Separability criterion of tripartite qubit systems

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In this paper, we present a method to construct full separability criterion for tripartite systems of qubits. The spirit of our approach is that a tripartite pure state can be regarded as a three-order tensor that provides an intuitionistic mathematical formulation for the full separability of pure states. We extend the definition to mixed states and give out the corresponding full separability criterion. As applications, we discuss the separability of several bound entangled states, which shows that our criterion is feasible.

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I. INTRODUCTION

Entanglement is an essential ingredient in quantum information and the central feature of quantum mechanics which distinguishes a quantum system from its classical counterpart. In recent years, it has been regarded as an important physical resource, and widely applied to a lot of quantum information processing (QIP): quantum computation [1], quantum cryptography [2], quantum teleportation [3], quantum dense coding [4], and so on.

Entanglement arises only if some subsystems ever interacted with the others among the whole multipartite system in physics, or only if the multipartite quantum state is not separable or factorable in mathematics. The latter provides a direct way to tell whether or not a given quantum state is entangled.

As to the separability of bipartite quantum states, partial entropy introduced by Bennett *et al.* [5] provides a good criterion of separability for pure states. Later, Wootters presents the remarkable concurrence for bipartite systems of qubits [6,7]. Based on the motivation of generalizing the definition of concurrence to higher dimensional systems, many attempts have been made [8–11], which provide good separability criteria for bipartite qubit systems under corresponding conditions, while Ref. [8] also presents an alternative method to minimize the convex hull for mixed states. As to multipartite quantum systems, several separability criteria have been proposed [12–17]. The most notable one is three-tangle for three qubits [13]. Recently, the result has been generalized to the higher dimensional systems [18]. Despite the enormous effort, the separability of quantum states especially in higher dimensional systems is still an open problem.

In this paper we construct the full separability criterion for arbitrary tripartite qubit system by a different method, i.e., a tripartite pure state can be defined by a three-order tensor. The definition provides an intuitionistic mathematical formulation for the full separability of pure states. Analogous to Ref. [8], we extend the definition to mixed states. More importantly, our approach is easily generalized to higher dimensional systems. As applications, we discuss separability of two bound entangled states introduced in Refs. [19,20], respectively.

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II. SEPARABILITY FOR PURE STATES

We start with the separability definition of tripartite qubit pure state $|\psi\rangle_{ABC}$. $|\psi_{ABC}\rangle$ is fully separable if

$$|\psi\rangle_{ABC} = |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C. \quad (1)$$

Consider a general tripartite pure state written by

$$|\psi\rangle_{ABC} = \sum a_{ijk}|i\rangle_A|j\rangle_B|k\rangle_C, \quad (2)$$

where $i, j, k=0, 1$, the coefficients a_{ijk}s can be arranged as a three-order tensor (tensor cube) [21] as shown in Fig. 1.

Note that the subscripts of a_{ijk} correspond to the basis $|i\rangle_A|j\rangle_B|k\rangle_C$. Every surface can be regarded as the tensor product of a single qubit and an unnormalized bipartite state. Hence if the two vectors (edges) of a surface are linear relevant (including one of the vectors is *zero* vector), then the bipartite state mentioned above can be factorized. The conclusion for diagonal plane is analogous. Considering all the planes, one can easily find that the tripartite state is fully separable if all the vectors which are parallel mutually shown in the cube are linear relevant, according to the fundamental linear algebra. I.e. the rank of every matrix composed of four coefficients on the corresponding surface and diagonal plane is 1. Equivalently, we can obtain the following lemma.

Lemma 1. A tripartite pure state $|\psi\rangle_{ABC}$ with the form of

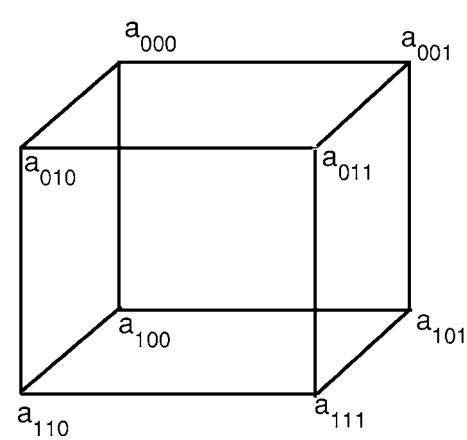

FIG. 1. Three-order tensor of the coefficients of a tripartite pure state.

