Casimir forces between defects in one-dimensional quantum liquids

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We discuss the effective interactions between two localized perturbations in one-dimensional quantum liquids. For noninteracting fermions, the interactions exhibit Friedel oscillations, giving rise to a Ruderman-Kittel-Kasuya-Yosida-type interaction familiar from impurity spins in metals. In the interacting case, at low energies, a Luttinger-liquid description applies. In the case of repulsive fermions, the Friedel oscillations of the interacting system are replaced, at long distances, by a universal Casimir-type interaction which depends only on the sound velocity and decays inversely with the separation. The Casimir-type interaction between localized perturbations embedded in a fermionic environment gives rise to a long-range coupling between quantum dots in ultracold Fermi gases, opening an alternative to couple qubits with neutral atoms. We also briefly discuss the case of bosonic quantum liquids in which the interaction between weak impurities turns out to be short ranged, decaying exponentially on the scale of the healing length.

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I. INTRODUCTION

Interactions between localized defects which are mediated by the continuum they are embedded in play an important role in many areas of physics. Typical examples are the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction between spins in a Fermi liquid or the interaction between vortices in superfluids. In the present work, we discuss interactions between impurities in one-dimensional (1D) quantum liquids. This study is motivated by the recent realization of strongly interacting "atomic quantum wires" with ultracold gases of both bosonic [1–3] and fermionic atoms [4] and the proposal [5] that single atoms in optical traps which are embedded in a superfluid reservoir allow one to realize an atomic analog of a quantum dot with a tunable coupling to the environment. Such quantum dots may be used to store qubits, which, under certain conditions, can be completely decoupled from their environment. Arrays of these dots thus appear as ideal candidates for quantum-information processing. It is therefore of considerable interest to study the induced interactions of such dots, mediated by the environment they are embedded in. Similar questions arise also for quantum dots in solid-state realizations—e.g., in carbon nanotubes [6]—where the interaction is mediated by electrons in the intervening wire.

Quite generally, for both bosons and fermions, the low-energy properties of a gapless 1D quantum liquid are described by the so-called Luttinger-liquid (LL) phenomenology [7,8]: the effective theory is a hydrodynamic energy functional characterized by the velocity of sound u and the Luttinger interaction parameter K. In particular, for fermions, $K=1$ corresponds to the noninteracting case, while $K<1$ for repulsion. For repulsive bosons, in turn, one has $K>1$, with $K\to\infty$ in the limit of weak interactions, where a Gross-Pitaevskii or Bogoliubov approximation applies. As shown by Kane and Fisher [9] (see also [10] for a recent discussion), the interaction of a single impurity with a LL depends crucially on the value of K: for $K>1$, the impurity is irrelevant for the low-energy properties. A 1D Bose liquid is therefore effectively superfluid, although there is no true condensate [9,11]. For $K<1$ the impurity changes the ground state of the liquid in a nonperturbative way, effectively cutting it into two disconnected parts. In this case, we will see that the induced interaction between two impurities is essentially a Casimir-like effect. Indeed, at low energies, two impurities at distance r define a box with reflecting boundary conditions for the phonon modes of the quantum liquid, which leads to an attractive Casimir interaction energy proportional to u/r with u the sound velocity. The case $K=1$ is marginal and corresponds to a noninteracting Fermi gas in 1D or, equivalently, a system of hard-core bosons: the Tonks-Girardeau gas [2,3,12]. In the following we will study the interactions mediated by the 1D quantum liquid between two impurities for the various cases, including fermions with spin. We focus our analysis on the case of *static* impurities, while the situation of a slow time dependence, relevant for atomic quantum dots, where the interactions depend on the internal states is only discussed qualitatively at the end of the paper.

II. ONE-DIMENSIONAL FERMIONIC LIQUID

A. Noninteracting fermions

Before considering the generic situation of impurities embedded in a sea of interacting particles, we first address the marginal case $K=1$ of noninteracting fermions. For simplicity we start from a gas of N noninteracting spinless fermions in the presence of two localized impurities separated by a distance r. Considering cold gases in atomic quantum wires, the solution of this problem is not just an academic exercise. Indeed, since fermions in a single hyperfine state have no s-wave interactions due to the Pauli principle, they realize an ideal Fermi gas at sufficiently low temperatures. We assume

that the particles are contained in a periodic box of length L with average density $\rho_0 \equiv N/L$. The (grand canonical) partition function of the liquid at a given temperature T may be expressed in terms of a functional integral over the Grassman fields $(\bar{\psi}, \psi)$ representing the fermions:

$$Z = \int \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp(-S_{\mathrm{FL}} - S_i). \tag{1}$$

The corresponding action of an ideal gas is

$$S_{\mathrm{FL}} = \int_0^\beta \mathrm{d}x\mathrm{d}\tau \left[\bar{\psi}\partial_\tau \psi - \left(\frac{1}{2m} \nabla \bar{\psi} \nabla \psi - \mu \bar{\psi}\psi \right) \right], \tag{2}$$

where τ is the imaginary time running from 0 to $1/T = \beta$ and μ is the chemical potential (we use units such that $\hbar = k_B = 1$). The fields are antiperiodic in imaginary time. For short-range interactions, appropriate for cold atoms, the interaction of the impurities with the liquid can be described by an additional contribution

$$S_i = \int_0^\beta \mathrm{d}\tau \sum_{\alpha=1,2} g_\alpha \bar{\psi}(x_\alpha)\psi(x_\alpha), \tag{3}$$

proportional to the local density at the impurity positions x_α. Here, the index $\alpha = 1,2$ labels the impurities, while the coupling constants g_α describe the strength of collisions between the atoms in the liquid and impurities. The expression for S_i is based on assuming an effective pseudopotential for the interaction between the impurity and quantum liquid. More precisely, the interaction should be replaced by a spatial integral of the detailed impurity potential with the microscopic density operator of the liquid. In the present section dealing with noninteracting fermions, there is no need of a high-energy cutoff, as one can directly work with the well-behaved microscopic theory. However, in order to discuss the low-energy behavior and to make contact with the following sections dealing with interacting fermions using the Luttinger-liquid phenomenology, we introduce a high-energy cutoff ω_c. Its value can be estimated as $\omega_c \sim \mathrm{Min}\{u/l_0, \mu\}$, where u is the characteristic velocity of excitations ($u = v_F$ in an ideal Fermi gas), l_0 is the impurity size, and $\mu = p_F^2/2m$ is the chemical potential (at zero temperature) with $p_F = mv_F \equiv \pi\rho_0$, the Fermi momentum. As will become clear from our results below, the coupling constants g_α are then—up to a factor v_F—identical with the dimensionless backscattering amplitudes $f_{1,2}(\pi)$ for fermions at the impurities. Microscopically, they are thus determined by a solution of the single-particle scattering problem off a single impurity. In practice, an appreciable value of the backscattering amplitude requires the impurity size to be smaller or of the order of the interparticle spacing, since otherwise the Fourier component of the potential at $2p_F$ is close to zero, and hence the dimensionless coupling constants $\gamma_\alpha \equiv g_\alpha/v_F$ vanish. Therefore, in the following, we will take $\omega_c \sim \mu$.

The Grassman fields $(\bar{\psi}, \psi)$ are free everywhere apart from the points $x = x_{1,2}$ and hence can be easily integrated out by the following standard trick: first we formally introduce four δ functions into the integrand:

$$Z = \int \mathrm{D}\bar{\psi}\mathrm{D}\psi \prod_{\alpha=1,2} \mathrm{D}\bar{\eta}_\alpha \mathrm{D}\eta_\alpha \delta[\psi(x_\alpha,\tau) - \eta_\alpha(\tau)] \times \delta[\bar{\psi}(x_\alpha,\tau) - \bar{\eta}_\alpha(\tau)] e^{-S_{\mathrm{FL}} - S_i}, \tag{4}$$

where $(\bar{\eta}_\alpha, \eta_\alpha)$ are the new Grassman variables describing the fermions at the location of the individual impurities. Then we introduce a set of auxiliary fields $(\bar{\kappa}_\alpha, \kappa_\alpha)$ using the identity $\delta(f) \sim \int \mathrm{D}\kappa \exp(i \int \kappa f \mathrm{d}\tau)$ to raise the δ functions into the action. Finally, we integrate out the fermionic fields $(\bar{\psi}, \psi)$, which appear only quadratically, by Fourier transformation:

$$Z = Z_{\mathrm{FL}}^0 \int \prod_{\alpha=1,2} \mathrm{D}\bar{\eta}_\alpha \mathrm{D}\eta_\alpha \mathrm{D}\bar{\kappa}_\alpha \mathrm{D}\kappa_\alpha e^{-S' - S_i(\bar{\eta},\eta)}, \tag{5}$$

where

$$S_i(\bar{\eta}, \eta) = \int_0^\beta \mathrm{d}\tau \sum_{\alpha=1,2} g_\alpha \bar{\eta}_\alpha \eta_\alpha \tag{6}$$

and

$$S' = \frac{L}{\beta} \sum_n \int \frac{\mathrm{d}p}{2\pi} \frac{\sum_{\alpha,\beta} \bar{\kappa}_\alpha \kappa_\beta e^{ip(x_\alpha - x_\beta)}}{-i\omega_n + \xi_p} - i \sum_{\alpha,n} (\kappa_\alpha \eta_\alpha + \bar{\kappa}_\alpha \bar{\eta}_\alpha), \tag{7}$$

