

Spin chains as perfect quantum state mirrors

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Quantum information transfer is an important part of quantum information processing. Several proposals for quantum information transfer along linear arrays of nearest-neighbor coupled qubits or spins were made recently. Perfect transfer was shown to exist in two models with specifically designed strongly inhomogeneous couplings. We show that perfect transfer occurs in an entire class of chains, including systems whose nearest-neighbor couplings vary only weakly along the chain. The key to these observations is the Jordan-Wigner mapping of spins to noninteracting lattice fermions that display perfectly periodic dynamics if the single-particle energy spectrum is appropriate. After a half-period of that dynamics, any state is transformed into its mirror image with respect to the center of the chain. The absence of fermion interactions preserves these features at an arbitrary temperature and allows for the transfer of nontrivially entangled states of several spins or qubits.

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Quantum information processing (QIP) [1] has been an increasingly important area of physics research over the past decade. The generic building block of QIP is the qubit, which is naturally realized as a spin-1/2 particle. A multitude of coupled spin-1/2 systems have been discussed as possible candidates for the quantum gates needed in quantum computing [2]. Only recently a new focal field of activity has developed, dealing not with the processing, but with the *transport* of quantum information. As most kinds of directed transport take advantage of one-dimensional structures it seems natural to explore the possibilities of one-dimensional arrays of coupled spin-1/2 systems as transmission lines for quantum information.

In [3] a sequence of external rf pulses was proposed to drive single-spin quantum information down a chain of Ising-coupled spins. Other studies proposed using the natural internal dynamics of coupled spins for the transfer of information. In a homogeneous ferromagnetic Heisenberg chain [4] initially in its ground state, a single-spin state generated at one end of the chain is transferred to the other end with reasonable (but not perfect) fidelity by means of spin waves. This approach is restricted to zero temperature and single spin-wave (and consequently single spin-flip) states since multiple spin-wave states are unstable under the Heisenberg interaction. This excludes the transport of entanglement, except for states of the type $\alpha|\uparrow\downarrow\rangle + \beta|\downarrow\uparrow\rangle$ (with the two spins initially located at fixed sites). The time evolution of these states was studied analytically [5] in an otherwise completely polarized infinitely long ferromagnetic Heisenberg chain. Initial states $\alpha|\uparrow\uparrow\rangle + \beta|\downarrow\downarrow\rangle$ were also studied analytically, but with the Heisenberg interaction changed to an XX interaction in order to exclude spin-wave interactions. Spin chain models for single-spin quantum information transport may be implemented as Josephson junction arrays [6]; another proposal [7] involves more general spin networks.

Gaussian spin wave packets (i.e., Gaussian-weighted superpositions of single-spin-flip states) may be more suitable for information transfer than single localized flipped spin states [8], because they may be tailored so as to occupy only the least dispersive part of the spin-wave dispersion relation

$\omega(k)$. Two [9,10] or more [11] spin chains may be used in parallel to enhance the fidelity of quantum information transfer via appropriate protocols. The dynamics of several entanglement-related quantities in the ground state of an infinitely long anisotropic XY chain with a supercritical magnetic field were also studied exhaustively [12], confirming the typical power-law decays found earlier in correlations of this model and related ones [13–20].

Other studies have looked for cases where a quantum state is transported *perfectly*, that is, without any decay or dissipation, along a spin chain. Perfect transmission is possible in a translationally invariant ring, if the time evolution operator for some particular time is equal to a lattice translation; it was found, however, that the 12-site antiferromagnetic Heisenberg ring does not fulfill the necessary conditions [21]. Perfect transmission was demonstrated [22] for an open-ended XX chain with *inhomogeneous* coupling: the amplitude for the transfer of a single flipped spin from one end of a completely spin-polarized chain to the opposite end is unity for certain times. This result is understood intuitively by observing that the, say, $2J+1$ sites of the chain can be mapped to the $2J+1$ eigenstates of J_z for a single fictitious particle with angular momentum quantum number J . With appropriately chosen matrix elements along the spin chain, the motion of the single flipped spin from one end of the chain to the opposite end, and back, corresponds to the motion of the fictitious particle's angular momentum in a transverse (x or y) magnetic field. That motion brings the fictitious particle from the $J_z=J$ state to the $J_z=-J$ state and back periodically. The perfect ground-state transport of more general states is also possible [24]. A state involving a number of spins in the left half of a symmetric quantum spin chain is *mirrored* about the middle of the chain so that the local information contained in it is effectively transported to the right half of the chain. The inhomogeneous XX chain of [22] shares this property with a similar system possessing an additional site-dependent magnetic field in the z direction.

