Vortices in condensate mixtures

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In a condensate made of two different atomic molecular species, Onsager's quantization condition implies that around a vortex, the velocity field cannot be the same for the two species. We explore some simple consequences of this observation. Thus, if the two condensates are in slow relative translation one over the other, the composite vortices are carried at a velocity that is a fraction of the single-species velocity. This property is valid for attractive interaction and below a critical velocity which corresponds to a saddle-node bifurcation.

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I. INTRODUCTION

One remarkable result in condensed-matter physics is the discovery by Onsager [1] that, in a superfluid, the vorticity can be present along narrow lines with a quantized circulation. Indeed, the integral $\int \mathbf{u}.\mathbf{ds}$ of the fluid velocity $\mathbf{u}$ taken along a circuit enclosing a vortex line must be a positive or negative integer multiple of h/m, where h is Planck's constant and m is the mass of the particles making the superfluid. This creates a striking analogy between the dynamics of the quantum vortices of a superfluid and the Kelvin vortices of a classical inviscid fluid, the quantization being present only to specify the value of the circulation, an arbitrary quantity in a classical fluid. This analogy between classical inviscid fluids and superfluids is at the heart of our understanding of superfluid mechanics, beginning with the Landau two-fluid theory [2] when there is no normal fluid. However, things are not so simple, just because the quantization of the circulation involves explicitly the mass of the particles. Therefore, if there is more than one species of particles with different masses, it is not obvious at all that classical fluid mechanics remains the right theory to describe the large-scale motion of this mixture with quantum vortices. This is because, in such a mixture, one does not know *a priori* which mass enters into Onsager's circulation condition.

This question seems to be irrelevant for superfluid ^{4}He, because it has no other stable bosonic isotope, and mixing it with any other atomic or molecular liquid is not possible at temperatures low enough to observe superfluidity. Mixtures of ^{4}He and ^{3}He (a fermion) can remain liquid, although the spin effects in ^{3}He make the whole picture quite different, but certainly extremely interesting from the present point of view (see Ref. [17]). We look at a situation that can be, presumably, realized in atomic vapors, namely, a mixture of two bosonic atoms (or eventually molecules) [3–5]. We consider the following general problem: given two species in the same gas, both condensed, what are the dynamical properties of the large-scale motion of this mixture? This problem has been looked at without vortices included (see [6,7]).

II. THE COUPLED GROSS-PITAEVSKIĬ EQUATIONS

By extrapolating from what is known about single-species condensates, one can think of many relevant issues such as the normal modes extending to mixtures in the Bogoliubov spectrum or the density profiles in harmonic traps. When looking at the fluid motion itself, one of the most interesting issues is the behavior of vortices.

We assume that the mixture is at zero temperature and that each molecular/atomic species of molecular mass m_j is described by a macroscopic wave function $\Psi_j(\mathbf{r},t)$, a complex valued function of the position $\mathbf{r}$ and of time t with the discrete index j being either 1 or 2, to denote the species under consideration. (One could deal as well with more than two species.) The equation of evolution of the coupled $\Psi_j(\mathbf{r},t) j=1,2$ is *a priori* an extension [6] of the familiar Gross-Pitaevskiĭ equation (GP)

$$i\hbar\frac{\partial\Psi_j}{\partial t} = -\frac{\hbar^2}{2m_j}\nabla^2\Psi_j + a_j|\Psi_j|^2\Psi_j + g|\Psi_{(j+1)}|^2\Psi_j. \qquad (1)$$

The proceding is for two coupled equations, with $j=1$ and $j=2$. In the interaction term, the index $(j+1)$ is computed mod 2, $1+1=2$, $2+1=1$. Last, the interaction real parameters a_j and g are such that the mixture is stable against collapse and against separation into two phases, one rich in 1, the other in 2. The stability depends on the minimum of the interaction part of the energy, the volume integral of

$$\left(\frac{a_1}{2}|\Psi_1|^4 + \frac{a_2}{2}|\Psi_2|^4 + g|\Psi_1|^2|\Psi_2|^2\right).$$