where $\xi_p = p^2/2m - \mu$ and the summation is over the fermionic Matsubara frequencies ω_n. The trivial prefactor Z_{FL}^0 arises from integration over the fermionic fields in the absence of impurities, giving the grand partition function of the homogeneous liquid. The fields $(\bar{\kappa}_\alpha, \kappa_\alpha)$ depend only on imaginary time τ, or frequency ω_n in the Fourier representation, and thus the integral over p can be easily calculated. Since for a sufficiently large separation $|x_1 - x_2| = r \gg p_F^{-1}$ the interaction energy is small, the relevant frequencies are small compared to the Fermi energy $\omega_c \sim \mu \sim v_F p_F$. An expansion to leading order in $\omega_n \ll \mu$ then gives

$$S' = -\frac{iL}{\beta v_F} \sum_n \sum_{\alpha,\alpha'} s_n \bar{\kappa}_\alpha \kappa_{\alpha'} e^{ip_F|x_\alpha - x_{\alpha'}| s_n} \times e^{-|\omega_n|(1-\delta_{\alpha,\alpha'})/\omega_r} - i \sum_{\alpha,n} (\kappa_\alpha \eta_\alpha + \bar{\kappa}_\alpha \bar{\eta}_\alpha), \tag{8}$$

where $\omega_r \equiv u/r \ll \mu$ and $s_n \equiv \mathrm{sgn}(\omega_n)$. The characteristic frequency ω_r will play an important role in our subsequent discussions. Physically it represents the inverse flight time for a characteristic excitation in the liquid between the locations of the two impurities, which naturally obeys the inequality $\omega_r \ll \omega_c$ provided the impurities are much farther apart than the average distance between two fermions in the liquid. It is also the quantization energy between the two impurities. In order to obtain the impurity interaction directly from the partition function, we integrate out the auxiliary fields $(\bar{\kappa}_\alpha, \kappa_\alpha)$. This results in an action

$$S'(\bar{\eta},\eta)=\frac{\beta v_F}{iL}\sum_n(\bar{\eta}_1,\bar{\eta}_2)\begin{pmatrix} f_n & -f_n e_n \\ -f_n e_n & f_n\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}, \quad (9)$$

which only depends on the four time-dependent Grassman fields $(\bar{\eta}_\alpha,\eta_\alpha)$ which describe the Fermi field at the impurity positions. The coefficients f_n and e_n are defined by

$$f_n\equiv\frac{s_n}{1-e_n^2},\quad e_n\equiv e^{is_n p_F r-|\omega_n|/\omega_r}. \quad (10)$$

Including the contribution (3) due to the interaction between the impurities and liquid, the complete expression for the statistical sum, Eq. (1), can now be written as

$$Z=Z_{\mathrm{FL}}^0 Z_\kappa\int\prod_{\alpha=1,2}\mathrm{D}\bar{\eta}_\alpha\mathrm{D}\eta_\alpha e^{-S'-S_i}, \quad (11)$$

where Z_κ comes from the integration of the auxiliary fields:

$$Z_\kappa=\prod_n\left(\frac{iL}{\beta v_F}\right)^2(1-e_n^2). \quad (12)$$

The total effective action $S'+S_i$ is quadratic in the Grassman fields $(\bar{\eta}_\alpha,\eta_\alpha)$, and thus the integration can be done exactly to yield $Z=Z_{\mathrm{FL}}^0 Z_\kappa Z_\eta$, with

$$Z_\eta=\prod_n\left(\frac{\beta v_{\mathrm{F}}}{iL}\right)^2[(f_n+i\gamma_1)(f_n+i\gamma_2)-(f_n e_n)^2]. \quad (13)$$

Here, the $\gamma_\alpha=g_\alpha/v_F$ are the dimensionless backscattering amplitudes, characterizing the interaction of the impurities and liquid. We can now obtain the free energy of the 1D Fermi gas in the presence of the impurities from $F=-\ln Z/\beta$ as follows:

$$F=F^0-\frac{1}{\beta}\sum_n\ln\{(1-e_n^2)\times[(f_n+i\gamma_1)(f_n+i\gamma_2)-(f_n e_n)^2]\}, \quad (14)$$

where $F^0=-\ln Z_{\mathrm{FL}}^0/\beta$ is the free energy of the undisturbed, homogeneous liquid. Expression (14) is ill defined as it stands, since it contains both the energy of zero-point fluctuations in the gas and the formally divergent self-energies of the separate impurities. The relevant interaction energy associated with a change of the separation of the two impurities is given by

$$\begin{aligned}V_{12}&\equiv F(\gamma_\alpha,r)-F(0,r)-[F(\gamma_\alpha,\infty)-F(0,\infty)]\\&=F(\gamma_\alpha,r)-F(\gamma_\alpha,\infty).\end{aligned} \quad (15)$$

The renormalization thus requires subtracting first the free energy of the liquid without the impurities ($\gamma_\alpha=0$, vacuum energy) and then the free energy of the system when the impurities are very far apart ($r\to\infty$, self-energy of the impurities). While both the vacuum energy and the individual self-energies are infinite in the absence of a cutoff, the renormalized interaction (15) is finite and independent of the cutoff (see also the discussion below in Sec. III A).

At low temperature $T\ll\omega_r$ we can switch from summation to integration according to $\mathrm{d}\omega=2\pi T\mathrm{d}n$, so that the effective interaction energy between the impurities can be expressed as

$$V_{12}=-\int_0^\infty\frac{\mathrm{d}\omega}{\pi}\ln\left|1+\frac{\gamma_1\gamma_2 e^{-2\omega/\omega_r+2ip_F r}}{1+i(\gamma_1+\gamma_2)-\gamma_1\gamma_2}\right|. \quad (16)$$

The integral can be performed analytically to yield our final result for the impurity interaction at $T=0$:

$$V_{12}=\frac{v_F}{2\pi r}\,\mathrm{Re}\,\mathrm{Li}_2\left(-\frac{\gamma_1\gamma_2 e^{2ip_F r}}{1+i(\gamma_1+\gamma_2)-\gamma_1\gamma_2}\right), \quad (17)$$

where Li_2 is the dilogarithmic function [13] and Re is the real part. Obviously the interaction quite generally falls off very slowly like $1/r$ with an amplitude, which is a strictly periodic function. Its period $\pi/p_F=\rho_0^{-1}$ is equal to the average interparticle distance. This is a characteristic property of degenerate fermions, essentially reflecting the well-known Friedel oscillations of the density (see below). Trivially, the interaction vanishes if one of the scattering amplitudes $\gamma_{1,2}$ is zero.

A simple expression for the renormalized interaction energy V_{12} is obtained in two limiting cases. First, if the interaction of the impurities with the liquid is weak, $\gamma_\alpha\ll 1$, we can expand the dilogarithm in Eq. (17) to obtain

$$V_{12}=-\gamma_1\gamma_2\frac{v_F}{2\pi r}\cos(2p_F r). \quad (18)$$

In the limit of strong impurities, $\gamma_\alpha\gg 1$, we find in turn the result

$$V_{12}=\frac{v_F}{2\pi r}\,\mathrm{Re}\,\mathrm{Li}_2(e^{i2p_F r}), \quad (19)$$

which is completely independent of the scattering amplitudes. In this case, the interaction energy V_{12} can be represented as $V_{12}=(v_{\mathrm{F}}/2\pi r)f(2p_{\mathrm{F}}r)$, where $f(x)\equiv\mathrm{Re}\,\mathrm{Li}_2(e^{ix})$ is a periodic function bounded as follows $f_{\min}\leqslant f\leqslant f_{\max}$ where

$$f_{\max,\min}=\mathrm{Li}_2(\pm 1)=\frac{\pi^2}{6},-\frac{\pi^2}{12}. \quad (20)$$

A simple way of understanding the slow $1/r$ decay and the oscillations with period π/p_F may be obtained in the weak-scattering limit, Eq. (18). Indeed, the density perturbation created by a single impurity of strength γ_1 at position x_1 is asymptotically given by

$$\rho_1(x)\approx\rho_0-\frac{\gamma_1}{2}\frac{\cos(2p_F|x-x_1|)}{2\pi|x-x_1|}. \quad (21)$$

This expression for the Friedel oscillations in a spinless 1D noninteracting Fermi gas is valid in the limit where $\gamma_1\ll 1$ and $|x-x_1|\gg\rho_0^{-1}$ [14]. Since the impurities couple to the local density, the interaction energy (excluding self-energies) of the system of two weak impurities is simply obtained from $U_{12}(\gamma_\alpha,r)=g_2\rho_1(x_2)+g_1\rho_2(x_1)$ where $\gamma_\alpha\ll 1$. When renormalized $V_{12}=U_{12}(\gamma_\alpha,r)-U_{12}(\gamma_\alpha,\infty)$, this interaction energy coincides with Eq. (18). Alternatively, the result may be de-

rived by using the random-phase approximation (RPA) as shown in Appendix A.

The analysis in this section is readily generalized to the case of a Fermi gas with spin. In fact for nonmagnetic impurities, such as considered in the current article, the two spin modes are decoupled and therefore the energy due to the presence of the two impurities is simply multiplied by a factor of 2.

