v) = \langle v^2\rangle - \langle v\rangle^2 = \int dx\, v^2(x)\rho(x) + \frac{\hbar^2}{m^2}U[\rho(x)] - \left[\int dx\, v(x)\rho(x)\right]^2, \tag{C2}$$

where we have omitted the irrelevant time variable t. Moreover, it can be readily verified that

$$\int dx\, v^2(x)\rho(x) - \left[\int dx\, v(x)\rho(x)\right]^2 = \int dx\left[v(x) - \int dq\, v(x)\rho(x)\right]^2\rho(x), \tag{C3}$$

the right-hand side of which is non-negative, so we get

$$\sigma^2(v) \geqslant \frac{\hbar^2}{m^2}U[\rho(x,t)]. \tag{C4}$$

Now, we consider the function $Z(\xi)$ defined by

$$Z(\xi) = \int dx\{[\partial_x - \xi(x - \langle x\rangle)]\sqrt{\rho(x,t)}\}^2, \tag{C5}$$

which is real-valued and non-negative [1,23]. Namely, by transforming the right-hand side of Eq. (C5) we obtain

$$Z(\xi) = \sigma^2(x)\xi^2 + \xi + U[\rho(x)], \tag{C6}$$

where $\sigma^2(x)$ is the variance of the system's position, which means that $Z(\xi)$ is quadratic with respect to ξ. Hence since $Z(\xi)$ is non-negative, we have

$$1 - 4\sigma^2(x)U[\rho(x)] \leqslant 0. \tag{C7}$$

Consequently, we get

$$\sigma^2(x)\sigma^2(v) \geqslant \frac{\hbar^2}{4m^2}, \tag{C8}$$

which follows from Eqs. (C4) and (C7). This means, however, that the Heisenberg principle for the system's position and velocity is satisfied.

APPENDIX D: PRINCIPLE OF CLASSICAL DISSIPATION

In the present appendix we argue that the term represented by the unknown quantity $\Xi[\rho(x,t),v(x,t)]$ in Eq. (45) ought to be equal to zero.

To start with, we show that the average energy of the system, given by Eq. (37), attains a global minimum that can be found. Namely, we first observe that the average energy can only attain its minimum for the velocity field equal to zero, since the contribution of the velocity field to the right-hand side of Eq. (37) is never negative. Now, if the velocity field is assumed to vanish, the remaining part of the average energy that we denote by $\langle E_s \rangle$ can be regarded as a functional of the system's configurational probability density $\rho(x,t)$. Obviously it is also a functional of the square root of $\rho(x,t)$ that we denote by $\varphi(x)$, omitting the irrelevant time variable t, i.e.,

$$\langle E_s \rangle = \int dx \left\{ V(x)\varphi^2(x) + \frac{\hbar^2}{2m}[\partial_x \varphi(x)]^2 \right\}. \qquad \text{(D1)}$$

To find the minimum of the average energy we need to determine the minima of $\langle E_s \rangle$ under the condition that the integral of $\varphi^2(x)$ over the whole system's configurational space must be equal to 1.

The necessary condition of the existence of such minima of $\langle E_s \rangle$ is the vanishing of the functional derivative with respect to $\varphi(x)$ of the functional

$$I[\varphi] = \int dx \left\{ [V(x) - E_\lambda]\varphi^2(x) + \frac{\hbar^2}{2m}[\partial_x \varphi(x)]^2 \right\}, \qquad \text{(D2)}$$

where the unknown constant E_λ has to be determined along with $\varphi(x)$. This functional derivative vanishes for $\varphi_\lambda(x)$ satisfying the Schrödinger equation without time,

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + V(x) \right] \varphi_\lambda(x) = E_\lambda \varphi_\lambda(x), \qquad \text{(D3)}$$

the solutions of which are well known [1,2,26]. Specifically, for a typical boundary condition it has a countable set of solutions $\varphi_\lambda(x)$ that generally correspond to different constants E_λ, and the set of all E_λ is lower-bounded. Moreover, we have

$$E_\lambda = \int dx \left\{ V(x)\varphi_\lambda^2(x) + \frac{\hbar^2}{2m}[\partial_x \varphi_\lambda(x)]^2 \right\}, \qquad \text{(D4)}$$