Here we analyze the ingredients necessary for perfect quantum state mirroring (for both pure and mixed states) and

we show how the two particular systems described in [24] can be generalized to an infinite number of cases. This opens up possibilities for quantum spin chain engineering that may be used for the design of spin chains with desirable properties in addition to a perfect state transfer. As an example we construct a spin chain without local magnetic fields and with only mildly inhomogeneous couplings. In fact, the necessary degree of inhomogeneity decreases with the growing chain length. We further show that the perfect mirroring property is stable against thermal fluctuations and we demonstrate how that property makes arbitrary spin autocorrelation functions perfectly periodic in time.

The model we consider is an inhomogeneous open-ended $(N+1)$ site $S=\frac{1}{2}$ XX chain with Hamiltonian

$$H = 2 \sum_{i=1}^N J_i (S_i^x S_{i-1}^x + S_i^y S_{i-1}^y) + \sum_{i=0}^N h_i \left(S_i^z + \frac{1}{2} \right), \quad (1)$$

where $S_i^\alpha (\alpha=x,y,z)$ are the usual $S=\frac{1}{2}$ operators with eigenvalues $\pm\frac{1}{2}$, and J_i and h_i are local couplings and magnetic fields, respectively. Equation (1) can be mapped to a Hamiltonian of noninteracting spinless lattice fermions,

$$H = \sum_{i=1}^N J_i (c_{i-1}^\dagger c_i + c_i^\dagger c_{i-1}) + \sum_{i=0}^N h_i c_i^\dagger c_i, \quad (2)$$

by means of the Jordan-Wigner transformation [25,26] between spin and fermion operators:

$$S_i^z = c_i^\dagger c_i - \frac{1}{2}, \quad (3)$$

$$S_i^+ = (-1)^{\sum_{k=0}^{i-1} c_k^\dagger c_k} c_i^\dagger = \prod_{k=0}^{i-1} (1 - 2c_k^\dagger c_k) c_i^\dagger. \quad (4)$$

Due to its bilinear nature, Eq. (2) can be diagonalized,

$$H = \sum_{\nu=0}^N \varepsilon_\nu c_\nu^\dagger c_\nu. \quad (5)$$

Here c_ν^\dagger creates a fermion in a single-particle eigenstate of energy ε_ν , whereas c_i^\dagger creates one at lattice site i . Once the ε_ν and the corresponding eigenstates are known, the dynamics generated by H can be calculated in detail, since every eigenstate of H is uniquely characterized by the fermion occupation numbers of the single-particle states.

The single-particle energies $\varepsilon_\nu (\nu=0, \dots, N)$ and the corresponding eigenstates are the eigenvalues and eigenvectors, respectively, of a symmetric tridiagonal matrix H_1 with diagonal elements (h_0, h_1, \dots, h_N) and subdiagonal elements (J_1, J_2, \dots, J_N) ; all J_i are strictly positive [27].

We further assume the system to possess a mirror symmetry, $h_i = h_{N-i}$ and $J_i = J_{N+1-i}$. The $(N+1)$ -dimensional eigenvectors \vec{x} of H_1 , $H_1 \vec{x} = \varepsilon_\nu \vec{x}$, then have definite parity, that is, every eigenvector is either even, with components $x_i = x_{N-i}$, or odd, $x_i = -x_{N-i}$. This resembles closely the situation in elementary one-dimensional (continuum) quantum mechanics, where for a symmetric potential the energy eigenfunctions have definite parity and are nondegenerate. From the well-