It should be emphasized that the calculation above can be immediately extended to the case of noninteracting fermions in two or three dimensions $d=2,3$, giving rise to an interaction energy for weak coupling of the form

$$V_{12} \sim f_1(\pi) f_2(\pi) \frac{p_F v_F}{(p_F r)^d} \cos(2p_F r), \tag{22}$$

where $f_\alpha(\pi)$ are the dimensionless backscattering amplitudes of the impurities. This result follows most simply by considering the density fluctuations $\delta\rho_1(\vec{x})$ induced by a single impurity at position $\vec{x}_1$. As discussed, e.g., in Ref. [15] they exhibit Friedel oscillations proportional to the dimensionless backscattering amplitude $f_1(\pi)$ at the Fermi energy. The resulting interaction energy is then simply given by $V_{12} \propto f_2(\pi)\delta\rho_1(\vec{x}_2)$. In fact, this is a special case of a general result [16] that the asymptotic interaction between two impurities is determined by the product of their backscattering amplitudes. In the presence of short-range repulsive interactions between the fermions, we expect that the result (22) remains qualitatively correct in the case of two and three dimensions. This is based on the existence of a Fermi-liquid description in $d=2,3$, which guarantees that the low-energy properties are qualitatively unchanged from those of a Fermi gas. For example, assuming that the static density response function at $2p_F$ is given by the particle-hole bubble [17], the renormalization factor $Z<1$ in the single-particle Green function will give rise to a Fermi-liquid correction factor Z^2 in V_{12}. This argument, however, neglects possible vertex corrections in the density response which may lead to an enhancement rather than a suppression of the amplitude of the Friedel oscillations. In fact this effectively happens in the one-dimensional case, where the vanishing Z factor gives rise to Friedel oscillations, which decay more slowly than in the noninteracting case (see below). While we are not aware of a quantitative calculation of the $2p_F$-density response in Fermi liquids, it is very likely that they will give rise only to finite, multiplicative corrections to Eq. (22). As we will see below, however, the situation in one dimension is quite different from that in $d=2,3$ in the sense that even qualitatively the asymptotic form of the interaction is *not* given by the Friedel oscillation picture, even for very weak impurities.

Finally, we mention a recent work dealing with neutron matter. Bulgac *et al.* [18] consider a neutron star crust, which is modeled as a degenerate noninteracting neutron gas (i.e., an ideal 3D Fermi gas) containing various kinds of defects or inhomogeneities (such as nuclei or bubbles) immersed in it. These authors compute the interaction energy between two defects resulting from the quantum fluctuations of the Fermi sea of neutrons. They obtain expressions similar to Eq. (22), which can be interpreted as RKKY-like interactions between defects. In addition, they discuss the influence of the shape of the defects and consider situations with more than two defects.

B. Spinless Fermi Luttinger liquid

Realistic Fermi systems consist of interacting particles. In three and also in two dimensions, it is possible to describe even strong interactions by Landau's Fermi-liquid theory. As is well known, however, this concept fails in one dimension. Here we consider fermions with repulsive short-range interaction. At low energy such a system exhibits a gapless excitation spectrum with a linear dispersion, characteristic of the universality class of Luttinger liquids [7,8,19].

For simplicity, we start by considering spinless fermions, for which the low-energy description is given by the following hydrodynamic action:

$$S_{\mathrm{LL}} = \frac{1}{2\pi K} \int \mathrm{d}x \mathrm{d}\tau \left[u(\partial_x \theta)^2 + \frac{1}{u}(\partial_\tau \theta)^2 \right]. \tag{23}$$

Here u is the sound velocity and K the Luttinger parameter. In a translationally invariant system, they obey the relation $uK=v_F$ [8], with $v_F=p_F/m=\pi\rho_0/m$ the Fermi velocity of the associated noninteracting spinless Fermi gas. We consider repulsive interactions for which $K<1$. Since the Luttinger-liquid description only applies at low energies, the fields have to be cut off at energy $\omega_c \sim \mu$, where μ is the chemical potential. The associated cutoff length $a \equiv u/\omega_c$ is of order $1/\rho_0$. Of course, for a quantitative calculation of the scale at which the low-energy description applies, a microscopic model is needed, which allows one to determine nonuniversal properties. For single impurities in Luttinger liquids this problem has only recently been discussed; see [20]. Since we are concerned with the interaction at distances much longer that the average interparticle separation, only the low-energy properties are relevant, which are well described by the hydrodynamic action (23). The corresponding field θ is related to the density of the liquid by

$$\rho(x) \approx \left(\rho_0 + \frac{\partial_x \theta}{\pi} \right) [1 + 2\cos(2\theta + 2p_F x)], \tag{24}$$

where ρ_0 is the equilibrium density and only the first harmonics are taken into account [8,9].

The interaction between the impurities and Luttinger liquid is taken to be of the form

$$S_i = \int_0^\beta \mathrm{d}\tau \sum_{\alpha=1,2} \tilde{g}_\alpha \rho(x_\alpha), \tag{25}$$

i.e., a coupling to the local density with phenomenological scattering amplitudes $\tilde{g}_\alpha$. Inserting the expansion (24) into this interaction, gives rise to four different terms. The first term is just the constant Hartree self-energy of the impurities, which, of course, does not contribute to the renormalized interaction energy V_{12}. In addition, there are terms containing $\partial_x\theta$ due to forward scattering. They describe quantum corrections to the self-energies but again are irrelevant for the interaction V_{12} between two widely separated impurities. The dominant term for this interaction is the contribution propor-

tional to $\cos(2\theta+2p_F x)$, which is due to backward scattering. In addition there are higher-order terms like $\partial_x\theta\cos(2\theta+2p_F x)$; however, these are less relevant in the renormalization group (RG) sense [21]. Taking only the most relevant part of the interaction, the coupling between the impurities and LL leads to the following nonlinear contribution to the action:

$$S_i[\Theta] \approx \int_0^\beta d\tau \sum_{\alpha=1,2} 2\tilde{g}_\alpha \rho_0 \cos[2\sqrt{\pi K}\Theta(x_\alpha) + 2p_F x_\alpha], \tag{26}$$

where we introduced the renormalized field Θ by $\theta = \Theta\sqrt{\pi K}$. The complete statistical sum of the system can again be represented by a functional integral. To perform the integration over the field Θ we use the same approach as for the ideal Fermi gas: first we introduce the new variables $\Theta(x_1,\tau)=\Theta_1(\tau)$ and $\Theta(x_2,\tau)=\Theta_2(\tau)$ and then insert the two δ functions into the functional integral:

$$Z = \int D\Theta \prod_{\alpha=1,2} D\Theta_\alpha \delta[\Theta(x_\alpha) - \Theta_\alpha] e^{-S_{\mathrm{LL}}-S_i}. \tag{27}$$

We then transform the δ functions into the functional integrals over auxiliary fields and perform the Gaussian integration

$$Z = Z_{\mathrm{LL}}^0 Z_\kappa \int D\Theta_1 D\Theta_2 e^{-S_{\mathrm{eff}}-S_i[\Theta_\alpha]}, \tag{28}$$

where the effective action for the real fields $\Theta_{1,2}$ is

$$S_{\mathrm{eff}} = \sum_n (\Theta_{1,-n}, \Theta_{2,-n}) \begin{pmatrix} f_n & -f_n e_n \\ -f_n e_n & f_n \end{pmatrix} \begin{pmatrix} \Theta_{1,n} \\ \Theta_{2,n} \end{pmatrix}, \tag{29}$$

with the summation occurring over the bosonic Matsubara frequencies ω_n, where $f_n \equiv \beta|\omega_n|/(1-e_n^2)$ and $e_n \equiv e^{-|\omega_n|/\omega_r}$ with $\omega_r \equiv u/r$. The factor Z_{LL}^0 comes from the integration over $\Theta(x,\tau)$ and is independent of the impurities, describing the homogeneous Luttinger liquid. By contrast, the factor

$$Z_\kappa = \prod_n (1-e_n^2)^{-1/2}, \tag{30}$$

which comes from the integration over the auxiliary fields, describes the change in the phonon modes due to the constraint on the Fermi fields at the positions of the two impurities. Similar to the noninteracting situation, this contribution depends on the associated characteristic frequency ω_r and is crucial in obtaining a finite interaction energy V_{12} which is independent of the cutoff.