Eq. (2) in $2\times2\times2$-dimensional Hilbert space is fully separable, if the following six equations hold:

$$\sum_{i=0}^{1}|(a_{i00}a_{i11}-a_{i01}a_{i10})|=0, \quad (3)$$

$$\sum_{i=0}^{1}|(a_{0i0}a_{1i1}-a_{0i1}a_{1i0})|=0, \quad (4)$$

$$\sum_{i=0}^{1}|(a_{00i}a_{11i}-a_{01i}a_{10i})|=0, \quad (5)$$

$$\sum_{i=0}^{1}(a_{0i0}a_{1j1}-a_{0j1}a_{1i0})=0, \quad (6)$$

$$\sum_{i=0}^{1}(a_{i00}a_{j11}-a_{j01}a_{i10})=0, \quad (7)$$

$$\sum_{i=0}^{1}(a_{00i}a_{11j}-a_{01j}a_{10i})=0, \quad (8)$$

where $i,j=0$, 1 and $i\oplus j=1$.

Proof (sufficient condition). If Eqs. (3)–(5) hold, then the rank of every matrix that the cubic surface corresponds to is 1. If Eqs. (6)–(8) hold, then the rank of every matrix that the cubic diagonal plane corresponds to is 1. Hence that Eqs. (3)–(8) hold simultaneously shows that the tripartite pure state can be fully factorized, i.e., it is fully separable.

(Necessary condition) If a given tripartite state is fully separable, one can easily obtain that the rank of every corresponding matrix is 1. Namely, Eqs. (3)–(8) hold.

Consider that $|\psi\rangle_{ABC}$ can be denoted by a vector in $2\times2\times2$-dimensional Hilbert space,

$$|\psi\rangle=(a_{000},a_{001},a_{010},a_{011},a_{100},a_{101},a_{110},a_{111})^T,$$

with the superscript T denoting transpose, we can write the above equations (3)–(8) in matrix notation by

$$\langle\psi^*|s^\alpha|\psi\rangle=0, \quad \alpha=1,2,\ldots,9,$$

where the star denotes complex conjugation, and $s^1=-\sigma_y\otimes\sigma_y\otimes I_1$, $s^2=-\sigma_y\otimes\sigma_y\otimes I_2$, $s^3=-\sigma_y\otimes I_1\otimes\sigma_y$, $s^4=-\sigma_y\otimes I_2\otimes\sigma_y$, $s^5=-I_1\otimes\sigma_y\otimes\sigma_y$, $s^6=-I_2\otimes\sigma_y\otimes\sigma_y$, $s^7=-Iv\otimes\sigma_y\otimes\sigma_y$, $s^8=-\sigma_y\otimes Iv\otimes\sigma_y$, $s^9=-\sigma_y\otimes\sigma_y\otimes Iv$, with

$$\sigma_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ I_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ I_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\text{ and } Iv=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Define a new vector $\boldsymbol{C}(\psi)$ by

$$\boldsymbol{C}(\psi)=\bigoplus_{\alpha=1}^{9}C^\alpha$$

with $C^\alpha=\langle\psi^*|s^\alpha|\psi\rangle$, then the length of the vector can be given by

$$|\boldsymbol{C}(\psi)|=\sqrt{\sum_\alpha (C^\alpha)^2}.$$

Therefore the full separability criterion for a tripartite state can be expressed by a more rigorous form as follows.

Theorem 1. A tripartite pure state ψ is fully separable if $|\boldsymbol{C}(\psi)|=0$.