known theorem relating the number of zeros of the wave function to the number of the energy eigenvalue (in ascending order), one can then conclude that even and odd eigenfunctions alternate as energy increases. It turns out that the same line of arguments is possible for the quantum lattice system of interest here, due to the following theorem [28]: *For a real symmetric tridiagonal matrix with only positive subdiagonal elements: (i) all eigenvalues are real and non-degenerate and (ii) the sequence of the components of the j th eigenvector (in ascending order of the eigenvalues, $j=0, 1, \dots$) shows exactly j sign changes.* This implies that the eigenvectors of H_1 are alternately even and odd. To achieve perfect mirror inversion of an arbitrary *many-particle* state by some operation M it is sufficient that M transforms all *single-particle* eigenstates into their mirror images. Note that M is not simply the reflection (parity) P , since P changes the sign of all odd single-particle states. That sign change must be compensated for by a dynamic phase factor $\exp[i\pi(2n+1)]$ (n is an arbitrary integer) to achieve perfect mirror inversion. This entails conditions on the energy spectrum ε_ν : a single-particle state $|i\rangle$ localized at site i evolves in time according to

$$e^{-iHt}|i\rangle = \sum_{\nu=0}^N e^{-i\varepsilon_\nu t} |\nu\rangle \langle \nu|i\rangle, \quad (6)$$

where $|\nu\rangle$ is a single-particle energy eigenstate with energy $\varepsilon_\nu (\hbar=1)$. The alternating parity implies $\langle N-i|\nu\rangle = (-1)^\nu \langle i|\nu\rangle$, and thus

$$|N-i\rangle = \sum_{\nu} |\nu\rangle \langle \nu|N-i\rangle = \sum_{\nu} (-1)^\nu |\nu\rangle \langle \nu|i\rangle. \quad (7)$$

Perfect quantum state mirroring occurs if, for some time τ , time evolution equals reflection,

$$e^{-iH\tau}|i\rangle = e^{i\phi_0}|N-i\rangle \quad (8)$$

(up to some global phase ϕ_0) for all i . This is obviously the case if $e^{-i\varepsilon_\nu \tau} = e^{-i(\pi\nu + \phi_0)}$, or equivalently

$$\varepsilon_\nu \tau = [2n(\nu) + \nu]\pi + \phi_0, \quad (9)$$

where $n(\nu)$ is an *arbitrary* integer function. Note that every spatially symmetric system whose single-particle energies obey (9) generates a perfect mirror image of *any* input state since it does so for all single-particle states.

The two systems discussed in [24] correspond to (i) a linear spectrum, $\varepsilon_\nu = \omega_0 + \nu\omega$, which, for $\tau = \pi/\omega$, leads to $\phi_0 = \pi\omega_0/\omega$ and $n(\nu) \equiv 0$, and (ii) a quadratic spectrum, $\varepsilon_\nu = \omega_0 + \nu[\nu+1+(2p+1)/q]\omega$, p and q positive integers, which, for $\tau = q\pi/\omega$, leads to $\phi_0 = \pi q\omega_0/\omega$ and $n(\nu) = q[\nu(\nu+1)/2] + p\nu$.

As the function $n(\nu)$ in (9) is completely arbitrary, there are infinitely many single-particle spectra suitable for quantum state mirroring. For a given nondegenerate single-particle spectrum there exists a unique symmetric tridiagonal Hamiltonian matrix with non-negative subdiagonal elements and with the additional spatial symmetry properties discussed above [29]. The actual construction of that matrix from its eigenvalues may proceed either via a direct algo-

rithm [30] or by simulated annealing [31]. The possibility of generalizing the work of [24] by solving an inverse eigenvalue problem was already pointed out in [32,33].

While most schemes for quantum information transfer in spin chains are restricted to states generated from the completely polarized ground state by manipulating a single spin, we stress that the class of models discussed here perfectly mirrors states involving an arbitrary number of lattice sites. This is due to the fact that the Jordan-Wigner transformation (3) and (4) maps the spins to *noninteracting* fermions. Since due to (9) all single-fermion states are mirrored perfectly at the same instant of time τ , so are all many-fermion states, pure or mixed, including finite-temperature density operators.