The remaining and now nontrivial functional integral over the time-dependent fields Θ_α is of the same form as the one which appears in the context of quantum Brownian motion in a periodic potential [22]. Indeed the effective action (29) basically describes two quantum particles subject to Ohmic dissipation of dimensionless strength $1/K$ which move in a periodic potential generated by the backscattering amplitude. As has been shown in [22] this problem leads to a localized ground state if $1/K>1$ with small fluctuations in the field Θ. For a quantum liquid with sufficiently strong interactions between the fermions $K \ll 1$ and strong impurities $\tilde{\gamma}_\alpha \equiv \tilde{g}_\alpha/v_F \gg 1$, the functional integral (28) over the time-dependent fields Θ_α can thus be calculated using the stationary-phase approximation (SPA): expanding the functions $\cos(2\sqrt{\pi K}\Theta_\alpha + 2p_F x_\alpha)$ from Eq. (26) to second order in the fields around one of its minimum, we approximate the interaction Lagrangian in the form

$$\tilde{S}_i = \int_0^\beta d\tau \sum_{\alpha=1,2} E_\alpha \Theta_\alpha^2, \tag{31}$$

where $E_\alpha = 4\pi K \tilde{g}_\alpha \rho_0$. Physically this means that the interaction between each of the impurities and the liquid is sufficiently strong to pin the local phase Θ near the value minimizing the potential energy. The quantities E_α play the role of effective frequencies for the evolution of the fields Θ_α. The approximation of the original Lagrangian (26) by the quadratic form (31) is equivalent to an adiabatic approximation which describes physical processes occurring slower than a time scale given by E_α^{-1}. As the typical frequency of interest is ω_r, the stationary-phase approximation is valid when $E_\alpha \gg \omega_r$. It is thus applicable in the case of strong impurities or, equivalently, long distances (see below) in a strongly repulsive liquid: $r\rho_0\tilde{\gamma}_\alpha \gg 1/K^2 \gg 1$.

Within the quadratic approximation (31), the full effective action in Eq. (28) is quadratic in Θ_α and hence can be evaluated exactly $Z = Z_{\mathrm{LL}}^0 Z_\kappa Z_\Theta$, where

$$Z_\Theta = \prod_n [(f_n+\beta E_1)(f_n+\beta E_2) - (f_n e_n)^2]^{-1/2}. \tag{32}$$

The associated free energy $F=-\ln Z/\beta$ is given by

$$F = F^0 + \frac{1}{2\beta}\sum_n \ln\{(1-e_n^2) \times [(f_n+\beta E_1)(f_n+\beta E_2) - (f_n e_n)^2]\}, \tag{33}$$

where $F^0 = -\ln Z_{\mathrm{LL}}^0/\beta$ again describes the undisturbed, homogeneous liquid. From the free energy we obtain the renormalized interaction energy between the two impurities V_{12} in precisely the same manner as in Eq. (15):

$$V_{12} = \frac{1}{\beta}\sum_{n>0} \ln\left(1 - \frac{E_1 E_2 e^{-2\omega_n/\omega_r}}{\omega_n^2 + \omega_n(E_1+E_2) + E_1 E_2}\right). \tag{34}$$

At sufficiently low temperatures $T \ll \omega_r$, the summation may be replaced by an integral over the real frequency ω:

$$V_{12} = \frac{1}{2\pi}\int_0^\infty d\omega \ln\left(1 - \frac{E_1 E_2 e^{-2\omega/\omega_r}}{\omega^2 + \omega(E_1+E_2) + E_1 E_2}\right). \tag{35}$$

In the limit of long distances (or strong impurities)—i.e., $E_\alpha \gg \omega_r$—the integral converges at $\omega \sim \omega_r$ and hence $\omega^2 \ll \omega E_\alpha \ll E_\alpha^2$. For sufficiently large separations, we thus obtain the simple universal interaction

$$V_{12} = \frac{u}{2\pi r}\int_0^\infty dy \ln(1-e^{-2y}) = -\frac{\pi}{24}\frac{u}{r}, \tag{36}$$

which decays inversely with distance. The short-distance regime, where $E_\alpha \ll \omega_r$, can, however, not be considered within

the stationary-phase approximation (31). Indeed, the latter is only justified if the characteristic energy scale $\omega \lesssim \omega_r$ of excitations involved in the interaction does not exceed the effective frequencies—i.e., if $E_\alpha \gg \omega_r$. As a result, for intermediate and short distances, the interaction energy V_{12} does not follow a simple $1/r$ behavior. In particular, it is impossible to describe the limit $r \to 0$ without properly including a cutoff or working in a microscopic model from the beginning. It is only within such a more complete calculation that the full energy $F(\tilde{\gamma}_\alpha, r) - F(0, r)$ of the two impurity problem approaches the self-energy of the doubled single-impurity case, as expected on physical grounds. For a single scalar field in 1D, as described by our hydrodynamic action (23), this calculation has recently been done by Jaffe [23].

The validity of the quadratic expansion (31) may be extended to the whole relevant range $K < 1$ of the Luttinger parameter with the help of the self-consistent harmonic approximation (SCHA) [22,24]. In the context of Friedel oscillations around a single impurity in a Luttinger liquid, this has been used by Egger and Grabert [25]. It is based on making a quadratic approximation (31) for the backscattering term, however with frequencies E_α which are determined from Feynman's variational principle. Using Eq. (31) as the trial action, one has to minimize the free energy

$$F_{\mathrm{var}} = -\frac{1}{\beta} \ln \tilde{Z} + \frac{1}{\beta} \langle S - \tilde{S} \rangle_{\tilde{S}}, \tag{37}$$

where $S = S_{\mathrm{eff}} + S_i$ and $\tilde{S} = S_{\mathrm{eff}} + \tilde{S}_i$. Taking E_α as variational parameters, we obtain

$$\frac{E_\alpha}{4\pi K \tilde{g}_\alpha \rho_0} = \left(1 + \frac{\omega_c}{E_\alpha}\right)^{-K}, \tag{38}$$

where ω_c is the high-energy cutoff. Following [25], we define the crossover scale r_0 by $E_\alpha \equiv u/r_0$ when both impurities have approximatively the same strength $\tilde{\gamma}_1 \approx \tilde{\gamma}_2 \approx \tilde{\gamma}$. The SCHA is a good approximation when $K < 1$ and $E_\alpha \gg \omega_r$—i.e., at long distances $r > r_0$.

In the limit of strong impurities, $\tilde{\gamma}_\alpha \gg 1$, the SCHA frequencies $E_\alpha \approx 4\pi K \tilde{g}_\alpha \rho_0$ are the same as those obtained within the stationary-phase approximation and the crossover scale is given by $r_0 \rho_0 \sim 1/K^2 \tilde{\gamma}$. In the opposite limit $\tilde{\gamma}_\alpha \ll 1$, they are given by

$$E_\alpha \approx 4\pi K \tilde{g}_\alpha \rho_0 \left(\frac{4\pi K \tilde{g}_\alpha \rho_0}{\omega_c}\right)^{K/(1-K)} \tag{39}$$

and the crossover scale is

$$r_0 \rho_0 \sim (K^2 \tilde{\gamma})^{-1/(1-K)} (\rho_0 a)^{-K/(1-K)}, \tag{40}$$

where $\rho_0 a \sim 1$ for repulsive fermions. Note the singular behavior of the crossover scale $r_0 \to \infty$ in the limit $K \to 1$ of noninteracting fermions. This implies that the regime of validity of the SCHA is moved out to extremely long scales r_0. Quite generally, therefore, the long-distance behavior $r \gg r_0$ of the interaction energy is always given by the Casimir-like expression (36) whenever $K < 1$. The scale, however, beyond which this simple result applies, strongly depends on the strength of the backscattering amplitude and the repulsive interaction.

In order to study the limit of weak impurities and weak interactions ($\tilde{\gamma}_\alpha \ll 1$ and $K < 1$ close to 1) in more detail, we use perturbation theory. At second order in $\tilde{\gamma}_\alpha$ and when $r \gg \rho_0^{-1}$, we find

$$V_{12} = -\tilde{\gamma}_1 \tilde{\gamma}_2 (\rho_0 a \pi)^2 \left(\frac{r}{a}\right)^{2(1-K)} \frac{v_F}{2\pi r} \cos(2p_F r) h(K), \tag{41}$$

where

$$h(K) \equiv \frac{K}{\sqrt{\pi}} \frac{\Gamma(K - 1/2)}{\Gamma(K)}$$

and $\Gamma(z)$ is the gamma function. The function $h(K)$ diverges as $K \to 1/2$ and approaches 1 at $K = 1$. The above equation was obtained under the assumption that $1/2 < K < 1$. When $K = 1$, it reproduces exactly Eq. (18) obtained for the noninteracting Fermi gas, provided that we choose the following relation between the microscopic and phenomenological impurity strengths: $\gamma_\alpha = \tilde{\gamma}_\alpha \rho_0 a \pi$. Perturbation theory breaks down when, in order of magnitude, the interaction energy (41) reaches the strong-impurity (or long distances) result (36): $|V_{12}| \sim \omega_r$. This occurs for $r\rho_0 \sim \tilde{\gamma}^{-1/(1-K)} (\rho_0 a)^{-K/(1-K)}$, in agreement with Eq. (40), because $1/2 < K < 1$. Therefore, the perturbative result (41) is valid for intermediate distances $\rho_0^{-1} \ll r \ll r_0$. At long distances $r \gg r_0$ it is replaced by the Casimir-type result (36).

Finally, it is worth mentioning that the presence of two impurities implies the possibility of a tunneling resonance [9,26]. The physics of such a resonance is, however, not captured by the SCHA. This issue will be discussed in more detail in Appendix B.

C. Spin-1/2 Fermi Luttinger liquid

In this section we generalize the above analysis by including the spin degree of freedom, again using a Luttinger-liquid description.

