Complex modes in unstable quadratic bosonic forms

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We discuss the necessity of using nonstandard boson operators for diagonalizing quadratic bosonic forms which are not positive definite and its convenience for describing the temporal evolution of the system. Such operators correspond to non-Hermitian coordinates and momenta and are associated with complex frequencies. As application, we examine a bosonic version of a BCS-like pairing Hamiltonian, which, in contrast with the fermionic case, is stable just for limited values of the gap parameter and requires the use of the present extended treatment for a general diagonal representation. The dynamical stability of such forms and the occurrence of nondiagonalizable cases are also discussed.

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Quadratic bosonic forms arise naturally in many areas of physics at different levels of approximation. Starting from the basic example of coupled harmonic oscillators, their ubiquity is testified to by their appearance in standard treatments of quantum optics [1], disordered systems [2], Bose-Einstein condensates [3–6], and other interacting many-body boson and fermion systems [7,8]. In the latter they constitute the core of the random-phase approximation (RPA), which arises as a first-order treatment in a bosonized description of the system excitations or, alternatively, from the linearization of the time-dependent mean-field equations of motion [time-dependent Hartree, Hartree-Fock (HF), or HF-Bogoliubov (HFB) [7,8]]. The ensuing forms are quite general and may contain all types of mixing terms (q_ip_j, q_iq_j, and p_ip_j) when expressed in terms of coordinates and momenta. Although the standard situation—i.e., that where the RPA is constructed upon a stable mean field (the Hartree, HF or HFB vacuum)—corresponds to a positive form, in more general treatments the RPA can also be made on top of unstable mean fields, as occurs in the study of instabilities in binary Bose-Einstein condensates [3–6], and even around nonstationary running mean fields, as in the case of the static path+RPA treatment of the partition function [9,10], derived from its path integral representation. In these cases the ensuing forms may not be positive and may lead, as is well known, to *complex* frequencies. Quadratic bosonic forms are also relevant in the study of dynamical systems [11–13], providing a basic framework for investigating diverse aspects such as integrals of motion and semiclassical limits.

Now, a basic problem with such forms is that while in the fermionic case they can always be diagonalized by means of a standard Bogoliubov transformation [7], in the bosonic case they may not admit a similar diagonal representation in terms of standard boson operators or in terms of the usual Hermitian coordinates and momenta. These cases can of course only arise in unstable forms which are not positive definite. The aim of this work is to discuss the diagonal representation of such forms in terms of nonstandard boson-like quasiparticle operators (or, equivalently, non-Hermitian coordinates and momenta), associated with complex normal modes. This requires the use of generalized Bogoliubov transformations since the usual one leads to a vanishing norm in the case of complex frequencies. The present treatment allows one then to identify the operators characterized by an exponentially increasing or decreasing evolution, providing a precise description of the dynamics and of the quadratic invariants in the presence of instabilities. It will also become apparent that an analysis of the dynamical stability based just on the Hamiltonian positivity may not be sufficient.

As an application, we will discuss a bosonic version of a BCS-type pairing Hamiltonian, which, in contrast with the fermionic case, exhibits a complex behavior, losing its positive definite character above a certain threshold value of the gap parameter and becoming dynamically unstable above a second higher threshold. In the presence of a perturbation it may even lead to a reentry of dynamical stability after an initial breakdown. This example illustrates the existence of simple quadratic forms which cannot be written in diagonal form in terms of standard boson operators or coordinates and momenta. Moreover, it also shows the existence of nondiagonalizable cases which do not correspond to a zero frequency (and hence to a free-particle term, in contrast with standard Goldstone or zero-frequency RPA modes arising from mean fields with broken symmetries [7]) and which are characterized by equations of motions which cannot be fully decoupled.

A general Hermitian quadratic form in boson annihilation and creation operators b_i and $b_i^\dagger$, can be written as

$$H=\sum_{i,j}A_{ij}\left(b_i^\dagger b_j+\frac{1}{2}\delta_{ij}\right)+\frac{1}{2}(B_{ij}b_i^\dagger b_j^\dagger+B_{ij}^*b_ib_j) \quad (1a)$$

$$=\frac{1}{2}Z^\dagger\mathcal{H}Z,\quad \mathcal{H}=\begin{pmatrix}A & B\\ B^* & A^t\end{pmatrix},\quad Z=\begin{pmatrix}b\\ b^\dagger\end{pmatrix}, \quad (1b)$$

where A is a Hermitian matrix, B is symmetric, and $Z^\dagger=(b^\dagger,b)$, with b and $b^\dagger$ arrays of components b_i and $b_i^\dagger$. The extended matrix $\mathcal{H}$ is Hermitian and satisfies in addition

$$\bar{\mathcal{H}}\equiv\mathcal{T}\mathcal{H}^t\mathcal{T}=\mathcal{H},\quad \mathcal{T}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \quad (2)$$