The mixture is then stable against collapse if a_1 and a_2 are positive and if $a_1 a_2 > g^2$. The linear stability against demixing is determined by the Bogoliubov spectrum of excitation. We obtain it by seeking the dispersion relation between the frequency ω and the wave number k of the linear perturbations around the homogeneous state of densities $\rho_j=|\Psi_j|^2$, respectively,

$$\left[\omega^2-\frac{k^2}{m_1}\left(a_1\rho_1+\frac{\hbar^2k^2}{m_1}\right)\right]\left[\omega^2-\frac{k^2}{m_2}\left(a_2\rho_2+\frac{\hbar^2k^2}{m_2}\right)\right]=\frac{g^2\rho_1\rho_2k^4}{m_1m_2}. \quad (2)$$

For uncoupled condensates $g=0$, we retrieve the Bogoliubov spectrum for each condensate. The condition for linear stability against demixing (ω real for all k) leads to the same criterion $a_1a_2>g^2$. The coupled equations (1) are Galilean invariant, and one can thus consider the flow of both condensates at the same constant velocity through Galilean boosts of the wave functions. Moreover, notice that for $g=0$, the uncoupled equation (1) are Galilean invariant separately, so one can consider a relative constant flow between each species. For weak coupling, one can then generalize this property and thus consider the relative flow of one species with respect to the other one. If the condensates are homogenous, such flow remains an exact solution of the equations and the model allows an extra "superfluid" property, that is, the two species can flow into each other without dissipation. For inhomogenous condensates, such as those containing vortices, for instance, the interaction between the two species generates a friction force, and the vortex dynamics are affected by the presence of the two species. The goal of the present paper is precisely to exhibit such motion for simple cases.

From the coupled equations (1), the vortices bear a double-integer index, denoting the numbers of the phase winding of each wave function around the core of the vortex. The solution of equations (1) for a vortex (n_1,n_2) is of the form $\Psi_j=e^{-iE_jt}e^{in_j\theta}\chi_j(r)$, where (r,θ) are the polar coordinates in the plane perpendicular to the vortex axis. The real functions $\chi_j(r)$ are solutions of two coupled ordinary differential equations

$$\hbar E_j\chi_j=-\frac{\hbar^2}{2m_j}\left(\chi_j''+\frac{1}{r}\chi_j'-\frac{n_j^2}{r^2}\chi_j\right)+a_j\chi_j^3+g\chi_{(j+1)}^2\chi_j, \quad (3)$$

where the primes stand for the derivative along the radius $r(\chi_j'=d\chi_j/dr)$. The asymptotic conditions are that, at very large r, χ_j tends to ρ_j^0, the uniform density of the species j, so that $\hbar E_j=a_j\rho_j^0+g\rho_{j+1}^0$. Moreover, χ_j behaves like $r^{|n_j|}$ at small r. We later restrict our study to the two-dimensional (2D) case, although three-dimensional (3D) dynamics should reveal interesting behavior (unzipping, Kelvin waves, etc.) and is postponed to further work. We assume that the multiply charged vortices (at least one of the $n_j>1$) are unstable and decompose into separated single-charged vortices, as is the case in general for uncoupled condensates [8,9]. Moreover, because of the coupling via the term proportional to g in the original equations, the vortices of the composite index (both $n_j=\pm1$) can be either stable, with a joint zero at the same location, or unstable, when such a composite vortex can decompose into one single-charged vortex in each condensate, not located at the same position. The interaction between vortices belonging to different species, like vortices (0, 1) and (1, 0), is short ranged, because it depends on the density distribution near the vortex core (the interaction between vortices of the same species is long-ranged, because of the velocity field decaying as $1/r$ at large distances from the core) [10]. A reasonable guess is to assume that for negative g, the vortices of different species attract each other (whatever their relative sign), although their interaction is repulsive for positive g. This is based on the fact that the "interaction energy" is, in a first approximation (that is for small g), represented by the integral of $g(\rho_1-\rho_1^0)(\rho_2-\rho_2^0)$, positive (repulsive) for positive g and negative (attracting) for negative g. Because of the Hamiltonian structure of the dynamics, this instability is very slow, since it manifests through radiation coming from the vortex acceleration [11]. Our numericas analysis is in complete agreement with this point: we observe that the two vortices stand at the same position for negative g while they describe a slow, outward-spiraling relative motion for positive g.