so for the system's configurational probability density equal to the square of $\varphi_\lambda(x)$ the energy $\langle E_s \rangle$ is equal to E_λ. Therefore the least of E_λ that we denote by E_0 can be regarded as the minimal energy of the system and the probability density $s_0(x,u) = \varphi_0^2(x)\delta(u)$ represents the state in which the system's average energy attains its minimum, or the system's *ground state*. Furthermore, each $\varphi_\lambda(x)$ corresponds to a local minimum of $\langle E_s \rangle$, because the coefficient multiplying the square of $\partial_x \varphi(x)$ in Eq. (D2) is positive and independent of x. Thus if the initial state of the system is represented by the probability density $s_\lambda(x,u) = \varphi_\lambda^2(x)\delta(u)$, its average energy cannot change and this state must be a stationary state of the system. In particular, the system's ground state, represented by $s_0(x,u)$, must be stationary.

We now see that Eq. (45) should be satisfied for the vanishing velocity field and $\rho(x,t)$ equal to $\varphi_\lambda^2(x)$ which obviously is time independent. Hence we have

$$0 = \partial_x \left\{ \frac{\hbar^2}{2m} \frac{\partial_x^2 \varphi_\lambda(x)}{\varphi_\lambda(x)} - V(x) \right\} - \frac{\gamma\hbar}{m} \Xi[\varphi_\lambda^2(x), 0], \qquad \text{(D5)}$$

which means that $\Xi[\varphi_\lambda^2(x),0]$ is equal to zero for any λ. However, this practically means that $\Xi[\rho(x,t),0]$ is always equal to zero, so $\Xi[\rho(x,t),v(x,t)]$ should depend on $v(x,t)$ in a significant way.

To investigate this further we compute the time derivative of the system's average energy given by

$$\frac{d}{dt}\langle E_q \rangle = -\gamma \int dx\, \rho(x,t)v(x,t) \left\{ v(x,t) + \frac{\hbar}{m}\Xi[\rho(x,t),v(x,t)] \right\}, \qquad \text{(D6)}$$

where we have used some observations made in Sec. II E. We notice that the right-hand side of this equation should never be positive, as we do not expect the system to accumulate energy. Moreover, the value of the integral on the right-hand side of it should not depend on the sign of the velocity field $v(x,t)$, because the change of the sign of $v(x,t)$ is equivalent to the change of the direction of the configurational coordinate axis. Hence we have $\Xi[\rho(x,t),-v(x,t)] = -\Xi[\rho(x,t),v(x,t)]$. Furthermore, $\Xi[\rho(x,t),v(x,t)]$ cannot be proportional to a negative power of $v(x,t)$, as it has to vanish for $v(x,t)$ equal to zero.

Of course, we can rewrite Eq. (D6) in the following form:

$$\frac{d}{dt}\langle E_q \rangle = -\gamma \int dx\, \rho(x,t)v^2(x,t) \left\{ 1 + \frac{\hbar}{m}\xi[\rho(x,t),v(x,t)] \right\}, \qquad \text{(D7)}$$

where the product of $(\hbar/m)$ and $\xi[\rho(x,t),v(x,t)]$ is dimensionless. Then, it follows from the above discussion that $\xi[\rho(x,t),v(x,t)]$ is symmetrical with respect to the velocity field. Moreover, we have

$$\lim_{\epsilon \to 0} \{\epsilon\, v(x)\xi[\rho(x), \epsilon\, v(x)]\} = 0, \qquad \text{(D8)}$$

$$\frac{\hbar}{m} \int dx\, \rho(x)v^2(x)\xi[\rho(x),v(x)] \geqslant -1, \qquad \text{(D9)}$$

for any smooth function $v(x)$ and for any probability density $\rho(x)$ for which the energy of concentration is finite. Also, the product of $(\hbar/m)$ and $\xi[\rho(x),v(x)]$ must vanish in the classical limit.