We consider N fermions in an equal mixture of spin-up and spin-down components—i.e., $N_\uparrow = N_\downarrow = N/2$. Here the (noninteracting) Fermi velocity is $v_F = p_F/m = \pi \rho_0 / 2m$ with $\rho_0 = N/L$. Taking spin into account the Euclidean action of the Luttinger liquid is generalized to

$$S_{\mathrm{LL}}[\theta_\mu] = \sum_{\mu=\rho,\sigma} \frac{1}{2\pi K_\mu} \int \mathrm{d}x \mathrm{d}\tau \left[u_\mu (\partial_x \theta_\mu)^2 + \frac{1}{u_\mu} (\partial_\tau \theta_\mu)^2 \right], \tag{42}$$

where $[\theta_\mu]$ is an abbreviation for $[\theta_\rho, \theta_\sigma]$. This corresponds to the usual "charge" θ_ρ and "spin" θ_σ fields that are linear combinations of the spin-up and -down fields: $\theta_{\rho/\sigma} = (1/\sqrt{2}) \times (\theta_\uparrow \pm \theta_\downarrow)$. The Luttinger-liquid parameters obey the relation $K_\rho u_\rho = v_F$ and in addition

$$K \equiv K_\rho < 1 \quad \text{and} \quad K_\sigma = 1, \tag{43}$$

where the first equation comes from considering repulsive interactions and the second from the SU(2) symmetry of the

model with spin-independent interactions [19]. In addition $K>1/2$ when considering contact interactions between fermions; see, e.g., [27]. The action term that describes the interaction with the impurities is still given by Eq. (25) where the density is now simply the sum of the densities of the two spin modes that have the same form as Eq. (24). As stated in the previous section we only keep the most relevant term, which corresponds to backscattering by the impurity, and hence obtain

$$S_i[\theta_\mu] = \sum_{\alpha=1,2} 2\tilde{g}_\alpha \rho_0 \int_0^\beta d\tau \cos[\sqrt{2}\theta_\rho(x_\alpha) + 2p_F x_\alpha] \times \cos[\sqrt{2}\theta_\sigma(x_\alpha)]. \quad (44)$$

Now we follow the same procedure as in previous sections and obtain

$$Z = Z_{\mathrm{LL}}^0 Z_\kappa \int \prod_{\alpha=1,2} \prod_{\mu=\rho,\sigma} D\Theta_{\mu\alpha} e^{-S_{\mathrm{eff}}[\Theta_{\mu\alpha}] - S_i[\Theta_{\mu\alpha}]}, \quad (45)$$

where we have rescaled the fields at the impurity positions by $\theta_{\mu\alpha} = \sqrt{\pi K_\mu}\Theta_{\mu\alpha}$. The effective action S_{eff} is given by

$$S_{\mathrm{eff}}[\Theta_{\mu\alpha}] = \sum_n \sum_{\mu,\alpha,\delta} I^\mu_{\alpha\delta} \Theta_{\mu\alpha,-n} \Theta_{\mu\delta,n}, \quad (46)$$

where the n refers to the Matsubara frequencies ω_n and $I^\mu_{\alpha\delta}$ are the elements of the matrix

$$I^\mu = f_{\mu n} \begin{pmatrix} 1 & -e_{\mu n} \\ -e_{\mu n} & 1 \end{pmatrix}, \quad (47)$$

with $f_{\mu n} = \beta|\omega_n|/(1-e_{\mu n}^2)$, $e_{\mu n} = e^{-|\omega_n|/\omega_\mu}$, and $\omega_\mu = u_\mu / r$.

In order to calculate the partition function, we again use the stationary-phase approximation, which corresponds to expanding S_i around its minima to second order in the fields, assuming that the impurities are strong—i.e., $\tilde{\gamma}_\alpha = \tilde{g}_\alpha / v_F \gg 1$. The nonlinear action S_i is thus replaced by a quadratic approximation

$$\tilde{S}_i = \beta \sum_{\mu\alpha} \sum_n E_{\mu\alpha} |\Theta_{\mu\alpha,n}|^2, \quad (48)$$

where $E_{\mu\alpha} = 2\pi K_\mu \tilde{g}_\alpha \rho_0$.

Since the charge and spin fields are now completely decoupled, we have $\tilde{Z} = Z_\rho Z_\sigma$, with

$$Z_\mu = \prod_n [(f_{\mu n} + \beta E_{\mu 1})(f_{\mu n} + \beta E_{\mu 2}) - (f_{\mu n} e_{\mu n})^2]^{-1/2}. \quad (49)$$

The total free energy $F = F_\rho + F_\sigma$ is then simply the sum of the charge and spin contributions.

After renormalization the free energy is given by

$$V_{12} = \frac{1}{\beta} \sum_\mu \sum_{n>0} \ln\left[1 - \frac{E_{\mu 1} E_{\mu 2} e^{-2\omega_n/\omega_\mu}}{(\omega_n^2 + E_{\mu 1})(\omega_n^2 + E_{\mu 2})}\right]. \quad (50)$$

At low temperature $T \ll \omega_\mu$ we can again replace the sum over Matsubara frequencies by an integral. In the limit of strong impurities, $\omega_\mu \ll E_{\mu a}$, we obtain

$$V_{12} = \sum_\mu u_\mu \int_0^\infty \frac{dy}{2\pi} \ln(1 - e^{-2y}) = -\frac{\pi}{24} \frac{u_\rho + u_\sigma}{r}. \quad (51)$$

This is the straightforward generalization for spin-1/2 fermions of the result obtained in the previous section; see Eq. (36). As discussed there, the SPA is valid only if $\omega_\mu \ll E_{\mu a}$. Hence we are unable to calculate the interaction at shorter distances, where $\omega_\mu \gg E_{\mu a}$.

As in the case of spinless fermions, we can go beyond the SPA regime, using the SCHA. In particular we use $S = S_{\mathrm{eff}} + S_i$ and $\tilde{S} = S_{\mathrm{eff}} + \tilde{S}_i$ in Eq. (37), and then we minimize F_{var} with respect to $E_{\mu a}$. In the case of identical impurities—i.e., $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma}$—the values of $E_{\mu\alpha}$ that minimize F_{var} are such that $E_{\mu 1} = E_{\mu 2} = E_\mu = \pi K_\mu E$. For large distances—i.e., for $\omega_\mu \ll E_\mu$—the SCHA is valid and E is given by

$$E = 2\tilde{g}\rho_0 \left(1 + \frac{\omega_c}{\pi K E}\right)^{-K/2} \left(1 + \frac{\omega_c}{\pi K E}\right)^{-1/2}, \quad (52)$$

where ω_c is the high-energy cutoff. As in the spinless case, we can define the crossover scale as $r_0 = \max(u_\mu / E_\mu) = v_F / \pi K^2 E$, since $K<1$. In the limit of very strong impurity backscattering $E_\mu \gg \omega_c$, we recover the SPA result—i.e., $E_\mu = 2\pi K_\mu \tilde{g}\rho_0$—and the crossover scale is $r_0 \rho_0 \sim 1/K^2 \tilde{\gamma}$. For intermediate impurity strength $\omega_\mu \ll E_\mu \ll \omega_c$, we obtain

$$E_\rho = E_\sigma / K = 2\pi \tilde{g}\rho_0 K^{1/(1-K)} \left(\frac{2\pi \tilde{g}\rho_0}{\omega_c}\right)^{(1+K)/(1-K)}, \quad (53)$$

and the crossover scale in this case is given by

$$\rho_0 r_0 \sim \tilde{\gamma}^{-2/(1-K)} K^{-3/(1-K)} (a\rho_0)^{-(1+K)/(1-K)}, \quad (54)$$

where $a \equiv v_F / K\omega_c$ is the short-distance cutoff and $a\rho_0 \sim 1$. Note that, as in the spinless case, the crossover scale diverges $r_0 \to \infty$ when $K \to 1$. For weak interactions, therefore, the result (51) is only valid at very large distances.

The regime of weak impurity strength can be studied using perturbation theory. At second order in $\tilde{\gamma}_\alpha$, we find

$$V_{12} = -\tilde{\gamma}_1 \tilde{\gamma}_2 (\rho_0 a\pi)^2 \left(\frac{r}{a}\right)^{1-K} \frac{v_F}{\pi r} \cos(2p_F r) h(K), \quad (55)$$

where

$$h(K) \equiv K^K \frac{\Gamma(K/2)}{\sqrt{\pi}\Gamma\left(\frac{K+1}{2}\right)} {}_2F_1\left(\frac{K}{2}, \frac{K}{2}; \frac{K+1}{2}; 1 - K^2\right)$$

and ${}_2F_1$ is the hypergeometric function [28]. The function $h(K)$ is a smooth, monotonically decreasing function of K which diverges like $h(K) \sim 2/\pi K$ as $K \to 0$ and approaches 1 at $K=1$. When $K=1$, the preceding result reproduces exactly Eq. (18) (with an extra factor of 2 due to the spin degeneracy) obtained for the noninteracting (spinless) Fermi gas, provided that we choose the following relation between the microscopic and phenomenological impurity coupling constants: $\gamma_\alpha = \tilde{\gamma}_\alpha \rho_0 a\pi$. The perturbative result (55) is valid for $r \ll r_0$, with r_0 given in Eq. (54). At the crossover scale $r = r_0$ and for $K \lesssim 1$, $|V_{12}| \sim \omega_r$ as in Eq. (51). In conclusion, therefore, the interaction between two impurities follows the behavior given by Eq. (55) only for intermediate distances

and weak interactions. By contrast, for large distances, there is a crossover to the universal Casimir-type interaction, Eq. (51), which depends only on the velocities u_ρ and u_σ.