The boson commutation relations $[b_i,b_j]=[b_i^\dagger,b_j^\dagger]=0$ and $[b_i,b_j^\dagger]=\delta_{ij}$ can be succinctly expressed as

$$ZZ^\dagger - (Z^{\dagger t}Z^t)^t = \mathcal{M}, \quad \mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

It is well known that if the matrix $\mathcal{H}$ possesses only strictly *positive* eigenvalues, the quadratic form (1) can be diagonalized by means of a standard linear Bogoliubov transformation for bosons preserving Eqs. (3) [7]. This is the standard situation where (1) represents a stable system with a discrete positive spectrum, such as a system of coupled harmonic oscillators. In general, however, and in contrast with the fermionic case, it is not always possible to represent Eqs. (1) as a diagonal form in standard boson operators. The physical reason is obvious. If $\mathcal{H}$ is not strictly positive, Eq. (1) may represent the Hamiltonian of systems like a free particle or a particle in a repulsive quadratic potential ($H \propto p^2 - q^2$) when expressed in terms of coordinates and momenta, which do not possess a discrete spectrum. Nonetheless, one may still attempt to write (1) as a convenient diagonal form in suitable operators, such that the ensuing equations of motion become decoupled and trivial to solve.

Let us consider for this aim a general linear transformation [7,8]

$$Z = \mathcal{W}Z', \quad Z' = \begin{pmatrix} b' \\ \bar{b}' \end{pmatrix}, \tag{4}$$

where $\bar{b}'_i$ is not necessarily the adjoint of b'_i, although b'_i and $\bar{b}'_j$ are still assumed to satisfy the same boson commutation relations—i.e., $Z'\bar{Z}' - (\bar{Z}'^t Z'^t)^t = \mathcal{M}$, where $\bar{Z}' \equiv (\bar{b}', b') = Z'^t\mathcal{T}$. Since $Z^\dagger = \bar{Z}'\bar{\mathcal{W}}$, with $\bar{\mathcal{W}} \equiv \mathcal{T}\mathcal{W}^t\mathcal{T}$, the matrix $\mathcal{W}$ should then fulfill

$$\mathcal{W}\mathcal{M}\bar{\mathcal{W}} = \mathcal{M}, \tag{5}$$

implying $\mathcal{W}^{-1} = \mathcal{M}\bar{\mathcal{W}}\mathcal{M}$. No conjugation is involved in Eq. (5). Note that $\bar{Z} \equiv Z^t\mathcal{T} = Z^\dagger$ while in general $\bar{Z}' \neq Z'^\dagger = \bar{Z}'\bar{\mathcal{W}}(\mathcal{W}^\dagger)^{-1}$. If $\bar{b}' = b'^\dagger$, then $\bar{\mathcal{W}} = \mathcal{W}^\dagger$ (and vice versa) and Eq. (4) reduces to a standard Bogoliubov transformation for bosons [7,8]. Equations (4) allow one to rewrite H as

$$H = \frac{1}{2}\bar{Z}'\mathcal{H}'Z', \quad \mathcal{H}' = \bar{\mathcal{W}}\mathcal{H}\mathcal{W} = \begin{pmatrix} A' & B' \\ \bar{B}' & A'^t \end{pmatrix}, \tag{6}$$

where relation (2) is preserved ($\bar{\mathcal{H}}' = \mathcal{H}'$, implying $B', \bar{B}'$ symmetric), although in general $\mathcal{H}'^\dagger \neq \mathcal{H}'$. Finding a representation where $\mathcal{H}'$ is diagonal implies then an eigenvalue equation with "metric" $\mathcal{M}$—i.e., $\mathcal{H}\mathcal{W} = \mathcal{M}\mathcal{W}\mathcal{M}\mathcal{H}'$—which can be recast as a standard eigenvalue equation for a *non-Hermitian* matrix $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}}\mathcal{W} = \mathcal{W}\tilde{\mathcal{H}}', \quad \tilde{\mathcal{H}} \equiv \mathcal{M}\mathcal{H} = \begin{pmatrix} A & B \\ -B^* & -A^t \end{pmatrix}. \tag{7}$$

This matrix is precisely that which determines the temporal evolution of the system when H is the Hamiltonian, as the Heisenberg equation of motion for b and $b^\dagger$ is

$$i\frac{dZ}{dt} = -[H, Z] = \tilde{\mathcal{H}}Z. \tag{8}$$

Its solution for a time-independent $\tilde{\mathcal{H}}$ is therefore

$$Z(t) = \mathcal{U}(t)Z(0), \quad \mathcal{U}(t) = \exp[-i\tilde{\mathcal{H}}t] \tag{9}$$