If all of the above-mentioned conditions are taken into consideration, it is really hard to build a function that would satisfy them all together using the quantities at our disposal, i.e., $\rho(x,t)$, $v(x,t)$, $V(x)$, $\hbar$, m, x, and any combination of derivatives with respect to it, except for the trivial function that is constant and equal to zero. Certainly, we could do this with the help of an additional constant of appropriate dimension, but then the interpretation of this constant and the whole $\xi[\rho(x,t),v(x,t)]$ would be very problematic. In particular, it would be necessary to relate this constant to the energy of concentration which is strictly related to the quantum-mechanical term in the evolution equations for $\gamma=0$. This would be clearly impossible, so we can safely assume that there is no $\Xi[\rho(x,t),v(x,t)]$ in Eq. (45), and this assertion is the principle of classical dissipation.

APPENDIX E: MULTIDIMENSIONAL PROBABILITY LIQUID IN CLASSICAL MECHANICS

In this appendix we derive the evolution equations for the classical-mechanical probability liquid in many dimensions.

For this purpose we consider an n-dimensional system for which the following Newton equations are satisfied in classical mechanics:

$$m_\alpha \frac{d^2 q_\alpha}{dt^2} = F_\alpha\left(\mathbf{q}(t), \frac{d\mathbf{q}}{dt}, t\right), \quad \alpha = 1,2,\ldots,n, \tag{E1}$$

where $\mathbf{q}(t)$ is the function the values of which represent the system's positions at different instants of time, and the total force acting on the system may depend on its position as well as on its velocity, and time. We note that these equations can be represented as a system of first-order differential equations with respect to time. Namely, if the system's velocity function $\mathbf{v}(t)=(d\mathbf{q}/dt)$ is introduced, we obtain the equations

$$\frac{d}{dt}\mathbf{q} = \mathbf{v}, \tag{E2}$$

$$\frac{d}{dt}v_\alpha = \frac{F_\alpha(\mathbf{q},\mathbf{v},t)}{m_\alpha}, \quad \alpha = 1,2,\ldots,n. \tag{E3}$$

We assume that the system's initial position and velocity are not known exactly, so its position at given instant of time t should be represented by the random variable $\tilde{\mathbf{x}}(t)$ with the probability distribution function $P(\mathbf{x},t)$. We denote the probability density of $\tilde{\mathbf{x}}(t)$, which is the system's configurational probability density, by $\rho(\mathbf{x},t)$ and assume that we know $\rho(\mathbf{x},t_0)$ for certain t_0. We also assume that we know the velocity field $\mathbf{v}(\mathbf{x},t_0)$ which assigns specific velocity of the flow of the probability liquid to each possible position of the system $\mathbf{x}$ at the time t_0.

In accordance with the discussion in Sec. II B we observe that the probability distribution function $P(\mathbf{x},t_0)$, the derivative of which is $\rho(\mathbf{x},t_0)$, can be approximated by the following staircase function:

$$P(\{\mathbf{x}_k\}_N,\mathbf{x},t_0) = \sum_{k=1}^{N+1} P_k \prod_{\alpha=1}^{n} \chi(x_\alpha - x_{k\alpha}), \tag{E4}$$

where the symbols $x_{k\alpha}$ denote the components of N fixed points $\mathbf{x}_k$ in the system's configurational space, and the coefficients P_k are defined in analogy with Eq. (5). By differentiating $P(\{\mathbf{x}_k\}_N,\mathbf{x},t_0)$ with respect to all of the system's Cartesian configurational coordinates we obtain the initial reduced configurational probability density of the system

$$\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t_0) = \sum_{k=1}^{N+1} P_k \delta(\mathbf{x}-\mathbf{x}_k). \tag{E5}$$

Next, we notice that if the points $\mathbf{x}_k$ along with the velocities $\mathbf{v}_k=\mathbf{v}(\mathbf{x}_k,t_0)$ are used as the initial conditions for the system's Newton equations, Eqs. (E1), we obtain N solutions of these equations, $\mathbf{q}_k(t)$, $k=1,2,\ldots,N$, of the following properties:

$$\mathbf{q}_k(t_0) = \mathbf{x}_k, \qquad \left.\frac{d\mathbf{q}_k}{dt}\right|_{t_0} = \mathbf{v}(\mathbf{x}_k,t_0). \tag{E6}$$

Hence we have

$$\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t) = \sum_{k=1}^{N+1} P_k \delta(\mathbf{x}-\mathbf{q}_k(t)), \tag{E7}$$

for $t \geqslant t_0$.