Proof. That $|\boldsymbol{C}(\psi)|=0$ is equivalent to that $C^\alpha=0$ holds for any α. According to Lemma 1, one can obtain that $|\boldsymbol{C}(\psi)|=0$ is the sufficient and necessary condition.

III. SEPARABILITY FOR MIXED STATES

A tripartite mixed state ρ is fully separable if there exists a decomposition $\rho=\sum_{k=1}^{K}\omega_k|\psi^k\rangle\langle\psi^k|$, $\omega_k>0$ such that ψ^k is fully separable for every k or equivalently if the infimum of the average $|\boldsymbol{C}(\psi^k)|$ vanishes, namely,

$$C(\rho)=\inf\sum_{k}^{K}\omega_k|\boldsymbol{C}(\psi^k)|=0, \quad (9)$$

among all possible decompositions. Therefore for any given decomposition

$$\rho=\sum_{k=1}^{K}\omega_k|\psi^k\rangle\langle\psi^k|, \quad (10)$$

according to the Minkowski inequality

$$\left[\sum_{i=1}\left(\sum_k x_i^k\right)^p\right]^{1/p}\leqslant\sum_k\left(\sum_{i=1}(x_i^k)^p\right)^{1/p}, \quad p>1, \quad (11)$$

one can get

$$C(\rho)=\inf\sum_k\omega_k|\boldsymbol{C}(\psi^k)|=\inf\sum_k\omega_k\sqrt{\sum_\alpha|\langle(\psi^k)^*|s^\alpha|\psi^k\rangle|^2}$$
$$\geqslant\inf\sqrt{\sum_\alpha\left(\sum_k\omega_k|\langle(\psi^k)^*|s^\alpha|\psi^k\rangle|\right)^2}. \quad (12)$$

Consider the matrix notation [8] of Eq. (10) as $\rho=\Psi W\Psi^\dagger$, where W is a diagonal matrix with $W_{kk}=\omega_k$, the columns of the matrix Ψ correspond to the vectors ψ^k, and the eigenvalue decomposition, $\rho=\Phi M\Phi^\dagger$, where M is a diagonal matrix whose diagonal elements are the eigenvalues of ρ, and Φ is a unitary matrix whose columns are the eigenvectors of ρ, associated with the relation $\Psi W^{1/2}=\Phi M^{1/2}U$, where U is a right-unitary matrix, inequality (12) can be rewritten as

$$C(\rho)\geqslant\inf\sqrt{\sum_\alpha\left(\sum_k|\Psi^TW^{1/2}s^\alpha W^{1/2}\Psi|_{kk}\right)^2}$$
$$=\inf_U\sqrt{\sum_\alpha\left(\sum_k|U^TM^{1/2}\Phi^Ts^\alpha\Phi M^{1/2}U|_{kk}\right)^2}. \quad (13)$$

In terms of the Cauchy-Schwarz inequality

$$\left(\sum_i x_i^2\right)^{1/2}\left(\sum_i y_i^2\right)^{1/2}\geqslant\sum_i x_iy_i, \quad (14)$$

the inequality

$$C(\rho) \geqslant \inf_U \sum_k \left| U^T\left(\sum_\alpha z_\alpha A^\alpha\right)U\right|_{kk} \quad (15)$$

is implied for any $z_\alpha = y_\alpha e^{i\phi}$ with $y_\alpha > 0$ and $\Sigma_\alpha y_\alpha^2 = 1$, where $A^\alpha = M^{1/2}\Phi^T s^\alpha \Phi M^{1/2}$. The infimum of Eq. (15) is given by $\max_{z\in\boldsymbol{C}} \lambda_1(z) - \Sigma_{i>1}\lambda_i(z)$ analogous to Ref. [8], with $\lambda_i(z)$s are the singular values, in decreasing order, of the matrix $\Sigma_\alpha z_\alpha A^\alpha$. $C(\rho)$ is as well expressed by

$$C(\rho) = \max\left\{0, \max_{z\in\boldsymbol{C}}\lambda_1(z) - \sum_{i>1}\lambda_i(z)\right\}. \quad (16)$$

One can easily see that $C(\rho)=0$ provides a necessary and even sufficient condition of full separability for tripartite mixed qubit systems, hence an effective separability criterion. However, it is so unfortunate that $C(\rho)$ cannot serve as a good entanglement measure, but only an effective criterion to detect whether a state is fully separable, because $\boldsymbol{C}(\psi)$ for pure states is not invariant under local unitary transformations.