(or in general $\mathcal{U}(t) = T\exp[-i\int_0^t \tilde{\mathcal{H}}(t')dt']$, where T denotes time ordering). The eigenvalues of $\tilde{\mathcal{H}}$ characterize then the temporal evolution and can be *complex* in unstable systems. Nevertheless, since $\tilde{\mathcal{H}}^\dagger = \mathcal{H}\mathcal{M} = \mathcal{M}\tilde{\mathcal{H}}\mathcal{M}$ and [Eq. (2)]

$$\mathcal{T}\tilde{\mathcal{H}}^t\mathcal{T} = -\mathcal{M}\tilde{\mathcal{H}}\mathcal{M}, \tag{10}$$

it is easily verified that the commutation relations (3) are always preserved $\forall t \in \Re$, as $\bar{\mathcal{U}}(t) \equiv \mathcal{T}\mathcal{U}^t\mathcal{T} = \mathcal{U}^\dagger(t)$ and $\mathcal{U}(t)\mathcal{M}\bar{\mathcal{U}}(t) = \mathcal{M}$. Moreover, the last identity remains valid also for *complex* times [although in this case $\bar{\mathcal{U}}(t) \neq \mathcal{U}^\dagger(t)$], so that Eq. (9) is a particular example of the general transformation (4), becoming a standard Bogoliubov transformation for bosons for $t \in \Re$.

Equation (10) implies that $\mathrm{Det}[\tilde{\mathcal{H}}^t - \lambda] = \mathrm{Det}[\tilde{\mathcal{H}} + \lambda]$, so that the eigenvalues of $\tilde{\mathcal{H}}$ (the same as those of $\tilde{\mathcal{H}}^t$) always come in pairs $(\lambda_i, \lambda_{\bar{i}})$ of *opposite* sign ($\lambda_{\bar{i}} = -\lambda_i$). Equation (10) also entails that the corresponding eigenvectors W_i (columns of $\mathcal{W}$) satisfy the orthogonality relations $\bar{W}_j\mathcal{M}W_i = -\bar{W}_i\mathcal{M}W_j = 0$ if $\lambda_i \neq -\lambda_j$, with $\bar{W}_i \equiv W_i^t\mathcal{T}$, which are those required by Eq. (5) (the required norm is $\bar{W}_{\bar{i}}\mathcal{M}W_i = 1$). In addition, for $\mathcal{H}$ Hermitian, $\mathrm{Det}[\tilde{\mathcal{H}} - \lambda]^* = \mathrm{Det}[\tilde{\mathcal{H}}^\dagger - \lambda^*] = \mathrm{Det}[\tilde{\mathcal{H}} - \lambda^*]$, so that if λ is an eigenvalue, so is λ^*. Combined with Eq. (10) this implies that if W_i is an eigenvector with eigenvalue λ_i, $\mathcal{W}_{\bar{i}^*} \equiv \mathcal{T}W_i^*$ is an eigenvector with eigenvalue $-\lambda_i^*$. For λ_i *real*, the required norm can then be reduced to the usual one for bosons [7], $W_i^\dagger\mathcal{M}W_i = 1$. However, for λ_i *complex*, the usual norm vanishes ($W_i^\dagger\mathcal{M}W_i = \bar{W}_{\bar{i}^*}\mathcal{M}W_i = 0$ as $\lambda_i \neq -\lambda_{\bar{i}^*} = \lambda_i^*$) while the present one does not in general. Note finally that the eigenvalues of $\tilde{\mathcal{H}}$ are the same as those of $\tilde{\mathcal{H}}_s \equiv \sqrt{\mathcal{H}}\mathcal{M}\sqrt{\mathcal{H}}$. When those of $\mathcal{H}$ are all *non-negative*, $\sqrt{\mathcal{H}}$ and hence $\tilde{\mathcal{H}}_s$ are *Hermitian*, so that all eigenvalues of $\tilde{\mathcal{H}}$ are *real*.

Let us assume now that the matrix $\tilde{\mathcal{H}}$ is *diagonalizable*, such that a nonsingular matrix $\mathcal{W}$ of eigenvectors exists. Then $\bar{\mathcal{W}}\mathcal{M}\mathcal{W}$ will be nonsingular, and due to the orthogonality relations can be set equal to $\mathcal{M}$ if eigenvectors are ordered and chosen such that $\bar{W}_{\bar{j}}\mathcal{M}W_i = \delta_{ij}$. The ensuing $\mathcal{W}$ will then satisfy Eqs. (5) and (7) with $\tilde{\mathcal{H}}'$ diagonal. Through the relation $\mathcal{H}' = \mathcal{M}\tilde{\mathcal{H}}'$ and Eq. (6) we obtain finally the diagonal representation

$$H = \sum_i \lambda_i\left(\bar{b}'_i b'_i + \frac{1}{2}\right), \tag{11}$$

where $b'_i = \bar{W}_{\bar{i}}\mathcal{M}Z$ and $\bar{b}'_i = Z^\dagger\mathcal{M}W_i$, with W_i and $W_{\bar{i}}$ the eigenvectors with eigenvalues λ_i and $-\lambda_i$ satisfying the present norm ($\bar{W}_{\bar{i}}\mathcal{M}W_i = 1$). If λ_i is real, we may choose $W_{\bar{i}}$