D. Discussion

As our main result, we have shown that for separations much larger than the interparticle spacing, the interaction energy of two impurities in a Luttinger liquid of repulsive fermions ($K<1$) is a Casimir-type interaction, given by a very simple universal relation, Eq. (36) [Eq. (51)] for spinless [spin-1/2] fermions. In contrast to the result Eq. (19) obtained for strong impurities in a noninteracting Fermi gas it does not contain Friedel oscillations and is independent of both the impurity strengths and the interaction parameter K. The physical origin of this long-range force is thus quite different from the $K=1$ case. In the noninteracting gas, the long-range force comes from the polarization of the ground state. In the strongly interacting case, the Friedel oscillations of the ground-state density around each of the independent impurity still exist [25] but they are not relevant for the impurity interaction at long distances. Instead the result, Eq. (36), is best understood as being the Casimir interaction energy of two mirrors—i.e., impenetrable impurities—in a phononic bath [29]. This interpretation is supported by the direct calculation of the interaction energy of two mirrors in the vacuum fluctuations of a 1D scalar field which represents the density modes of the intervening quantum liquid; see, e.g., Ref. [30]. In fact, a similar result has previously been found for the force between two infinite-mass beads on a string [31]. The Friedel oscillations are relevant for the interaction between two impurities only in the noninteracting case or at intermediate distances in the interacting Luttinger liquid ($K<1$); see Eqs. (41) and (55).

The resulting picture is consistent with the RG calculation of Kane and Fisher for a single impurity [9]: when $K<1$ and $\tilde{\gamma}>0$, the backscattering amplitude is renormalized to strong coupling in the low-energy (or long-distance $r\gg r_0$) limit. The liquid is thus effectively cut into pieces, with the impurities acting like perfect mirrors for the acoustic modes, resulting in a Casimir force between them. The scale on which the impurities flow to strong coupling depends on (i) the initial strength of the impurities $\tilde{\gamma}$ and (ii) the flow velocity given by $1-K$; see below. When the impurities are strong and the liquid is strongly interacting, the impurities flow quickly to strong coupling. The associated crossover scale r_0 is thus of the order of the interparticle distance. By contrast, when the impurities are weak and the liquid is almost noninteracting, it takes very long distances to reach the asymptotic regime. Qualitatively, the crossover scale r_0 for two weak impurities can already be obtained from the scaling theory for a single impurity. Indeed, the perturbative flow equation of Kane and Fisher [9] gives as the running impurity strength

$$\tilde{\gamma}_{\mathrm{eff}} \approx \tilde{\gamma}(r/a)^{1-K}. \tag{56}$$

For spin-1/2 fermions, the preceding equation holds provided that the exponent $1-K$ is replaced by $(1-K)/2$. The crossover scale r_0 then corresponds to the distance at which the running impurity strength $\tilde{\gamma}_{\mathrm{eff}}$ becomes of order 1—i.e., $r_0\rho_0 \sim \tilde{\gamma}^{-1/(1-K)}$—in agreement with Eq. (40), because $\rho_0 a \sim 1$ and $1/2<K<1$ for repulsive fermions with contact interactions. For longer distances, the impurity reaches strong coupling and cuts the liquid into disconnected pieces.

III. ONE-DIMENSIONAL BOSE LIQUID

In this section, we discuss the case of two identical impurities $g\delta(x_\alpha)$ in a one-dimensional Bose liquid. In particular, we consider 1D bosons with short-range repulsive interactions $g_B\delta(x)$. As was first shown by Lieb and Liniger [32], the dimensionless interaction parameter $\gamma_B \equiv mg_B/\rho_0$ in this problem is inversely proportional to the density. The strong-coupling, Tonks-Girardeau limit $\gamma_B \gg 1$ is thus reached either for strong repulsion or at low densities. Within a low-energy effective Luttinger-liquid description, the Luttinger parameter K for interacting bosons is larger than 1. It is related to the sound velocity u by $u=\pi\rho_0/mK$. In the weakly interacting, Gross-Pitaevskii limit $\gamma_B \ll 1$, the Luttinger parameter diverges like $K \approx \pi/\sqrt{\gamma_B} \to \infty$. For strong coupling $\gamma_B \gg 1$ in turn, one finds $K \approx 1+4/\gamma_B \to 1$. The singular case of noninteracting bosons ($\gamma_B=0$) is discussed separately in Appendix C.

A. Classical ground-state energy and vacuum fluctuations

Before starting the explicit calculation of the renormalized impurity interaction energy $V_{12}(r)$ in the case of bosons, we discuss a limitation of our quantum hydrodynamic approach when treating the Bose case. We will shortly see why this limitation was not discussed in the context of fermions. In quantum hydrodynamics, the ground-state energy E_0 is obtained as the sum of two terms: the classical ground-state energy E_0^{cl} and the quantum vacuum fluctuations E_0^{qfl}; see, e.g., Ref. [33]. The classical ground-state energy $E_0^{cl}[\rho_0(x)]$ is a functional of the density profile $\rho_0(x)$. Now, the presence of impurities or boundaries in a quantum liquid modifies the density profile $\rho_0(x)$ over distances of the order of the healing length $\xi \equiv 1/\sqrt{m\mu}$. This in turns modifies the classical ground-state energy in addition to the change in the quantum vacuum fluctuations, which are responsible for the Casimir-type interaction energy. In the LL approach, the classical ground-state energy is usually neglected and only the fluctuations *above* the classical ground state are taken into account. In a homogeneous system, the classical ground-state energy is just a constant in the Hamiltonian and can therefore be safely ignored. Provided that the healing length is much smaller than the system size, the effect of the boundaries disappears for bulk properties. In fermionic quantum liquids, the healing length is of the order of the interparticle distance $1/\rho_0$. Therefore, introducing two impurities with a separation $r \gg 1/\rho_0$ leaves the classical ground-state energy unchanged. However, in the case of bosons, the healing length is much larger than the interparticle spacing, diverging like $\xi \sim K/\rho_0$ for weak interactions $\gamma_B \to 0$. The condition $r \gg \xi$ for neglecting the contribution of the classical ground-state energy in the calculation of the renormalized interaction energy between impurities V_{12} thus becomes increasingly restrictive

for small γ_B. As we will see, this leaves us only with an exponentially small interaction, quite in contrast to the case of fermions.

B. Weak impurities and weakly interacting bosons

In the case of a weakly interacting Bose gas $\gamma_B \ll 1$ (corresponding to *high* densities) the Bogoliubov approach is quantitatively applicable, as was shown long ago by Lieb and Liniger [32]. The physical reason for that is that the healing length $\xi \approx 1/\rho_0\sqrt{\gamma_B}$ in this limit is much larger than the average interparticle spacing. The situation is therefore essentially equivalent to the weak-coupling limit of a 3D Bose-Einstein condensate at *low* densities, where interactions between two impurities have recently been discussed by Klein and Fleischhauer [34] (the interaction energy V_{12} is called the conditional energy shift and denoted by Δ in this work). The explicit calculation is based on the Bogoliubov approach and assumes that the dimensionless impurity strength $\gamma \equiv mg/\pi\rho_0$ is much smaller than one. Adapting the calculation of Ref. [34] to a one-dimensional gas, we obtain

$$V_{12}(\gamma, r) \approx -\gamma^2 \rho_0 \xi \frac{\pi^2 \rho_0^2}{m} \exp(-2r/\xi) \tag{57}$$

to lowest order in γ. The interaction energy between widely separated impurities in a weakly interacting Bose gas thus vanishes exponentially on the scale set by the healing length ξ. As discussed above, this calculation assumes that the impurities modify the density profile only locally, which is also the physical reason for the vanishing interaction at $r \gg \xi$.