Since mirroring twice reproduces the initial state, the time evolution of the system should be periodic with period 2τ . This can, in fact, be verified by considering the equilibrium time autocorrelation function of an arbitrary observable $A = A^\dagger$:

$$\begin{aligned} \langle A(t)A \rangle &= Z^{-1} \sum_n \langle n | e^{-\beta H} e^{iHt} A e^{-iHt} | n \rangle \\ &= Z^{-1} \sum_{n,m} e^{-\beta E_n} e^{i(E_n - E_m)t} |\langle n | A | m \rangle|^2, \end{aligned} \quad (10)$$

where $Z = \sum_n e^{-\beta E_n}$ with $\beta = (k_B T)^{-1}$ is the canonical partition function; $|n\rangle$ is a many-particle eigenstate of H . Since all $(E_n - E_m)$ are multiples of some energy, $\langle A(t)A \rangle$ is a periodic function of t . See Fig. 2 below, for an example.

The freedom in the choice of $n(\nu)$ in (9) may be used for *quantum spin chain engineering*, that is, the design of systems with desirable additional properties. As an example of these possibilities we show how the well-studied homogeneous XX chain [$J_i \equiv J$ in (1)] may be modified to display the perfect mirroring property. The system proposed in [22] has J_i growing from the boundary to the center of the system, with a maximum value proportional to the chain length, if the time for perfect transfer is kept constant. If the maximum achievable J_i is fixed by physical restrictions, the J_i toward the boundary of the system decrease $\sim N^{-1/2}$ as the chain length N grows and the transmission time grows $\sim N$, implying a constant signal transmission velocity [23]. The very weak couplings close to the ends of the chain would make such a system extremely susceptible to external perturbations. It is therefore interesting to know whether the same performance (constant signal velocity) can be achieved with more homogeneous “wires,” that is, less variation in the J_i . Note that $J_i \equiv J$ leads to the familiar spin wave dispersion relation $\varepsilon_\nu = \omega(k) = 2J \cos k$ [$k = \pi(\nu+1)/(N+2)$] for the single-particle states. For small k , thus, $\varepsilon_\nu \sim -2J - k^2$, which fulfills the spectral condition (9) for perfect mirror inversion. It is possible, by only slightly distorting the cosine dispersion, to fulfill that condition throughout the entire k range. If the unit of energy (say, the energy gap above the lowest single-particle eigenvalue) is kept constant, the coupling obviously scales with chain length as $J \sim N^2$. In that case the transmission time is constant, regardless of the chain length. For an easy comparison of energy eigenvalues and periodicities between chains of different lengths, we decided to

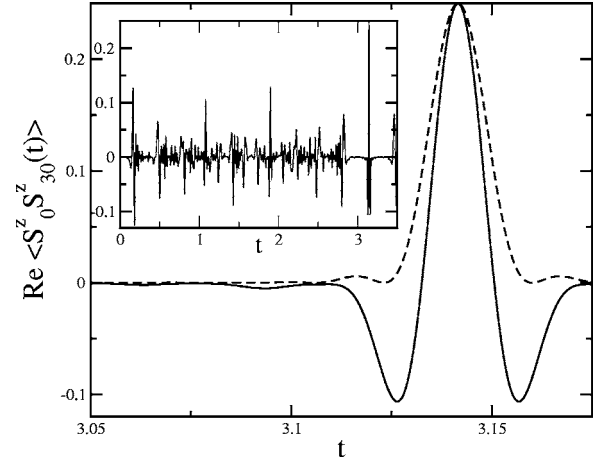


FIG. 1. z spin correlation function (real part) between the two ends of a 31-spin chain at temperatures $T=0$ (solid line) and $T=1000$ (dashed line), in the vicinity of $t=\pi$. The maximum possible value $1/4$ of the correlation at $t=\pi$ demonstrates perfect state transfer. Inset: same correlation for $T=0$ over an extended time range. (Note the symmetry with respect to $t=\pi$; the correlation function is periodic with period 2π .)

choose that possibility. The eigenvalues ε_ν of a chain with 31 sites, for example, may be chosen as $\varepsilon_\nu = 0, \pm 21, \pm 40, \pm 61, \pm 80, \pm 97, \pm 116, \pm 131, \pm 146, \pm 161, \pm 172, \pm 183, \pm 192, \pm 199, \pm 204, \pm 207$. The corresponding nearest-neighbor couplings J_i then vary between 101.5 and 108.5, a relative variation of ± 3.3 percent. With growing length the variation in the J_i decreases; for a 50-site system, for example, it amounts to only about one percent.