$=\mathcal{T}W_i^*$ such that $\bar{W}_{\bar{i}}=W_i^\dagger$ (with $W_i^\dagger\mathcal{M}W_i=1$) and hence $\bar{b}_i'=b_i'^\dagger$. Nonetheless, for complex λ_i, $\bar{b}_i'\neq b_i'^\dagger$. Equation (11) remains, however, physically meaningful, as the eigenvalues λ_i determine the temporal evolution. We immediately obtain from Eqs. (11) and (9) the decoupled evolution

$$b_i'(t)=e^{-i\lambda_i t}b_i'(0),\quad \bar{b}_i'(t)=e^{i\lambda_i t}\bar{b}_i'(0), \tag{12}$$

in all cases, together with the quadratic invariants $\bar{b}_i'b_i'=Z^\dagger\mathcal{M}W_i\bar{W}_{\bar{i}}\mathcal{M}Z$. If all eigenvalues λ_i are real and positive (with $\bar{b}_i'=b_i'^\dagger$), we have the standard case of a positive-definite quadratic form. If all λ_i are real but some of them are *negative* (with $\bar{b}_i'=b_i'^\dagger$), the system is unstable in the sense that H is no longer positive and does not possess a minimum energy, but the spectrum is still discrete and the temporal evolution (9) remains stable. Finally, when some of the λ_i are complex, the evolution becomes unbounded, with $b_i'(t)[\bar{b}_i'(t)]$ increasing [decreasing] exponentially for $\text{Im}(\lambda_i)>0$ and increasing t. In these cases the sign of λ_i in Eqs. (12) depends on the choice of operators and can be changed with the transformation $b_i'\to-\bar{b}_i'$, $\bar{b}_i'\to b_i'$ (which preserves the commutation relations) such that $\bar{b}_i'b_i'+\frac{1}{2}\to-(\bar{b}_i'b_i'+\frac{1}{2})$ (for λ_i real the sign can be fixed by the additional condition $\bar{b}_i'=b_i'^\dagger$). Cases where $\tilde{\mathcal{H}}$ is not diagonalizable (which may arise when its eigenvalues are not all different) are also dynamically unbounded as the temporal evolution determined by Eq. (9) will contain terms proportional to some power of t (times some exponential; see example).

We may also express (1) in terms of hermitian coordinates $q=(b+b^\dagger)/\sqrt{2}$ and momenta $p=(b-b^\dagger)/(\sqrt{2}i)$, satisfying $[p_i,p_j]=[q_i,q_j]=0$, $[q_i,p_j]=i\delta_{ij}$, as

$$H=\frac{1}{2}\sum_{i,j}T_{ij}p_ip_j+V_{ij}q_iq_j+U_{ij}q_ip_j+U^t_{ij}p_iq_j \tag{13a}$$

$$=\frac{1}{2}R^t\mathcal{H}_cR,\quad \mathcal{H}_c=\begin{pmatrix}V & U\\ U^t & T\end{pmatrix},\quad R=\begin{pmatrix}q\\ p\end{pmatrix}, \tag{13b}$$

where $V,T=\text{Re}(A\pm B)$ and $U=\text{Im}(B-A)$, with T, V, and $\mathcal{H}_c$ *symmetric*. The corresponding transformation is

$$Z=\mathcal{S}R,\quad \mathcal{H}_c=\mathcal{S}^\dagger\mathcal{H}\mathcal{S}, \tag{14}$$

where

$$\mathcal{S}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}$$

is unitary and satisfies $\mathcal{S}^\dagger=\mathcal{S}^t\mathcal{T}$. The commutation relation for R reads

$$RR^t-(RR^t)^t=\mathcal{M}_c,\quad \mathcal{M}_c=\mathcal{S}^\dagger\mathcal{M}\mathcal{S}=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \tag{15}$$

and the transformation (4) becomes

$$R=\mathcal{W}_cR',\quad \mathcal{W}_c\mathcal{M}_c\mathcal{W}_c^t=\mathcal{M}_c, \tag{16}$$

where $\mathcal{W}_c=\mathcal{S}^\dagger\mathcal{W}\mathcal{S}$ and $R'=\binom{q'}{p'}$ satisfies Eq. (15). Note that q',p' *will not be Hermitian* if $\mathcal{W}_c$ is complex. *Standard* linear canonical transformations among Hermitian coordinates and momenta correspond to $\mathcal{W}_c$ *real*, which is equivalent to the condition $\bar{\mathcal{W}}=\mathcal{W}^\dagger$ in Eq. (5).