The preceding result can again be qualitatively understood with the help of the scaling theory of Kane and Fisher [9] for a single weak impurity in a LL. Indeed, when $K > 1$, the effective coupling of a single impurity is renormalized to zero in the low-energy limit. The RG flow thus starts at high energy $\omega_c = u/a \sim \mu$ (corresponding to a short distance cutoff $a \sim \xi$) and ends at a much lower energy $\omega_r = u/r$ which is set by the separation r of the two impurities. Similar to the case of fermions, we define a crossover scale r_0 by the condition that the effective dimensionless impurity strength at this scale is of order 1:

$$\gamma_{\mathrm{eff}} \approx \gamma (a/r)^{K-1} \sim 1 \Leftrightarrow r_0 \approx \xi \gamma^{1/(K-1)} \lesssim \xi. \tag{58}$$

Since $\gamma \ll 1$ for weak impurities and $K = \pi/\sqrt{\gamma_B} \to \infty$ due to the weak interaction condition, we find $r_0 \approx \xi$. This confirms that there is no interaction between two weak impurities embedded in a weakly interacting Bose gas when they are farther apart than a distance of the order of the healing length. More generally, the scaling theory indicates that there is no long-range interaction between weak impurities even in a strongly interacting Bose gas, in which $K \gtrsim 1$. Of course the limiting case $K = 1$ of hard-core bosons is special as is the case $K = \infty$ of no interaction at all. The latter case is treated in Appendix C and shows that there is no interaction between the impurities whatever the impurity strength. In strong contrast to that, the limit of a Tonks-Girardeau gas of hard-core bosons is equivalent to the case of noninteracting fermions for properties depending only on the modulus of the ground-state wave function like the density distribution. On the basis of the calculations in Sec. II A, one thus expects a long-range interaction of the form (17) between impurities in a Tonks-Girardeau gas which exhibits Friedel-like oscillations. In view of the fact that the momentum distribution of hard-core bosons is quite different from that of a free Fermi gas, showing no jump at p_F, this is a quite remarkable result [35]. Another singular limit, where long-range interactions appear in a Bose liquid, is that of impenetrable impurities $\gamma = \infty$. For arbitrary values $\infty > K > 1$ of the interactions, the interacting Bose liquid is then cut into three disconnected pieces. The impurities thus act as perfect mirrors for the low-energy phonon excitations of the intervening Bose liquid, giving rise to a Casimir interaction energy precisely as in Eq. (36) for spinless fermions. It appears, however, that the limit of impenetrable impurities at arbitrary energies is nonphysical, imposing strict Dirichlet boundary conditions on a scalar field [23]. The Casimir force is thus expected to be restricted to $\gamma = \infty$, while for any finite γ only short-range interactions should survive. Describing these crossovers in detail clearly requires a quantitative theory of impurity interactions in 1D Bose liquids at arbitrary values of K and γ, an interesting problem for further study.

IV. EXPERIMENTAL REALIZATION AND DETECTION OF THE CASIMIR-LIKE FORCE

The recent realization of one-dimensional ultracold Fermi gases in a strong 2D optical lattice [4] provides a novel opportunity to study Luttinger-liquid effects in a setup with cold gases—e.g., spin-charge separation [36]. In order to study whether the Casimir interactions discussed here might be observed in these systems, we consider an atomic gas of fermions in two hyperfine states. These two internal states play the role of (iso)spin-1/2 states. In principle both the sign and strength of the interaction can be controlled using scattering resonances—e.g., a confinement induced resonance as shown by Olshanii [37]. For simplicity, we assume that the two spin states are equally populated, $N_\uparrow = N_\downarrow = N/2$. We note that in order to have $K < 1$, one needs to consider a two-component atomic Fermi gas. Indeed, in a single-component Fermi gas, Pauli's principle forbids s-wave collisions, implying that $K = 1$ for spinless fermions interacting via a contact potential.

Following several recent ideas [5,34,38] which involve trapping single atoms in ultracold gases, we consider an atomic-quantum-dot- (AQD-) like configuration, which consists of single atoms confined in a tight trap created either magnetically or optically—e.g., by an additional optical lattice. We assume that the confining potential can be adjusted in such a way that it does not affect the atoms of the bath. The impurity atom, which is trapped in a certain internal state $|a\rangle$, interacts with the atoms of the bath through s-wave collisions. In the case where two such AQD's are embedded in the bath and both impurity atoms are in state $|a\rangle$, the system precisely realizes the situation of two localized impurities interacting via a 1D quantum liquid. Provided the liquid consists of spin-1/2 repulsive fermions, we expect that for distance r much larger than the average interparticle spacing, the interaction is of the Casimir form given in Eq.

(51). In principle, using a scattering resonance may allow one to reach the strong-impurity regime $\tilde{\gamma} \gg 1$ where the crossover scale r_0 is even smaller than the interparticle distance $1/\rho_0$.

A possible way to detect the interaction energy $V_{12}(r)$ is to do spectroscopy of a single trapped atom as a function of the distance r to a neighboring trapped atom. In addition to the mean-field line shifts modifying the internal levels of the impurity atom, the Casimir interaction produces a line shift depending on distance as $1/r$. For a quantitative estimate of this effect, we compute the energy (51) for the experimental situation realized in Ref. [4]. There, about $N \sim 100$ ^{40}K atoms (per tube) form an atomic wire of length $L \sim 10$ μm. The temperature can be as low as $T \sim 50$ nK, which is about one-tenth of the Fermi temperature. The Fermi velocity is of order $v_F \sim 2 \times 10^{-2}$ m/s and we take $u_\sigma + u_\rho \sim 2v_F$. As the tube length is of the order of 10 μm and the interparticle distance $1/\rho_0 \sim 0.1$ μm, we assume the interimpurity distance to be $r \sim 1$ μm, which is larger than the crossover length for strong impurities. This gives a Casimir-related line shift of the order of 1 kHz, which is in an experimentally accessible range. With the parameters given above, the characteristic frequency ω_r is of the order of $\sim 3T$, which is not much larger than T as required for the validity of the zero-temperature limit in which the Casimir force is obtained.

V. CONCLUSION

In conclusion, we studied the long-range interaction between two impurities mediated by the 1D quantum liquid in which they are embedded. We found that for repulsive fermions, the impurities interact via a RKKY-like interaction at intermediate distances and via a Casimir-like force at large distances. The crossover scale separating these two regimes depends on the strength of the impurities and on the interactions between fermions. We proposed an experimental realization of such a system with atomic quantum dots in an ultracold atomic gas and suggested a way to detect the Casimir-type interaction by spectroscopy of a single atom in an AQD.

An issue which is still open is to understand the interaction between impurities in a strongly interacting Bose liquid. In particular, how does the short-range interaction (on the scale of the healing length) turn into a long-range interaction featuring Friedel oscillations in the Tonks-Girardeau limit? Another issue is to assess the validity of the self-consistent harmonic approximation used to discuss spin-1/2 fermions in Sec. II C. Indeed, it is not obvious that the variational ansatz, which assumes decoupling of the charge and spin modes in presence of the impurities, is a valid starting point.

In this paper, we studied static impurities. The situation becomes even more interesting if one has dynamic impurities, as in an AQD. First, we discuss the possibility of *internal* dynamics for the AQD. We have seen that the characteristic frequencies of vacuum modes (excitations) responsible for the long-range interactions between AQD's are limited by ω_r. This means that the effective interaction potential between static impurities can be used (in an adiabatic approximation) for time-dependent impurity strengths $\tilde{\gamma}_\alpha(t)$ provided that the interaction properties change slowly compared with the time scale ω_r^{-1}. Consider, for instance, the configuration with two AQD's described previously; see Fig. 1 (a similar scheme for two impurities in a 3D Bose-Einstein condensate is discussed in Ref. [34]). The two-level impurity atoms can be described as isospins 1/2 or qubits. A laser can drive transitions (equivalent to a single-qubit gate) between the two internal levels $|a\rangle$ and $|b\rangle$. In the adiabatic approximation, the AQD variables are slow and can be taken out of the integrals so that our previous treatment to calculate the interaction between two impurities applies. Thus one can easily write an effective Hamiltonian for the two AQD's in the form

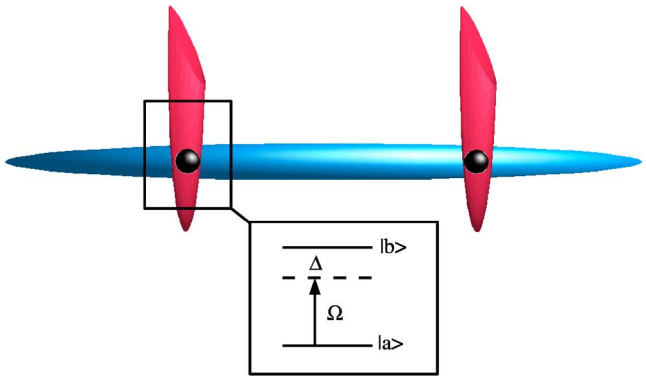

FIG. 1. (Color online) Schematic setup of two AQD's coupled to a 1D atomic reservoir. The impurity atoms (see text) in a tightly confining potential interact with the bath when their internal level is $|a\rangle$. Here δ is the renormalized detuning and Ω is the Rabi frequency coming from a laser-induced coupling; see Sec. V.

$$H_{\text{eff}} = \sum_{\alpha=1,2} \left(-\frac{\delta}{2}\sigma_z^{(\alpha)} + \Omega\sigma_x^{(\alpha)} \right) + \tfrac{1}{2} V_{12}(\sigma_z^{(1)} + 1)(\sigma_z^{(2)} + 1), \tag{59}$$

where δ is the (renormalized) detuning, Ω is the (effective) Rabi frequency coming from the laser induced coupling [5], and $\sigma_x^{(\alpha)}$, $\sigma_y^{(\alpha)}$, and $\sigma_z^{(\alpha)}$ are the Pauli matrices describing the isospin 1/2 of each AQD, $\alpha = 1,2$. The long-range potential V_{12} depends, as we have seen, strongly on the characteristics of the bath and on the distance between the AQD's. In addition, the case of impurities with internal dynamics embedded in a spin-1/2 fermionic liquid is of course also relevant for discussing the RKKY interaction in Luttinger liquids, as discussed perturbatively in [39].

It is also possible to imagine *external* motion or dynamics for the impurities. The argument of adiabaticity also holds in the case where the distance between the impurities changes sufficiently slowly. This means that the Casimir-like interaction could be used, for example, to create long-range attractive forces in mixtures of 1D fermionic gases. However, when the external dynamics of the impurities are taken into account on the same footing as the bath dynamics other effects could reduce, if not wash out, the Casimir force [40].