By differentiating $\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t)$ with respect to time we get

$$\partial_t \rho(\{\mathbf{x}_k\}_N,\mathbf{x},t) = -\sum_{\alpha=1}^{n} \partial_\alpha j_\alpha^{(1)}(\{\mathbf{x}_k\}_N,\mathbf{x},t), \tag{E8}$$

where the current $\mathbf{j}^{(1)}(\{\mathbf{x}_k\}_N,\mathbf{x},t)$ is given by

$$\mathbf{j}^{(1)}(\{\mathbf{x}_k\}_N,\mathbf{x},t) = \sum_{k=1}^{N+1} P_k \frac{d\mathbf{q}_k}{dt} \delta(\mathbf{x}-\mathbf{q}_k(t)), \tag{E9}$$

which means that

$$\mathbf{j}^{(1)}(\{\mathbf{x}_k\}_N,\mathbf{x},t_0) = \rho(\{\mathbf{x}_k\}_N,\mathbf{x},t_0)\mathbf{v}(\mathbf{x},t_0). \tag{E10}$$

By differentiating the αth component of the current $\mathbf{j}^{(1)}(\{\mathbf{x}_k\}_N,\mathbf{x},t)$ with respect to time we obtain

$$\begin{aligned}\partial_t j_\alpha^{(1)}(\{\mathbf{x}_k\}_N,\mathbf{x},t) = {}& \sum_{k=1}^{N+1} \frac{1}{m} F_\alpha\left(\mathbf{x}, \frac{d\mathbf{q}_k}{dt}, t\right) P_k \delta(\mathbf{x}-\mathbf{q}_k(t)) \\ & - \sum_{\beta=1}^{n} \partial_\beta j_{\alpha\beta}^{(2)}(\{\mathbf{x}_k\}_N,\mathbf{x},t),\end{aligned} \tag{E11}$$

where the quantity $j_{\alpha\beta}^{(2)}(\{\mathbf{x}_k\}_N,\mathbf{x},t)$ is given by

$$j_{\alpha\beta}^{(2)}(\{\mathbf{x}_k\}_N,\mathbf{x},t) = \sum_{k=1}^{N+1} \left(\frac{dq_{k\alpha}}{dt}\right)\left(\frac{dq_{k\beta}}{dt}\right) P_k \delta(\mathbf{x}-\mathbf{q}_k(t)), \tag{E12}$$

which follows from Eqs. (E1).

Now, in order to avoid unnecessary complications, we require *a priori* that the system's state probability density be semipure, i.e.,

$$s(\mathbf{x},\mathbf{u},t) = \rho(\mathbf{x},t)\delta(\mathbf{u}-\mathbf{v}(\mathbf{x},t)). \tag{E13}$$

This means that the velocity of the flow of the probability liquid is defined unambiguously for any point $\mathbf{x}$ in the system's configurational space. In other words, the initial state of the system is such that for any selection of the points $\mathbf{x}_k$ the time derivative of $\mathbf{q}_k(t)$ taken at any time t can always be replaced with the value of the system's velocity field for $\mathbf{x}=\mathbf{q}_k(t)$, i.e.,

$$\frac{d\mathbf{q}_k}{dt} = \mathbf{v}(\mathbf{q}_k(t),t), \quad k = 1,2,\ldots,N. \tag{E14}$$

Hence we have

$$\partial_t[v_\alpha(\mathbf{x},t)\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t)] = \frac{F_\alpha(\mathbf{x},\mathbf{v}(\mathbf{x},t),t)}{m}\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t) - \sum_{\beta=1}^{n}\partial_\beta[v_\alpha(\mathbf{x},t)v_\beta(\mathbf{x},t)\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t)], \tag{E15}$$

which is a consequence of Eqs. (E11) and (E12).