We introduce here a convenient dimensionless version of the model. Rescaling space and time by the factors $(m_1m_2a_1a_2)^{1/4}/\hbar$, and $(a_1a_2)^{1/2}/\hbar$, respectively, we obtain the following set of two coupled equations:

$$i\frac{\partial\Psi_j}{\partial t}=-\frac{\alpha_j}{2}\nabla^2\Psi_j+\beta_j|\Psi_j|^2\Psi_j+g|\Psi_{(j+1)}|^2\Psi_j, \quad (4)$$

with $\alpha_1=1/\alpha_2=\sqrt{m_2/m_1}$ and $\beta_1=1/\beta_2=\sqrt{a_1/a_2}$. Figures 1 and 2, will illustrate the case $\alpha_1=\beta_1=1$, since no new effects appear when considering different α and β (but staying in the domain where the mixture remains thermodynamically stable). For uncoupled condensates ($g=0$), we note that the vortex solution $\Psi_j^0(\mathbf{r})$ for each condensate is determined by a single function f

$$\Psi_j^0(\mathbf{r})=\sqrt{\rho_j^0}f\left(\sqrt{\frac{\beta_j\rho_j^0}{\alpha_j}}r\right)e^{i(\epsilon_j\theta-\beta_j\rho_j^0t)}, \quad (5)$$

where $\epsilon_j=\pm1$ describes the sign of the circulation of each vortex, and with f the real solution for the equation (see Ref. [12])

$$-\frac{1}{2}\left[f''(r)+\frac{f'(r)}{r}-\frac{f(r)}{r^2}\right]+[f^2(r)-1]f(r)=0.$$

III. NUMERICAL RESULTS

Consider now the effect of a flow on a composite vortex, when g is negative (that is, when this composite vortex is stable). Because of the Galilean invariance of the coupled equations, we have only to study the relative flow of one species (say, 1) with respect to the other at constant speed (say, along direction 1, $\mathbf{v_1}=v_1\mathbf{e_1}$). Numerical analysis shows that at low speed, the single composite vortex splits first into two vortices, a (0, 1) vortex and a (1, 0) vortex. Then the two vortices move together almost at a constant speed that is a fraction (as explained later) of the speed of the moving species. Oscillations are observed around this stationary motion, and the two vortices are also oriented to each other in the direction orthogonal to the imposed velocity. At higher speeds, the vortices split apart, one being carried by the fluid of the same species, the other one remaining immobile, unaffected by the velocity of the other species. Figure 1 illus-