We may now rewrite Eqs. (13) as $H=\frac{1}{2}R'^t\mathcal{H}_c'R'$, where $\mathcal{H}_c'=\mathcal{W}_c^t\mathcal{H}_c\mathcal{W}_c$ is symmetric although not necessarily real. Finding a representation with $\mathcal{H}_c'$ *diagonal* implies then the nonstandard eigenvalue problem

$$\tilde{\mathcal{H}}_c\mathcal{W}_c=\mathcal{W}_c\tilde{\mathcal{H}}_c',\quad \tilde{\mathcal{H}}_c=\mathcal{M}_c\mathcal{H}_c=i\begin{pmatrix}U^t & T\\ -V & -U\end{pmatrix}, \tag{17}$$

with $U'=0$ and V',T' *diagonal* in $\tilde{\mathcal{H}}_c'=\mathcal{M}_c\mathcal{H}_c'$, which leads to the coupled equations $\tilde{\mathcal{H}}_cW_{ci}=-iV_i'W_{c\bar{i}}$ and $\tilde{\mathcal{H}}_cW_{c\bar{i}}=iT_i'W_{ci}$, for the columns of $\mathcal{W}_c$. The required norm [Eqs. (16)] is again $\bar{W}_{c\bar{i}}\mathcal{M}W_{ci}=1$. The matrix $\tilde{\mathcal{H}}_c$ determines the evolution of q,p, as $idR/dt=\tilde{\mathcal{H}}_cR$, and its eigenvalues are of course *the same* as those of $\tilde{\mathcal{H}}$, as $\tilde{\mathcal{H}}_c=\mathcal{S}^\dagger\tilde{\mathcal{H}}\mathcal{S}$. If a matrix $\mathcal{W}_c$ (real or complex) satisfying Eqs. (16) and (17) exists, we obtain the diagonal form

$$H=\frac{1}{2}\sum_i(T_i'p_i'^2+V_i'q_i'^2),\quad T_i'V_i'=\lambda_i^2, \tag{18}$$

where $p_i'=-\bar{\mathcal{W}}_{ci}\mathcal{M}R$, $q_i'=\bar{\mathcal{W}}_{c\bar{i}}\mathcal{M}R$, and λ_i are the eigenvalues of $\tilde{\mathcal{H}}$ or $\tilde{\mathcal{H}}_c$. For $\lambda_i\neq0$ we may always set $T_i'=V_i'=\lambda_i$ by a scaling $p_i'\to s_ip_i'$, $q_i'\to q_i'/s_i$, where $s_i=\sqrt[4]{V_i'/T_i'}$ can be complex, in which case we may choose $W_{ci}=\mathcal{S}^\dagger(W_i+W_{\bar{i}})/\sqrt{2}$ and $W_{c\bar{i}}=i\mathcal{S}^\dagger(W_i-W_{\bar{i}})/\sqrt{2}$, with W_i and $W_{\bar{i}}$ the eigenvectors of $\tilde{\mathcal{H}}$ with eigenvalues $\pm\lambda_i$ satisfying $\bar{W}_{\bar{i}}\mathcal{M}\mathcal{W}_i=1$, such that $p_i'^2+q_i'^2=2\bar{b}_i'b_i'+1$. The ensuing operators p_i',q_i' will not be Hermitian when λ_i is complex, but their evolution will still be given by the usual expressions $q_i'(t)=q_i'(0)\cos(\lambda_it)+p_i'(0)\sin(\lambda_it)$ and $p_i'(t)=p_i'(0)\cos(\lambda_it)-q_i'(0)\sin(\lambda_it)$.