In Fig. 1 we demonstrate the perfect transfer of a single-site state between the first and last sites of a chain with $N=30$. The correlation $\langle S_0^z(0) S_{30}^z(t) \rangle$ is zero at $t=0$ and reaches its maximum possible value $1/4$ at $t=\tau=\pi$. Thus, an S^z eigenstate initially prepared at site 0 can be retrieved at the

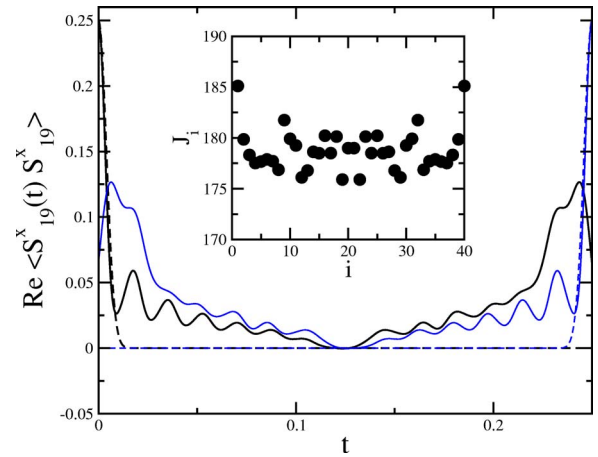


FIG. 2. (Color online) x spin autocorrelation function (real part) at site 19 of a 41-spin chain at temperatures $T=0$ (solid line) and $T=10^4$ (dashed line). The maximum at the left edge of the graph is the one at $t=0$, that at the right edge (slightly thinner blue lines) is the one at $t=48\pi$, after 24 periods of the correlation function (t axis shifted). Inset: nearest-neighbor exchange constants J_i .

initially uncorrelated site 30 after time τ at any temperature. The inset of Fig. 1 gives an impression of the irregular behavior of the correlation during the whole time range between 0 and π .

Figure 2 shows the autocorrelation function of the x spin component at site 19 in a 41-site chain, demonstrating the periodicity, Eq. (10). It is important to note that by the Jordan-Wigner transformation (4) a two-spin correlation $\langle S_k^x(0)S_k^x(t) \rangle$ becomes (in fermionic terms) a complicated many-particle correlation involving lattice sites 0 through k . The single-fermion eigenstates and eigenvalues enter the correlation function in an entirely nontrivial way; nevertheless it is perfectly periodic. The figure shows the initial decay of the correlation at $t=0$ and its 24th “revival” at $t=48\pi$. Note the rapid decay and the absence of oscillations at high T . In fact, for the homogeneous XX chain ($J_i \equiv J$) the x autocorrelation is known [34–36] to be a Gaussian at $T=\infty$ in the thermodynamic limit, while all nonlocal x correlations vanish identically. The inset shows the weakly varying nearest-neighbor exchange constants J_i . Note that the correlation functions

displayed in Figs. 1 and 2 are not particularly significant for quantum information processing; they were picked more or less arbitrarily to illustrate our main result, the perfect transfer of many-qubit states.

To conclude, we have found an infinitely large class of inhomogeneously coupled spin chain systems capable of perfect quantum information transfer. The freedom of choice within that class relaxes the stringent constraints on the local spin couplings and magnetic fields that previous models [22,24] had to meet. We have demonstrated perfect state transfer over fairly long distances in a chain with almost homogeneous exchange coupling and without an external magnetic field. While many previous proposals have been restricted to the transfer of single-spin states at zero temperature, the systems discussed here are capable of transferring genuinely entangled states involving several qubits, at an arbitrary temperature. Sensitivity to perturbations like noise and imperfections [6,10,37] will be the subject of further research.

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