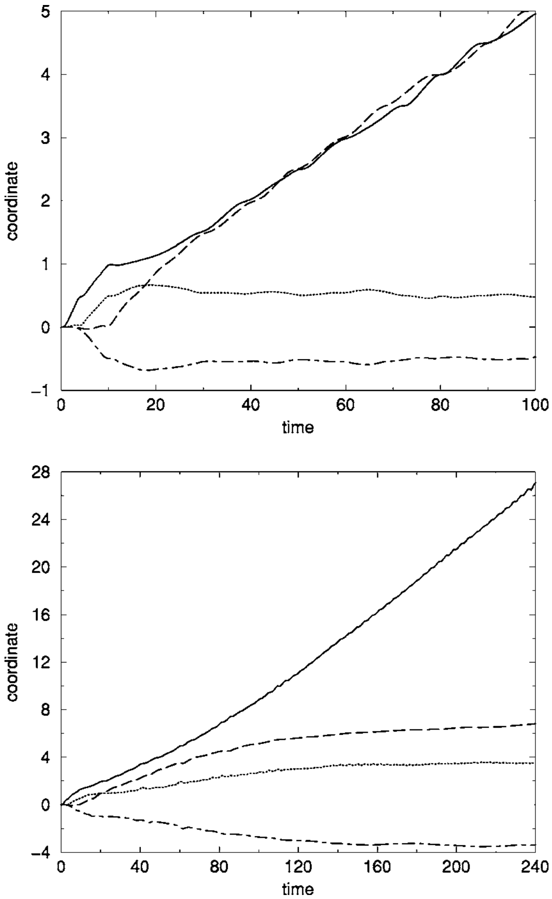

FIG. 1. Coordinates of the two vortices, one in the moving condensate and one in the condensate at rest, for a drift velocity (a) $v=0.1$ and (b) $v=0.14$, with $\alpha_1=\beta_1=1$ and $\rho_1^0=\rho_2^0$. The bold lines and dashed lines represent the coordinates parallel to the drift velocity for the vortex in the moving and the static condensate, respectively. Similarly, the dotted and dashed-dotted lines represent the transversal coordinates. The first vortex moves at the drift velocity until it pulls with it the vortex in the static condensate. Meanwhile, the two vortices move apart in the transversal direction. In the first case (a), the two vortices reach a constant drift velocity which is half of the imposed velocity as predicted by the theory. On the other hand, for higher velocity (b), the second vortex cannot follow the moving one, and they both reach the drift velocity of their condensate.

trates these major effects. We use a pseudospectral scheme that allows for simple rotations of the wave function in real (for the nonlinear part) and Fourier (for the linear terms) spaces. An efficient fast Fourier transform (FFT) [13] is used, and the Yoshida scheme for the Hamiltonian system is employed to improve the efficiency. Initial conditions are taken as square periodic patterns of alternate-sign vortices (to allow for periodic boundary conditions), located at the same position for both species. An imposed velocity $\mathbf{v_1}$ is applied to the first condensate only.

IV. SOLVABILITY CONDITIONS

To understand this condition, the simplest thing is to solve the original equations (1) by perturbation, assuming the coupling term to be small and the uniform velocity of species 1 and species 2 as being motionless. The zero-order solution is a pair of (1, 0) and (0, 1) vortices, the first one being located at $\mathbf{r}_1(t)$, the other at $\mathbf{r}_2(t)$. Without flow speed and without interactions, both vortices remain where they are. As the speed of species 1 and the interaction is turned on, one finds by an expansion in powers of a unique small parameter the velocity and the strength of the interaction, the time-dependent solution of the equations. The equations of motion for $\mathbf{r_1}(t)$ and $\mathbf{r_2}(t)$ follow from a solvability condition in this expansion. We in fact seek solutions in the form

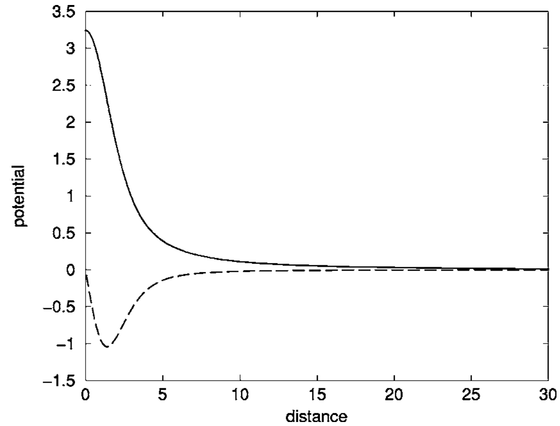

FIG. 2. The potential $U(r)$ (bold line) and its derivative (dashed line) calculated in the case $\alpha_1=\beta_1=1$.