When $\tilde{\mathcal{H}}$ is *diagonalizable*, Eq. (18) is obviously equivalent to Eq. (11) (with $Z'=\mathcal{S}R'$ for $T'=V'$). However, Eq. (18) is more general since it may also contain *free-particle terms* $\frac{1}{2}T_i'p_i'^2$ when $\lambda_i=0$, which *cannot* be written in the form (11). In these cases the matrix $\tilde{\mathcal{H}}$ *is not diagonalizable*, as easily recognized from the ensuing linear evolution $p_i'(t)=p'(0)$ and $q_i'(t)=q_i'(0)+tT_i'p_i'(0)$, having a degenerate eigenvalue 0. Nonetheless, it should be emphasized that *it is not always possible* to represent Eq. (13) in the diagonal form (18), as nondiagonalizable cases where *no* eigenvalue of $\tilde{\mathcal{H}}$ vanishes *also exist* (see example). Let us also remark that if one considers just *Hermitian* q_i' and p_i' in Eq. (18), with T_i' and V_i' *real*, the eigenvalues λ_i of $\tilde{\mathcal{H}}$ are either real ($T_i'V_i'\geqslant0$) or purely imaginary ($T_i'V_i'<0$). Thus, quadratic forms whose matrix $\tilde{\mathcal{H}}$ possesses full complex eigenvalues (see example) *cannot be written in the diagonal form (18) unless non-Hermitian coordinates and momenta q',p' are admitted.*

The following example clearly illustrates the previous situations. Let us consider the Hamiltonian

$$H=\sum_{\nu=\pm}\varepsilon_\nu\left(b_\nu^\dagger b_\nu+\frac{1}{2}\right)+\Delta(b_+b_-+b_+^\dagger b_-^\dagger) \tag{19a}$$

$$=\frac{1}{2}\sum_{\nu=\pm}\varepsilon_\nu(p_\nu^2+q_\nu^2)+\Delta(q_+q_- - p_+p_-), \qquad (19b)$$

which represents two boson modes interacting through a BCS-like pairing term. We assume $\varepsilon_+>\varepsilon_->0$, and write $\varepsilon_\pm=\varepsilon\pm\gamma$, with $\varepsilon>0$, $0<\gamma<\varepsilon$. The eigenvalues of the ensuing matrix $\mathcal{H}$ (or $\mathcal{H}_c$), twofold degenerate, are

$$\sigma_\pm=\varepsilon\pm\sqrt{\gamma^2+\Delta^2}, \qquad (20)$$

which are both positive only for $|\Delta|<\sqrt{\varepsilon^2-\gamma^2}=\sqrt{\varepsilon_+\varepsilon_-}$ [the condition for a positive mass and potential tensor in Eq. (19b)]. However, the four eigenvalues of $\tilde{\mathcal{H}}=\mathcal{M}\mathcal{H}$ are

$$\lambda_\nu^\pm=\pm[\nu\gamma+\sqrt{\varepsilon^2-\Delta^2}], \quad \nu=\pm, \qquad (21)$$

which are real for $|\Delta|\leq\varepsilon=(\varepsilon_++\varepsilon_-)/2$. Thus, if $\sqrt{\varepsilon^2-\gamma^2}<|\Delta|<\varepsilon$, H is no longer positive definite ($\sigma_-<0$), but all eigenvalues $\lambda_\nu^\pm$ remain *real* (and distinct), implying that the temporal evolution is still *bounded* (quasiperiodic). However, for $|\Delta|>\varepsilon$, all eigenvalues are complex (with *nonzero* real part if $\gamma\neq 0$) and the evolution becomes unbounded.

Let us obtain now the diagonal representation of H. It is sufficient to consider in Eq. (5) a BCS-like transformation for bosons of the form

$$b_\nu=ub'_\nu-v\bar{b}'_{-\nu}, \quad b_\nu^\dagger=u\bar{b}'_\nu-vb'_{-\nu}, \qquad (22)$$

which correspond to $q_\nu=uq'_\nu-vq'_{-\nu}$ and $p_\nu=up'_\nu+vp'_{-\nu}$. The commutation relations are preserved if $u^2-v^2=1(\mathcal{W}\mathcal{M}\bar{\mathcal{W}}=\mathcal{M})$ and the inverse transformation ($\mathcal{M}\bar{\mathcal{W}}\mathcal{M}$) is obtained for $v\to -v$ ($b'_\nu=ub_\nu+vb_{-\nu}^\dagger$, $\bar{b}'_\nu=ub_\nu^\dagger+vb_{-\nu}$). Now, for

$$\begin{pmatrix}u\\v\end{pmatrix}=\sqrt{\frac{\varepsilon\pm\alpha}{2\alpha}}, \quad \alpha=\sqrt{\varepsilon^2-\Delta^2}, \qquad (23)$$

where we assume $\alpha\neq 0$ ($|\Delta|\neq\varepsilon$) and signs in square roots are to be chosen such that $2\alpha uv=\Delta$, we may express H as a sum of two independent modes,

$$H=\sum_{\nu=\pm}\lambda_\nu\left(\bar{b}'_\nu b'_\nu+\frac{1}{2}\right)=\frac{1}{2}\sum_{\nu=\pm}\lambda_\nu(p'^2_\nu+q'^2_\nu), \qquad (24)$$