We notice that Eq. (E15) is linear with respect to $\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t)$ and, on the other hand, it should be satisfied for any N and for any selection of the points $\mathbf{x}_k$. Therefore we can replace $\rho(\{\mathbf{x}_k\}_N,\mathbf{x},t)$ in this equation with $\rho(\mathbf{x},t)$, which leads to the following result:

$$\partial_t[v_\alpha(\mathbf{x},t)\rho(\mathbf{x},t)] = \frac{F_\alpha(\mathbf{x},\mathbf{v}(\mathbf{x},t),t)}{m}\rho(\mathbf{x},t) - \sum_{\beta=1}^{n}\partial_\beta[v_\alpha(\mathbf{x},t)v_\beta(\mathbf{x},t)\rho(\mathbf{x},t)]. \tag{E16}$$

Similarly, we can obtain the continuity equation, Eq. (2), from Eq. (E8), independently of the fact that it must always be satisfied for the probability liquid. Moreover, we can use Eq. (2) to transform Eq. (E16) so that we get

$$\partial_t v_\alpha(\mathbf{x},t) + \sum_{\beta=1}^{n} v_\beta(\mathbf{x},t)\partial_\beta v_\alpha(\mathbf{x},t) = \frac{F_\alpha(\mathbf{x},\mathbf{v}(\mathbf{x},t),t)}{m}, \tag{E17}$$

and this equation has to be satisfied for each $\alpha=1,2,\ldots,n$. Thus we have obtained the required system of evolution equations for the probability liquid consisting of Eqs. (2) and (75).

APPENDIX F: CIRCULATION OF THE PROBABILITY VELOCITY FIELD

In the present appendix we examine the evolution of the circulation of the probability velocity field $\mathbf{v}(\mathbf{x},t)$ of a multidimensional quantum-mechanical system such that

$$\begin{aligned}\partial_t v_\alpha(\mathbf{x},t) + \sum_{\beta=1}^{n} v_\beta(\mathbf{x},t)\partial_\beta v_\alpha(\mathbf{x},t) &= \frac{1}{m_\alpha}[-\gamma v_\alpha(\mathbf{x},t) + F_\alpha(\mathbf{x})] \\ &\quad + \frac{1}{m_\alpha}\sum_{\beta=1}^{n} v_\beta(\mathbf{x},t)[\partial_\alpha A_\beta(\mathbf{x}) - \partial_\beta A_\alpha(\mathbf{x})] \\ &\quad + \frac{\hbar^2}{2m_\alpha}\partial_\alpha\left[\sum_{\beta=1}^{n}\frac{1}{m_\beta}\frac{\partial_\beta^2\sqrt{\rho(\mathbf{x},t)}}{\sqrt{\rho(\mathbf{x},t)}}\right],\end{aligned} \tag{F1}$$

for $\alpha=1,2,\ldots,n$, where the force $\mathbf{F}(\mathbf{x})$ is potential, and the constant γ is non-negative.

First, we introduce the quantity

$$\Gamma_C(t) = \oint_C d\mathbf{s}\cdot[\mathbf{p}(\mathbf{s},t) + \mathbf{A}(\mathbf{s})], \tag{F2}$$

where $p_\alpha(\mathbf{x},t)=m_\alpha v_\alpha(\mathbf{x},t)$, the scalar product is defined in the standard way, i.e., $\mathbf{a}\cdot\mathbf{b}=\Sigma_\beta a_\beta b_\beta$, and C is a closed contour in the system's configurational space. Next, we compute the difference,

$$\Gamma_C(t+\Delta t) - \Gamma_C(t) = \oint_{C(t+\Delta t)} d\mathbf{s}\cdot[\mathbf{p}(\mathbf{s},t+\Delta t) + \mathbf{A}(\mathbf{s})] - \oint_C d\mathbf{s}\cdot[\mathbf{p}(\mathbf{s},t) + \mathbf{A}(\mathbf{s})], \tag{F3}$$

where the contour C is transformed into the contour $C(t+\Delta t)$ during the system's evolution between the instants of time t and $t+\Delta t$ [28].

We notice that if the vectors $\mathbf{x}$ and $\mathbf{y}$ represent two points located very close to each other in the contour C and $d\mathbf{s}=\mathbf{x}-\mathbf{y}$, then after the time Δt the elements of the probability liquid located initially at $\mathbf{x}$ and $\mathbf{y}$ will move to $\mathbf{x}+\mathbf{v}(\mathbf{x},t)\Delta t$ and $\mathbf{y}+\mathbf{v}(\mathbf{y},t)\Delta t$, respectively. Thus we have

$$\begin{aligned}d\mathbf{s}(t+\Delta t) &= d\mathbf{s} + [\mathbf{v}(\mathbf{x}+d\mathbf{s},t) - \mathbf{v}(\mathbf{x},t)]\Delta t \\ &\approx d\mathrm{s} + \Delta t\left(\sum_{\alpha=1}^{n} ds_\alpha\partial_\alpha\right)\mathbf{v}(\mathbf{x},t).\end{aligned} \tag{F4}$$