$$\Psi_j(r,t)=\Psi_j^0[\mathbf{r}-\mathbf{r}_j(t)]+\omega_j(r)e^{-i\beta_j\rho_j^0},$$

where $\omega_j(r)$ is a small correction to the unperturbed vortex solution. At the first order, one has to solve a linear equation for the perturbation to the basic solution, with an inhomogeneous term coming both from the external speed of species 1 and from the interaction

$$\mathcal{L}_1\cdot\omega_1=i\left(\mathbf{v}_1-\frac{d\mathbf{r}_1}{dt}\right)\cdot\,\boldsymbol{\nabla}\,\psi_1^0-g|\psi_2^0|^2\psi_1^0,$$

$$\mathcal{L}_2\cdot\omega_2=-i\frac{d\mathbf{r}_2}{dt}\cdot\,\boldsymbol{\nabla}\,\psi_2^0-g|\psi_1^0|^2\psi_2^0, \tag{6}$$

where $\mathcal{L}_j$ is the linearized operator of Eq. (4) around the vortex solution $\psi_j^0[\mathbf{r}-\mathbf{r}_j(t)]$ for $g=0$, and

$$\mathcal{L}_j=-\frac{\alpha_j}{2}\nabla^2+\beta_j[2|\psi_j^0|^2-\rho_j^0+(\psi_j^0)^2\hat{T}],$$

where $\hat{T}$ is the complex-conjugation operator. The kernel of these operators is nontrivial, since it contains elements coming from the symmetries of the problem. In particular, $\nabla\Psi_j^0$ belongs to ker($\mathcal{L}_j$), because the ground solution is invariant under the translation of the position of each vortex. The linear equations (6) have, in general, no solution precisely because the homogeneous piece has a nontrivial kernel. Although the operators $\mathcal{L}_j$ are not self-adjoint, the solvability condition in the expansion can be found [14]: the scalar product of both equations (6) are taken with each component of $\nabla \psi_j^0$, respectively, and are added with their complex conjugate. We then formally retrieve the adjoint operators and obtain the equation

$$\langle \mathcal{L}_1 \cdot \nabla \psi_1^0 | \omega_1 \rangle + \langle \omega_1 | \mathcal{L}_1 \cdot \nabla \psi_1^0 \rangle = \left\langle \nabla \psi_1^{0*} \middle| i\left(\frac{d\mathbf{r}_1}{dt} - \mathbf{v}\right) \cdot \nabla \psi_1^{0*} \right\rangle - \left[\left\langle \nabla \psi_1^0 \middle| i\left(\frac{d\mathbf{r}_1}{dt} - \mathbf{v}\right) \cdot \nabla \psi_1^0 \right\rangle + g\frac{\partial}{\partial_1} \|\psi_1^0 \psi_2^0\|^2 \right], \tag{7a}$$

$$\langle \mathcal{L}_2 \cdot \nabla \psi_2^0 | \omega_2 \rangle + \langle \omega_2 | \mathcal{L}_2 \cdot \nabla \psi_2^0 \rangle = -\left\langle \nabla \psi_2^0 \middle| i\frac{d\mathbf{r}_2}{dt} \nabla \psi_1^0 \right\rangle + \left\langle \nabla \psi_1^{0*} \middle| i\frac{d\mathbf{r}_2}{dt} \nabla \psi_2^{0*} \right\rangle - g\frac{\partial}{\partial_2} \|\psi_1^0 \psi_2^0\|^2, \tag{7b}$$

where $\|a\|^2 = \langle a | a \rangle$, and using the usual scalar product $\langle \cdot | \cdot \rangle$

$$\langle a | b \rangle = \int \mathbf{dr} a^*(\mathbf{r}) b(\mathbf{r}) = \int \mathbf{dr} \hat{T}[a(\mathbf{r})] b(\mathbf{r}).$$