where $\lambda_\nu\equiv\lambda_\nu^+$. If $|\Delta|<\varepsilon$, u and v are both real, so that $\bar{b}'_\nu=b'^\dagger_\nu$, with q'_ν and p'_ν, Hermitian, while if $|\Delta|>\varepsilon$, u and v are *complex*, implying $\bar{b}'_i\neq b'^\dagger_i$ and q'_i,p'_i no longer Hermitian. Instead, $(\lambda_\nu)^*=-\lambda_{-\nu}$ and $u^*=iv$ [with $\mathrm{Im}(\alpha)>0$ for $\Delta>0$], entailing $b'^\dagger_\nu=ib'_{-\nu},\bar{b}'^\dagger_\nu=i\bar{b}'_{-\nu}$ and $q'^\dagger_\nu=iq'_{-\nu}$, $p'^\dagger_\nu=-ip'_{-\nu}$. Note that in this case the usual norm vanishes ($|u|^2-|v|^2=0$) but the present one remains unchanged ($u^2-v^2=1$ still holds).

If $|\Delta|<\sqrt{\varepsilon^2-\gamma^2}$, $\lambda_\pm>0$, so that both modes have a discrete positive spectrum. However, if $\sqrt{\varepsilon^2-\gamma^2}<|\Delta|<\varepsilon$, $\lambda_+>0$ but $\lambda_-<0$, so that the spectrum of the lowest mode, though still discrete, becomes *negative*, implying that H has no longer a minimum energy. Care should be taken here to select the correct eigenvalue in Eq. (21), as $\tilde{\mathcal{H}}$ still has two positive eigenvalues ($\lambda_-^->0$). Note also that for $|\Delta|=\sqrt{\varepsilon^2-\gamma^2}$, $\lambda_-^\pm=0$, reflecting the onset of the instability, but $\tilde{\mathcal{H}}$ is still *diagonalizable*, as u and v remain finite. The lowest mode in Eq. (24) has here a single degenerate eigenvalue 0.

Finally, for $|\Delta|>\varepsilon$, the operators b'_ν and $\bar{b}'_\nu$ represent complex modes with an exponentially increasing or decreasing evolution. The evolution of the original operators b_ν and $b_\nu^\dagger$ for any $|\Delta|\neq\varepsilon$ can be immediately obtained from Eqs. (12) and (22) and is given by

$$b_\nu(t)=e^{-i\lambda_\nu t}[b_\nu+v(1-e^{2i\alpha t})(vb_\nu+ub_{-\nu}^\dagger)], \qquad (25)$$

where $b_\nu\equiv b_\nu(0)$ and $b_\nu^\dagger\equiv b_\nu^\dagger(0)$, with $b_\nu^\dagger(t)=[b_\nu(t)]^\dagger$. It becomes clearly unbounded for $|\Delta|>\varepsilon$.

For $|\Delta|=\varepsilon$, $\tilde{\mathcal{H}}$ *is not diagonalizable*, even though its eigenvalues $\lambda_\nu^\pm$ are in this case *all real and nonzero* (but degenerate), and H cannot be written in the form (24). However, the time evolution can still be obtained from Eq. (25) taking the limit $\alpha\to 0$, which leads to

$$b_\nu(t)=e^{-i\nu\gamma t}[(1-it\varepsilon)b_\nu-it\Delta b_{-\nu}^\dagger]. \qquad (26)$$

The factor t confirms that the evolution equations cannot be fully decoupled in this case, while the exponential multiplying this factor shows that they do not arise from a free-particle term either. We may, however, rewrite H in this case (assuming, for instance, $\Delta=\varepsilon$) as

$$H=\gamma(\bar{b}_+^s b_+^s-\bar{b}_-^s b_-^s)+2\Delta\bar{b}_-^s\bar{b}_+^s, \qquad (27)$$

where $b_\nu=(b_\nu^s+\bar{b}_{-\nu}^s)/\sqrt{2}$ and $b_\nu^\dagger=(\bar{b}_\nu^s-b_{-\nu}^s)/\sqrt{2}$, with $b_\nu^{s\dagger}=-b_{-\nu}^s$, $\bar{b}_\nu^{s\dagger}=\bar{b}_{-\nu}^s$, also satisfy boson commutation relations. In the form (27), H is "maximally decoupled," in the sense that the evolution equations for $\bar{b}_\nu^s$ are fully *decoupled*, while those of b_ν^s are coupled just to $\bar{b}_{-\nu}^s$. This leads to $\bar{b}_\nu^s(t)=e^{i\nu\gamma t}\bar{b}_\nu^s$ and $b_\nu^s(t)=e^{-i\nu\gamma t}(b_\nu^s-2it\Delta\bar{b}_{-\nu}^s)$. Equation (26) can also be obtained from these expressions. The associated invariants in this case are $\bar{b}_-^s\bar{b}_+^s$ and $\bar{b}_+^s b_+^s-\bar{b}_-^s b_-^s$—i.e., the two terms in Eq. (27)—which are mutually commuting.