Moreover, the difference between $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t+\Delta t)$ is approximately equal to the total derivative of $\mathbf{v}(\mathbf{x},t)$ multiplied by Δt, and analogously for the quantity $\mathbf{A}(\mathbf{x})$. Therefore Eq. (F3) can be transformed into the following equation:

$$\Gamma_C(t+\Delta t) - \Gamma_C(t) \approx \Delta t\oint_C\left[d\mathbf{s}\cdot\left(\frac{d\mathbf{p}}{dt} + \frac{d\mathbf{A}}{dt}\right) + (\mathbf{p}+\mathbf{A})\cdot\left(\sum_{\alpha=1}^{n} ds_\alpha\partial_\alpha\right)\mathbf{v}\right], \tag{F5}$$

where we have omitted the terms proportional to $(\Delta t)^2$.

The integral on the right-hand side of Eq. (F5) can be simplified if Eqs. (F1) and the definition of the total derivative, Eq. (72), are taken into consideration. Namely, Eqs. (F5), (F1), and (72) together lead to the following result:

$$
\begin{aligned}
&\Gamma_C(t+\Delta t)-\Gamma_C(t)\\
&\quad\approx \Delta t\oint_C d\mathbf{s}\cdot[\mathbf{F}-\gamma\mathbf{v}]+\Delta t\oint_C\left(\sum_{\alpha=1}^{n}ds_\alpha\partial_\alpha\right)\\
&\quad\times\left[\sum_{\beta=1}^{n}\left(\frac{m_\beta v_\beta^2}{2}+\frac{\hbar^2}{2m_\beta}\frac{\partial_\beta^2\sqrt{\rho}}{\sqrt{\rho}}\right)\right]\\
&\quad+\Delta t\oint_C\left\{\sum_{\alpha=1}^{n}ds_\alpha\left[\sum_{\beta=1}^{n}v_\beta(\partial_\alpha A_\beta-\partial_\beta A_\alpha)\right]\right\}\\
&\quad+\Delta t\oint_C d\mathbf{s}\cdot\left(\sum_{\beta=1}^{n}v_\beta\partial_\beta\right)\mathbf{A}\\
&\quad+\Delta t\oint_C\left\{\sum_{\alpha=1}^{n}ds_\alpha\left[\sum_{\beta=1}^{n}(\partial_\alpha v_\beta)A_\beta\right]\right\},
\end{aligned}
\tag{F6}
$$

and it is straightforward to verify that the sum of the last three terms on the right-hand side of this equation is zero. Moreover, the second term on its right-hand side vanishes, because it is a closed-path integral of a gradient of a single-valued function [28]. The term related to $\mathbf{F}(\mathbf{x})$ vanishes for the same reason, and we get

$$
\frac{d}{dt}\Gamma_C(t)=-\gamma\oint_C d\mathbf{s}\cdot\mathbf{v}(\mathbf{x},t).
\tag{F7}
$$

The above result leads to some important conclusions. First, if γ is equal to zero, which means that the system is nondissipative, the quantity $\Gamma_C(t)$ is a constant of motion, so in that case, if Eq. (104) is satisfied at the initial instant of time, it is possible to define the system's wave function and use the Schrödinger equation for determining its evolution. Second, if $\mathbf{A}(\mathbf{x})$ vanishes and the circulation of the velocity field is equal to zero at the initial instant of time, the quantity $\Gamma_C(t)$ is a constant of motion and we can define the system's wave function in analogy with Eq. (86). In that case a wave equation can be used for describing the system too. However, if both γ and $\mathbf{A}(\mathbf{x})$ are different to zero, the system's wave function cannot be defined and the Schrödinger equation cannot be used.

APPENDIX G: STATIONARY STATES OF NONDISSIPATIVE SYSTEMS

In this appendix we consider the stationary states of quantum-mechanical nondissipative systems.

We have shown in Sec. V A that the stationary states of a quantum-mechanical dissipative system can be determined with the help of Eq. (97). Now, we are going to show that the very same equation can be used for determining the stationary states of a nondissipative system which is not subject to any forces given by Eq. (103), provided that its velocity field satisfies Eq. (101) for certain integer k, including $k=0$, for any closed contour C in the system's configurational space.