The expansion can be done by imposing the orthogonality of the inhomogeneous term with the kernel of the adjoint operator of the homogeneous equation. This yields eventually a pair of coupled equations for the time derivative of the two vortex positions. They read

$$\rho_1^0 \epsilon_1 \frac{d\mathbf{r}_1(t)}{dt} = \rho_1^0 \epsilon_1 \mathbf{v}_1 + \frac{g}{2\pi} \mathbf{e}_z \times \frac{dV(r_{12})}{d\mathbf{r_{12}}} \tag{8}$$

and

$$\rho_2^0 \epsilon_2 \frac{d\mathbf{r}_2(t)}{dt} = -\frac{g}{2\pi} \mathbf{e}_z \times \frac{dV(r_{12})}{d\mathbf{r_{12}}}, \tag{9}$$

where $\mathbf{r_{12}} = \mathbf{r_2} - \mathbf{r_1}$, $\times$corresponds to the vector product, $\mathbf{e_z}$ is the unit vector perpendicular to the 2D plane and $g \cdot V(|\mathbf{r_{12}}|)$ is the potential energy of the interaction between the two vortices, depending on the norm of $\mathbf{r_{12}}$ only,

$$V(|\mathbf{r_{12}}|) = \int d\mathbf{r} [|\psi_2^0(\mathbf{r} - \mathbf{r_{12}}, t)|^2 - \rho_2^0] \cdot [|\psi_1^0(\mathbf{r}, t)|^2 - \rho_1^0].$$

Using the vortex profile (5), the equation of motion simplifies into the set of equations

$$\frac{d\mathbf{r_{12}}}{dt} = -\mathbf{v_1} - \frac{g\epsilon_1\epsilon_2}{2\pi}(\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0)\mathbf{e_z} \times \frac{dU(r_{12})}{d\mathbf{r_{12}}},$$

$$\rho_1^0 \epsilon_1 \frac{d\mathbf{r_1}}{dt} + \rho_2^0 \epsilon_2 \frac{d\mathbf{r_2}}{dt} = \rho_1^0 \epsilon_1 \mathbf{v_1}, \tag{10}$$

the first one for the relative motion between the vortices, and the second one giving the momentum conservation. Moreover, the rescaled potential U is defined through the function f only

$$U(|\mathbf{r_{12}}|) = \int d\mathbf{r} \left[f^2\left(\sqrt{\frac{\beta_2 \rho_2^0}{\alpha_2}} |\mathbf{r} - \mathbf{r_{12}}| \right) - 1 \right] \cdot \left[f^2\left(\sqrt{\frac{\beta_1 \rho_1^0}{\alpha_1}} r \right) - 1 \right].$$

The equations of motion keep the Hamiltonian structure of the coupled G-P equations

$$\rho_j^0 \epsilon_j \frac{d\mathbf{r_j}(t)}{dt} = -\mathbf{e_z} \times \frac{\delta \mathcal{H}}{\delta \mathbf{r_j}}, \tag{11}$$

with $\mathcal{H} = \rho_1^0 \epsilon_1 \mathbf{e_z} \cdot (\mathbf{v_1} \times \mathbf{r_1}) + (g\rho_1^0 \rho_2^0 / 2\pi) U(r_{12})$.

However, wave radiations coming from any accelerated motion of the vortices have to be added to the dynamics. To account for these dissipative effects (for the vortex motion only; the full set of equations are still Hamiltonian), one needs to estimate the radiative terms coming from nonuniform vortex motions. Such complicated calculations have been done for a pair of corotating vortices, and they show that the dynamics slowly deviate from the Hamiltonian dynamics, with the decreasing value of the energy [11,14]. The stability of the composite vortex at zero velocity (stability for negative g only) relies on this argument. Moreover, this effect disappears for the nonradiating equilibrium states moving at constant speed. They can thus be determined by the Hamiltonian dynamics, their stability being determined using arguments on the radiative losses. The trajectories of the Hamiltonian system are as follows:

$$\frac{g\epsilon_1^0 \epsilon_2}{2\pi} U(r_{12}) - \frac{v_1}{\epsilon_1 \rho_1^0 + \epsilon_2 \rho_2^0} y_{12} = K, \tag{12}$$

where K is a motion constant deduced from the Hamiltonian dynamics (11).