If b_ν and $b_\nu^\dagger$ were fermion operators, Eq. (19a) would represent essentially a generic term of the standard BCS approximation to a pairing Hamiltonian [7] [$H_{BCS}=\sum_{k,\nu}\varepsilon_{k\nu}b_{k\nu}^\dagger b_{k\nu}+\sum_k\Delta_k(b_{k+}b_{k-}+b_{k-}^\dagger b_{k+}^\dagger)$, where $k\pm$ denote time-reversed states, Δ_k the BCS gap, $b_{k\nu}$ and $b_{k\nu}^\dagger$ fermion operators, and the splitting between $\varepsilon_{k\pm}$ may represent the effect of a Zeeman coupling to a magnetic field]. In the fermionic case, Eq. (19a) (with $\frac{1}{2}\to-\frac{1}{2}$) can be written as $\sum_\nu\lambda_\nu(b'^\dagger_\nu b'_\nu-\frac{1}{2})\forall\Delta$, where $\lambda_\nu=\nu\gamma+\alpha$, with $\alpha=\sqrt{\varepsilon^2+\Delta^2}$, are the quasiparticle energies and $b'_\nu,b'^\dagger_\nu$ quasiparticle fermion operators defined by $b_\nu=ub'_\nu+\nu vb'^\dagger_{-\nu}$, with $u,v=\sqrt{(\alpha\pm\varepsilon)/2\alpha}$. The analogous boson problem is, in contrast, stable just for limited values of Δ, as the latter *decreases* (rather than increases) the "quasiparticle energies" λ_ν. The onset of complex frequencies occurs finally when $\lambda_-=-\lambda_+$.

Let us also mention that in general, when $\mathcal{H}$ is not positive regions of dynamical stability may also arise between fully unstable regions. For instance, if a perturbation $\kappa(b_+^\dagger b_-+b_-^\dagger b_+)$ is added to Eq. (19), the eigenvalues of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ become $\sigma_\nu^\pm=\varepsilon+\nu\sqrt{\gamma^2+(\Delta\pm\kappa)^2}$ and $\lambda_\nu^\pm=\pm\sqrt{\tilde{\lambda}_\nu^2-\kappa^2(\varepsilon^2/\gamma^2-1)}$, with $\tilde{\lambda}_\nu=\nu\gamma+\sqrt{\Delta_c^2-\Delta^2}$ and $\Delta_c=\varepsilon^2(1+\kappa^2/\gamma^2)$. Those of $\mathcal{H}$ are split, and assuming κ small such that $\mathcal{H}$ is positive at $\Delta=0$, the two lowest ones $\sigma_{-}^{\pm}$ become negative at different values $\Delta_{c\pm}=\sqrt{\varepsilon^2-\gamma^2}\pm|\kappa|$. In such a case $\lambda_{-}^{\pm}$ becomes *imaginary* for $\Delta_{c-}<|\Delta|<\Delta_{c+}$, but returns again to *real values* for $\Delta_{c+}<|\Delta|<\Delta_c$ if $|\kappa|<\gamma^2/\sqrt{\varepsilon^2-\gamma^2}$, exhibiting a reentry of dynamical stability. Finally, both $\lambda_{\pm}$ become fully complex for $|\Delta|>\Delta_c$. A diagonal representation of the general form (24) is feasible except at the critical values $\Delta_{c\pm}$ and Δ_c.

In summary, we have extended the standard methodology employed for diagonalizing an Hermitian quadratic bosonic form, employing generalized quasiparticle-boson-like operators for describing unstable cases with arbitrary complex frequencies. In this way the operators exhibiting an exponentially increasing or decreasing temporal evolution are explicitly identified, together with the associated quadratic invariants, allowing for a precise characterization of the system evolution in the presence of general instabilities. While positive- definite forms can be considered completely stable, those which are not positive, but whose matrix $\tilde{\mathcal{H}}$ is diagonalizable and has only *real* eigenvalues, can still be considered *dynamically* stable, as the temporal evolution remains quasiperiodic, in contrast with the case where $\tilde{\mathcal{H}}$ has complex eigenvalues or is nondiagonalizable. Finally, we have seen that a BCS-like Hamiltonian for bosons can be completely stable, just dynamically stable, or unstable depending on the values of the gap parameter and requires the generalized approach for a diagonal representation valid for large gaps. Moreover, it also shows that cases where $\tilde{\mathcal{H}}$ is nondiagonalizable are not necessarily associated with zero frequencies or free-particle terms and may arise even if all its eigenvalues are nonzero. For such cases the evolution equations cannot be fully decoupled.

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