IV. EXAMPLES

At first, consider the complementary states to SHIFTS UPB [19]. SHIFTS UPB is the set of the following four product states:

$$\{|0,1,+\rangle, |1,+,0\rangle, |+,0,1\rangle, |-,-,-\rangle\} \quad (17)$$

with $\pm = (|0\rangle \pm |1\rangle)/\sqrt{2}$. The corresponding bound entangled (complementary) state is given by

$$\bar{\rho} = \frac{1}{4}\left(1 - \sum_{i=1}^{4} |\psi_i\rangle\langle\psi_i|\right) \quad (18)$$

with $\{\psi_i : i=1,\ldots,4\}$ corresponding to the SHIFTS UPB. In Ref. [19], it is stated that this complementary state has the curious property that not only is it two-way ppt, it is also two-way separable. The numerical result based on our criterion can show a *nonzero* (0.1469) entanglement for $C(\bar{\rho})$, which is consistent with Ref. [19].

Let us consider the second example, the Dür-Cirac-Tarrach states [20]

$$\rho_{DCT} = \begin{pmatrix} \frac{a+b}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a-b}{2} \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ a-b2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a+b}{2} \end{pmatrix}; \quad (19)$$

we can also show the nonzero $C(\rho_{DCT})$ (0.3747) for $a=\frac{1}{3}$; $c=d=\frac{1}{6}$; $b=e=0$. The conclusion is also implied in Ref. [20].

The above numerical tests are operated as follows. In order to show the nonzero $C(x)$ with $x=\bar{\rho}$ or ρ_{DCT}, we choose 10^5 random vectors $(z_1, z_2, \ldots, z_9)$ generated by MATLAB 6.5 for a given x, then substitute these vectors into $\Sigma_\alpha z_\alpha A^\alpha$ and obtain 10^5 matrices. We can get 10^5 $([\lambda_1(z) - \Sigma_{i>1}\lambda_i(z)]$ by singular value decomposition for the matrices. The maximal $\lambda_1(z) - \Sigma_{i>1}\lambda_i(z)$ among the matrices is assigned to $C(x)$. Due to the whole process, it is obvious that our numerical approach is more effective to test the nonzero $C(\rho)$. It can only provide a reference for the zero $C(x)$. If a standard numerical process is needed, we suggest that the approach introduced in Ref. [17] be preferred.

V. DISCUSSION AND CONCLUSION

As a summary, we have shown effective criterion for tripartite qubit systems by the pioneering application of the approach to define a tripartite pure state as a three-order tensor. However, although our criterion can be reduced to Wootters' concurrence [7] for bipartite systems, as mentioned above, the criterion cannot serve as a good entanglement measure. Therefore it is not necessary to find out the concrete value of $C(\rho)$, but whether $C(\rho)$ are greater than zero, as can be found in our examples. Based on the tensor treatment for a tripartite pure state, if a more suitable $\boldsymbol{C}(\psi)$ that can serve as a good entanglement measure can be found, it will be interesting. It deserves our attention that our approach can be easily extended to test the full separability of multipartite systems in arbitrary dimension, which will be given out in the forthcoming works.

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[21] Different from the previous definition of tensors, all the quantities with indices, such as T_{ijk}, are called three-order tensors here. Therefore the set of all one-order tensor is the set of vectors, and the set of all two-order tensors is the set of matrices. Three-order tensors T_{ijk} are matrices correpsonding to the planes in the cube (vectors corresponding to the edges) when any one (two) of their three indices is (are) fixed.