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APPENDIX A: RPA FOR WEAK IMPURITIES IN AN IDEAL FERMI GAS

In this Appendix, we give another derivation of the weak coupling result (18) using the random-phase approximation. The interaction between the two impurities may be written as the sum of two contributions: a direct interaction (which is zero in the present case due to the short-range nature of the impurities) and an indirect interaction induced via the polarization of the medium. The polarization operator in the Fourier representation $\Pi(\omega_n,q)$ has a singularity at $q=2p_F$, which leads to the appearance of the long-range force. Specifically, we calculate the interaction energy of the two impurities using the RPA, which works well whenever only states with very few particle-hole pairs are excited by the perturbation—i.e., in the limit $\gamma_\alpha \ll 1$. For an ideal (spinless) Fermi gas the RPA Lagrangian can be represented as ($T=0$ and here we use the real-time formalism from the start)

$$S^{\mathrm{RPA}} = \frac{1}{2}\int \mathrm{d}t \sum_k \sum_p (\dot{\phi}^\dagger_{pk}\dot{\phi}_{pk} - \omega^2_{pk}\phi^\dagger_{pk}\phi_{pk}), \qquad \text{(A1)}$$

where $\rho_k=\Sigma_p(2\omega_{pk})^{1/2}\phi_{pk}$ is the RPA expression for the fermion density, $\omega_{pk}=(p+k)^2/2m-p^2/2m$ is the energy of the electron-hole pair, and the summation over the momentum p is limited by the conditions $|p|<p_F$ and $|p+k|>p_F$. The interaction of the AQD's with the liquid can be written as

$$S^{\mathrm{RPA}}_i = \frac{1}{2}\sum_{pk}(2\omega_{pk})^{1/2}(\phi_{pk}V_k + \phi^\dagger_{pk}V_{-k}), \qquad \text{(A2)}$$

where $V_k=g_1+g_2\exp(ikr)$. The total RPA action $S^{\mathrm{RPA}}+S^{\mathrm{RPA}}_i$ is quadratic, and hence the fields ϕ_{pk} can be integrated out, so that the effective interaction between the impurities is given by $V_{12}=-\frac{1}{2}\Sigma_k|V_k|^2\Pi(\omega=0,k)$, where the polarization operator is

$$\Pi(0,k) = -\frac{2}{\pi}\int_{-p_\mathrm{F}}^{p_\mathrm{F}} \mathrm{d}p \frac{1}{2pk+k^2} = -\frac{1}{\pi k}\ln\left|\frac{k+2p_\mathrm{F}}{k-2p_\mathrm{F}}\right|. \qquad \text{(A3)}$$

Substituting V_k, performing the integration by transforming the integral along the real k axis into the integrals along the branch cut corresponding to the singularity of the logarithmic-function, and removing the self-energies of the separated impurities (renormalizing the interaction), we find

$$V_{12} = -\gamma_1\gamma_2\frac{v_F}{2\pi r}\cos(2p_F r), \qquad \text{(A4)}$$

in agreement with Eq. (18).

APPENDIX B: COULOMB BLOCKADE AND RESONANT TUNNELING

The present paper discusses the interaction energy between two impurities in an atomic quantum wire (i.e., a 1D quantum liquid made of atoms). In this Appendix, we wish to make contact with related subjects in the field of mesoscopic conductors (or solid-state quantum wires): namely, Coulomb blockade and resonant tunneling of electrons in a quantum wire with two tunnel junctions or barriers (see, e.g., [9,10,26]). For simplicity, we only consider the case of spinless fermions.

We first consider an atomic quantum wire with two impurities. When the impurities are strong (or very far apart), the atomic quantum wire is cut into three disconnected pieces and we may picture the low-energy behavior of the system as the following structure: left wire/impurity/island(or central wire)/impurity/right wire. The Casimir effect occurring in this system is directly related to the energy cost of transferring a supplementary particle from one of the wires to the island. In our case (neutral atoms interacting via a short-range potential), this energy cost is a finite-size energy [41] equal to

$$\pi v_F/2K^2 r. \qquad \text{(B1)}$$

With the help of the approximate relation $K^{-2}\approx 1+g_f/\pi v_F$ [9], this energy cost can be seen as the sum of two contributions: a kinetic energy cost $\pi v_F/2r$ and an interaction energy cost, resulting from the local interaction with the other particles on the island, $g_f/2r$, where g_f is the forward-scattering coupling constant (in the standard notation of the g-ology, $g_f=g_2=g_4$; see [19], e.g.). Note that the finite-size energy is of the order of the zero-point kinetic energy of a phonon on the island, $\pi v_F/2Kr$.

In the case of electrons in a solid-state quantum wire with two tunnel junctions (corresponding to the structure left electrode/barrier/island/barrier/right electrode), the Coulomb blockade is due to the energy cost of transferring a single electron from an electrode to the island [9,26]. However, this energy cost is not only due to the finite-size energy $\pi v_F/2K^2 r$ (with $K^{-2}\approx 1+e^2/\epsilon\pi v_F$, where e is the electron charge and ϵ is an appropriate dielectric constant) but also gets an additional contribution from the charging energy $e^2/2C$ of the capacitors (i.e., the two tunnel junctions), where C is the sum of the capacitance of each tunnel junction [26]. The total energy cost is equal to $\pi v_F/2K^2 r+e^2/2C$. The charging energy is due to the long-range part of the interaction between electrons and therefore does not arise in the case of cold atoms interacting via a short-range potential.

Another subject of comparison between the atomic quantum wire and the solid-state quantum wire is the possibility of tunneling resonances across the double-barrier structure [9,26] (see [19] for review). Indeed, in an atomic quantum wire with two impurities, when $\cos(p_F r)=0$ [9,26] a tunnel-

ing resonance occurs: the energy cost to add a particle on the island vanishes and particles can therefore tunnel through the impurities. The liquid is no longer cut into disconnected pieces, and we do not expect a Casimir effect to occur. Tunneling resonances do not appear in our calculations because they are not captured by the self-consistent harmonic approximation we used, as discussed in Ref. [25], for example. However, such resonances are infinitely sharp at zero temperature [9,26], and therefore they do not play a major role and should be easy to avoid experimentally.

APPENDIX C: IMPURITIES IN AN IDEAL BOSE GAS

The ideal Bose gas ($\gamma_B=0$) is a singular case: the healing length ξ diverges and boundaries are therefore felt over macroscopic distances. Here using quantum hydrodynamics makes no sense, but of course one can exactly solve the problem from first principles (Schrödinger equation). At zero temperature, all bosons are in the same single-particle wave function ψ_0 (the corresponding many-body wave function is just a product or Hartree state), which is the ground state of the single-particle Schrödinger equation

$$-\frac{1}{2m}\partial_x^2\psi_0(x)+g[\delta(x-r/2)+\delta(x+r/2)]\psi_0(x)=\epsilon_0\psi_0(x), \tag{C1}$$

where g is the impurity coupling constant. We assume that the particles are on a ring of length L. The ground-state energy E_0 for N bosons is given by $N\epsilon_0$. For $g\geqslant 0$, it is bounded as follows:

$$\epsilon_0(g=0,r)\leqslant\epsilon_0(g,r)\leqslant\epsilon_0(g=\infty,r), \tag{C2}$$

which is a direct consequence of the Schrödinger equation. In the following we will show that both bounds are going to zero in the thermodynamic limit, implying that $E_0(g,r)=0$ for all $g\geqslant 0$. Therefore, the interaction energy $V_{12}(g,r)\equiv E_0(g,r)-E_0(g,r\to\infty)$ vanishes whatever the distance between the impurities.

On the one hand, when the impurity strength is zero $g=0$, the ground state of the single-particle Schrödinger equation with periodic boundary conditions is just the constant wave function, which has zero energy $\epsilon_0(r,g=0)=0$. On the other hand, when $g=\infty$, the wave function has to vanish on the location of the impurities, implying some bending of the wave function and a corresponding cost in kinetic energy. The ground-state wave function is

$$\psi_0(x)=\psi_m\sin\left(\frac{\pi(|x|-r/2)}{L-r}\right)\quad(\text{if } |x|>r/2)$$
$$=0\quad(\text{if } |x|\leqslant r/2), \tag{C3}$$

where $\psi_m\equiv\psi_0(x=\pm L/2)$ is the maximum value of the wave function and the energy is

$$\epsilon_0(g=\infty,r)=\frac{1}{2m}\left(\frac{\pi}{L-r}\right)^2. \tag{C4}$$

This quantity vanishes in the thermodynamic limit ($L\to\infty$ at fixed density $\rho_0=N/L$) such that $L\gg r$. Therefore $\epsilon_0(g=\infty,r)=0$ in the thermodynamic limit for all r such that $r\ll L$. Of course, this conclusion does not hold for an ideal Fermi gas, because fermions have to occupy different single-particle states (following the Pauli principle) and therefore the average energy per particle does not vanish in the thermodynamic limit.

In conclusion, there is no interaction energy between two impurities in an ideal Bose gas, provided that the distance r is much smaller than the ring size L, which is always satisfied in the thermodynamic limit.

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