For this purpose we first observe that the velocity field of the system corresponding to any stationary state for which $k=0$ must vanish, because the divergence of its product with the configurational probability density has to be equal to zero. Thus if $k=0$, we can repeat the discussion presented in Sec. V A, which leads to the conclusion that the real-valued solutions of Eq. (97) correspond to the stationary states of the considered system for $k=0$.

Assuming that $k\neq 0$ and that the configurational probability density $\rho(\mathbf{x})$ and probability velocity field $\mathbf{v}(\mathbf{x})$ correspond to a stationary state of the system, we can define a complex-valued function

$$
\phi(\mathbf{x})=\sqrt{\rho(\mathbf{x})}\exp\left[\frac{i}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}}d\mathbf{s}\cdot\mathbf{p}(\mathbf{s})\right],
\tag{G1}
$$

where $\mathbf{x}_0$ is an arbitrary fixed point in the system's configurational space. Then, $\rho(\mathbf{x})$ is the product of $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$, and the components of $\mathbf{v}(\mathbf{x})$ each satisfy Eq. (102) with $\phi(\mathbf{x})$ substituted for $\phi_\lambda(\mathbf{x})$. Thus we can express the average energy of the system, given by Eq. (93), in terms of $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$, i.e.,

$$
\langle E\rangle=\int d^nx\left\{\sum_{\alpha=1}^{n}\frac{\hbar^2}{2m_\alpha}[\partial_x\phi^*(\mathbf{x})][\partial_x\phi(\mathbf{x})]+V(\mathbf{x})\phi^*(\mathbf{x})\phi(\mathbf{x})\right\},
\tag{G2}
$$

and we can regard it as a functional of $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$. Accordingly, the necessary condition of the existence of minima of $\langle E\rangle$ is the vanishing of the functional derivatives with respect to $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$ of the functional

$$
\begin{aligned}
I[\phi(\mathbf{x}),\phi^*(\mathbf{x})]=\int d^nx\Bigg\{&\sum_{\alpha=1}^{n}\frac{\hbar^2}{2m_\alpha}[\partial_x\phi^*(\mathbf{x})][\partial_x\phi(\mathbf{x})]\\
&+[V(\mathbf{x})-E_\lambda]\phi^*(\mathbf{x})\phi(\mathbf{x})\Bigg\},
\end{aligned}
\tag{G3}
$$

where the constant E_λ has to be determined along with $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$. Hence we find that the local minima of the average energy correspond to $\phi_\lambda(\mathbf{x})$ that satisfy Eq. (97). We must remember, however, that these minima have been determined under specific assumption regarding the system's velocity field. Thus, formally, there could have been other minima of the right-hand side of Eq. (93) attained for the velocity fields that do not satisfy this condition and therefore are treated as nonphysical [1].

Now, if the right-hand side of Eq. (G1) is substituted for $\varphi_\lambda(\mathbf{x})$ in Eq. (97), we get

$$
\sum_{\alpha=1}^{n}\partial_\alpha[\rho(\mathbf{x})v_\alpha(\mathbf{x})]=0,
\tag{G4}
$$

$$
\sum_{\beta=1}^{n}v_\beta(\mathbf{x})\partial_\beta v_\alpha(\mathbf{x})=\frac{F_\alpha(\mathbf{x})}{m_\alpha}+\frac{\hbar^2}{2m_\alpha}\partial_\alpha\left[\sum_{\beta=1}^{n}\frac{1}{m_\beta}\frac{\partial_\beta^2\sqrt{\rho(\mathbf{x})}}{\sqrt{\rho(\mathbf{x})}}\right],
\tag{G5}
$$

where $F_\alpha(\mathbf{x})=-\partial_\alpha V(\mathbf{x})$, and $\alpha=1,2,\ldots,n$, which means that the time derivatives of $\rho(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ are identically equal to zero. Thus the states of the system that correspond to complex-valued solutions of Eq. (97) are all stationary.