V. VORTEX DYNAMICS

An amazing consequence of the equations of motion (8) and (9) is that there is a possible equilibrium solution at constant speed of the two vortices, such that the force of interaction is balanced by a kind of Kutta-Joukovsky force on each vortex. When this is possible, the joint velocity of motion is

$$\frac{\epsilon_1 \rho_1^0}{\epsilon_1 \rho_1^0 + \epsilon_2 \rho_2^0} \mathbf{v_1}, \tag{13}$$

a simple looking result. However, this joint drift of the two vortices cannot happen if the flow speed is too large. From Eqs. (8) and (9), we obtain the following relation between $\mathbf{v_1}$, $\mathbf{r_{12}}$ and the potential if the equilibrium solution exists:

$$\mathbf{v_1} = -\frac{g\epsilon_1\epsilon_2}{2\pi}(\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0) U'(r_{12}) \mathbf{e_z} \times \frac{\mathbf{r_{12}}}{r_{12}}. \tag{14}$$

From this equation, we deduce first that the separation vector $\mathbf{r_{12}}$ is orthogonal to the imposed velocity $\mathbf{v_1}$ and that such a solution can only be found for low-enough velocity.

Indeed, the interaction potential has a monotonic behavior with a zero derivative both at zero distance and at infinity, as shown on Fig. 2, so that the derivative has a maximum between. It determines in particular the maximum value v_m of the drift velocity for which a vortex couple can travel at the same speed (13)

$$v_m = \frac{|g(\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0)|}{2\pi} \max|U'|.$$

For a velocity smaller than this maximum, two solutions exist. The two solutions collapse for the maximum velocity, and no more equilibrium positions exist above that. Figure 3 shows the Hamiltonian structure of the dynamics for different cases $v_1 < v_m$, $v_1 = v_m$, and $v_1 > v_m$. The stability of the equilibrium solutions depends then also on the sign of g. For $g > 0$, every solution is linearly unstable, and no steady flow can be obtained. For $g < 0$, we have a classical saddle-node bifurcation structure: at low speed when we have two equilibrium positions, one of the solutions is linearly stable (the one of smaller r_{12}) and the other is unstable. For $v_1 = 0$, the trajectories are circular, and depending on the sign of g, the vortex dynamics correspond to a slow converging (diverging) spiraling motion toward (away from) the origin.

Moreover, to be able to reach effectively the equilibrium solution (14), where the two vortices move together at the same speed, we need to determine whether the trajectory (12) initiated by the initial condition $r_{12} = 0$ encloses the equilibrium solution instead of approaching infinity. This is the case only if the velocity v_1 is in fact below the critical velocity $v_c < v_m$, which is determined by the value of r_c such that

$$U(r_c) - U(0) = U'(r_c)r_c,$$

which gives, following the calculation of $U(r)$ shown in Fig. 2,

$$v_c = \frac{|g(\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0)U'(r_c)|}{2\pi}.$$

Note this surprising effect: if $\epsilon_2 = -\epsilon_1$ and $\rho_2^0 > \rho_1^0$, then the constant speed of the two paired vortices is in the opposite direction of the imposed flow velocity. Such counterflow vortex dynamics have been observed in the numerical analysis's as well. Moreover, in the case of $\epsilon_2 = -\epsilon_1$ and $\rho_2^0 = \rho_1^0$, the only equilibrium solution is for $\mathbf{v_1} = 0$.