If a quantum-mechanical nondissipative system is subject to the force given by Eq. (103), its stationary states cannot be determined with the help of Eq. (97), since the satisfaction of Eq. (104) is assumed for it [1]. However, the assumed satisfaction of Eq. (104) allows us to state that the complex-valued function $\phi(\mathbf{x})$ corresponding to a stationary state of the system should be given by [1]

$$\phi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}\exp\left\{\frac{i}{\hbar}\int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{s}\cdot[\mathbf{p}(\mathbf{s})+\mathbf{A}(\mathbf{s})]\right\}. \tag{G6}$$

Thus we have

$$\rho(\mathbf{x})v_\alpha(\mathbf{x}) = \frac{i\hbar}{2m_\alpha}[\phi(\mathbf{x})\partial_\alpha\phi^*(\mathbf{x}) - \phi^*(\mathbf{x})\partial_\alpha\phi(\mathbf{x})] - \frac{1}{m_\alpha}A_\alpha(\mathbf{x})\phi^*(\mathbf{x})\phi(\mathbf{x}), \tag{G7}$$

so we can express the average energy of the system, given by Eq. (93), in terms of $\phi(\mathbf{x})$, $\phi^*(\mathbf{x})$, and $\mathbf{A}(\mathbf{x})$ i.e.,

$$\begin{aligned}\langle E\rangle = {} & \int d^n x\left\{\sum_{\alpha=1}^{n}\frac{\hbar^2}{2m_\alpha}[\partial_\alpha\phi^*(\mathbf{x})][\partial_\alpha\phi(\mathbf{x})]\right\} \\ & - \int d^n x\left\{\sum_{\alpha=1}^{n}\frac{i\hbar}{2m_\alpha}[\phi(\mathbf{x})\partial_\alpha\phi^*(\mathbf{x}) \right. \\ & \left. - \phi^*(\mathbf{x})\partial_\alpha\phi(\mathbf{x})]A_\alpha(\mathbf{x})\right\} + \int d^n x\left[\sum_{\alpha=1}^{n}\frac{A_\alpha^2(\mathbf{x})}{2m_\alpha} + V(\mathbf{x})\right] \\ & \times\phi^*(\mathbf{x})\phi(\mathbf{x}),\end{aligned} \tag{G8}$$

and treat it as a functional of $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$. Next, we can observe that the necessary condition of the existence of minima of $\langle E\rangle$ is the vanishing of the functional derivatives with respect to $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$ of the functional

$$I[\phi(\mathbf{x}),\phi^*(\mathbf{x})] = \langle E\rangle - E_\lambda\int d^n x\,\phi^*(\mathbf{x})\phi(\mathbf{x}), \tag{G9}$$

where $\langle E\rangle$ is given by Eq. (G8), and the unknown constant E_λ has to be determined along with $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$. Hence we find that the system's average energy attains minima for $\phi_\lambda(\mathbf{x})$ that satisfy Eq. (105). Yet, again, these minima have been obtained under specific conditions regarding the system's velocity field, so formally there may be other minima of the right-hand side of Eq. (93) attained for the velocity fields that do not satisfy this condition, although they need not correspond to any physically relevant states of the system.

If the right-hand side of Eq. (G6) is substituted for $\phi_\lambda(\mathbf{x})$ in Eq. (105), we find that Eq. (G4) is satisfied and

$$\begin{aligned}& \sum_{\beta=1}^{n} v_\beta(\mathbf{x})\partial_\beta v_\alpha(\mathbf{x}) \\ & = \frac{1}{m_\alpha}\left\{F_\alpha(\mathbf{x}) + \sum_{\beta=1}^{n} v_\beta(\mathbf{x})[\partial_\alpha A_\beta(\mathbf{x}) - \partial_\beta A_\alpha(\mathbf{x})]\right\} \\ & \quad + \frac{\hbar^2}{2m_\alpha}\partial_\alpha\left[\sum_{\beta=1}^{n}\frac{1}{m_\beta}\frac{\partial_\beta^2\sqrt{\rho(\mathbf{x})}}{\sqrt{\rho(\mathbf{x})}}\right], \quad \alpha = 1,2,\ldots,n,\end{aligned} \tag{G10}$$

where $F_\alpha(\mathbf{x}) = -\partial_\alpha V(\mathbf{x})$, which confirms that the system's states represented by the functions $\phi_\lambda(\mathbf{x})$ corresponding to the local minima of its average energy are stationary.

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