VI. CONCLUSIONS

The experimental consequences are simple to state in atomic vapors, where one can manipulate the condensate by optical methods in particular. It would be interesting also to see such behavior in superfluid mixtures of ^{4}He and ^{3}He, if the two kinds of atoms can move independently. [17] Another last remark concerns how the two rotating species condensate. With a single species, the equilibrium state is a triangular lattice of like-sign vortices with a mesh size of order $(\hbar/m\Omega)^{1/2}$, where Ω is the angular frequency. Therefore, if the two coupled condensates are subject to the same angular speed Ω, each bosonic species should, in the absence of coupling $g = 0$, exhibit a lattice of vortices of mesh $(\hbar/m_j\Omega)^{1/2}$. A nonzero coupling will most probably induce deformations of the two lattices. Even with the weak coupling limit, no formal theory seems available for this kind of situation. During the submission process for this paper, a paper has been published showing experimental evidence of this strong interplay between vortex lattices [15]. The mixture is obtained by coherently transferring a fraction of the condensate into a different atomic state, and the interaction between the two condensates is repulsive. Another instance where coupled vortices would be present is in nonlinear optical fields in the classical approximation [16]. There, the role of time in the G-P equation is represented by the direction parallel to the beam propagation, and the Laplacian accounts for the 2D variation perpendicular to it. The equations of propagation of

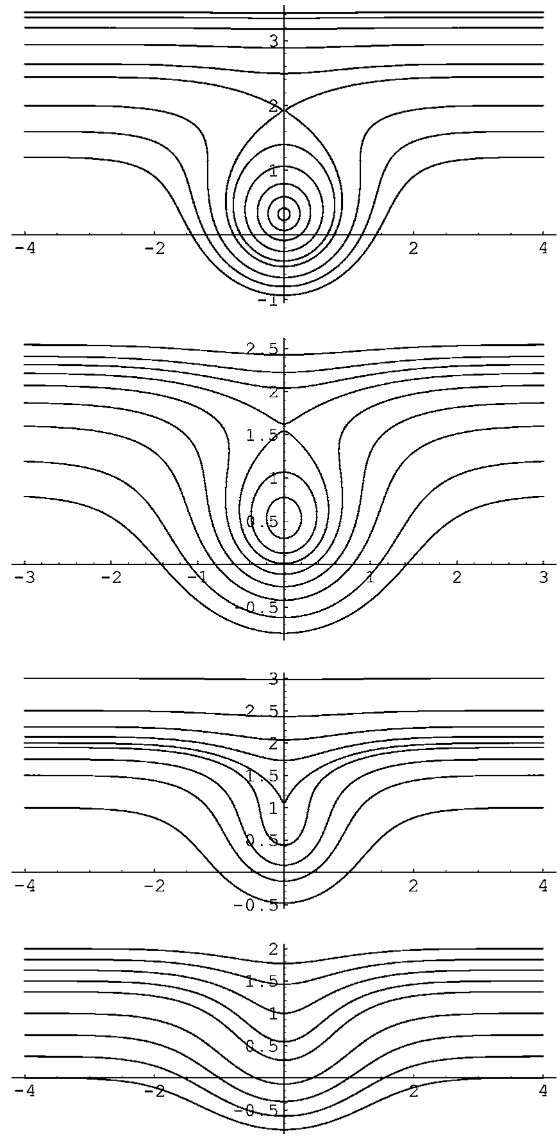

FIG. 3. Trajectories of the relative vortex motion for different velocities (a) $v_1 < v_c < v_m$, (b) $v_1 = v_c < v_m$, (c) $v_1 = v_m$, and (d) $v_1 > v_m$, neglecting the radiative effects. These trajectories correspond to the constant value of the Hamiltonian.

two parallel light beams of different frequency in the same nonlinear material are very similar to the coupled G-P equations. Therefore, we expect in this case the occurence of phenomena roughly similar to those described here.

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