Elementary formula for entanglement entropies of fermionic systems

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An elementary formula for the von Neumann and Rényi entropies describing quantum correlations in two-fermionic systems having four single-particle states is presented. An interesting geometric structure of fermionic entanglement is revealed. A connection with the generalized Pauli principle is established.

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I. INTRODUCTION

During the past decade the characterization of inseparable quantum correlations, or entanglement, has become one of the most active research fields. The reason for this flurry of activity is twofold. First the attitude of regarding entanglement like energy as a resource paved the way to the appearance of quantum information science including such exciting applications as teleportation [1], quantum cryptography [2], and more importantly quantum computing [3]. Second, entanglement as the "characteristic trait of quantum mechanics" [4] is of fundamental importance for a deeper understanding of the conceptual foundations of quantum theory.

The main problem is how to quantify entanglement. In this respect as far as entanglement of distinguishable particles is concerned a large number of useful results exists. Entanglement measures for bipartite [5] and multipartite [6] pure states have been defined and used in a wide variety of interesting physical applications. However, the challenging problem of quantifying also mixed-state entanglement is still at its infancy. Although the development in this field is apparent, apart from systems [7–9] of two qubits and a qubit and a qutrit, no *simple* sufficient and necessary conditions are known for deciding whether a state is entangled or not.

In accordance with the ideas of the founding fathers of quantum mechanics [4] the notion of pure-state entanglement in a quantum system consisting of two subsystems has to be related to the impossibility of assigning a complete set of properties to both constituents. Alternatively, we can define a composite system to be *nonentangled* if and only if both constituents possess a complete set of properties. After giving a precise mathematical meaning to this definition, in [10] it has been shown that for distinguishable particles it is equivalent to the separability of the wave function representing the composite system in Hilbert space.

This physically motivated definition can also be used to clarify the meaning of entanglement for systems with indistinguishable (identical) constituents [10,11]. The crucial idea in this case is that for nonentangled states we can attribute to *at least* one of the particles (it is meaningless to ask which one) the complete set of properties associated with the considered set of observables. For fermions in terms of representative state vectors in a Hilbert space this definition was shown to be mathematically equivalent to the possibility of writing the state vector as a *single* Slater determinant—i.e., the antisymmetrization of a product state. This approach established a nice connection with earlier results of Schliemann *et al.* [12] who characterized and classified quantum correlations in two-fermion systems having $2K$ single-particle states. For pure states they introduced in analogy to the Schmidt decomposition, a decomposition in terms of Slater determinants. States with Slater rank (i.e., the number of Slater determinants occurring in the canonical form) greater than 1 are called entangled. Slater rank-1 states are expressible in terms of a single Slater determinant; hence, according to the definition above they are nonentangled. For two fermions having four single-particle states ($K=2$) a measure $0 \leqslant \eta \leqslant 1$ was introduced [12]. It was shown that a state is Slater rank 1 (nonentangled) if and only if $\eta=0$. Slater rank-2 states with maximal entanglement correspond to $\eta=1$. The quantity η in many respects behaves similarly to the well-known concurrence [13] $0 \leqslant \mathcal{C} \leqslant 1$ quantifying two-qubit entanglement for distinguishable particles. In a special case they can in fact be related [14]. A sufficient and necessary condition for a state having an arbitrary value for K to be nonentangled was established later [15].

However, some problems arise when we calculate the reduced (single-particle) density matrix. Regarding the von Neumann entropy S as a good correlation measure for fermions [16] raises the following puzzling issue. S attains its minimum value $S_{\min}=1$ corresponding to Slater rank 1—i.e., nonentangled states. This situation is to be contrasted with the case familiar for two distinguishable particles where for nonentangled Schmidt rank-1 states one has $S_{\min}=0$. However, as was shown in [11] this contradiction arises from the fact that we have not taken correctly into account the physical meaning of the von Neumann entropy. Indeed, the minimum value $S_{\min}=1$ of the von Neumann entropy reflects our unavoidable ignorance concerning the identity of the fermions. In this respect this nonzero value indicates the presence of correlations due to the exchange properties of the indistinguishable fermions. Since these correlations cannot be used to implement a teleportation process or to violate Bell's inequality, they cannot be regarded as manifestations of entanglement.

The aim of the present paper is to study these issues by *explicitly* working out the example of two correlated fermions having four single-particle states. Motivated by geometric considerations after employing a special representation for the complex amplitudes of our fermionic wave function we explicitly calculate the reduced density matrix and show

that the von Neumann and Rényi entropies can be expressed in terms of the measure η via a simple formula. Our elementary formula is entirely analogous to the one known for distinguishable particles using the concurrence $\mathcal{C}$. Moreover, our construction yields the canonical form (Slater decomposition) explicitly. Unlike, however, the canonical form of [12] where the expansion coefficients are *complex*, in our form they are *non-negative real* numbers. This decomposition with real non-negative expansion coefficents is similar to the Schmidt decomposition well known for distinguishable particles. Next by calculating the von Neumann and Rényi entropies it is also shown that the residual entropy $S_{\min}=1$ reflecting the exchange properties of the fermions can be reinterpreted as a manifestation of the generalized Pauli exclusion principle. As an advantage of having real non-negative numbers in the Slater decomposition we show that these numbers can be related to lengths of geodesic segments from the Slater states to the manifold of nonentangled states. Finally it is shown that the residual entropy can be given a nice geometric interpretation in terms of the nonseparability of a quadric surface representing the manifold of nonentangled fermionic states.

We remark in closing that in quantum information science it is usually presumed that really indistinguishable qubits can be addressed. In this respect at first sight it is not at all clear how fermionic entanglement can be used as a resource. We also stress here that the well-known correlations between fermions being far apart from each other are not the ones we are having in mind for such an application. We would rather conform with the ideas of [12,15,17] to use fermionic correlations at *short distances*. These correlations can indeed be regarded as a resource via the use of quantum dots—for example, to implement the $\sqrt{\mathrm{SWAP}}$ operation.

The organization of this paper is as follows. In Sec. II, using a convenient representation the structure of the density matrix is elucidated and the canonical form with real expansion coefficients is achieved. In Sec. III the von Neumann and Rényi entropies are calculated and the limiting cases are discussed. Here the connection with the generalized Pauli exclusion principle is established. In Sec. IV the interesting geometry of fermionic entanglement is elucidated. Some comments and the conclusions are left for Sec. V.

II. DENSITY MATRIX

As a starting point let us assume that the Hilbert space $\mathcal{H}$ describing the quantum correlations of two fermionic systems with four single-particle states is of the form $\mathcal{H}\equiv\mathcal{A}(\mathbb{C}^4\otimes\mathbb{C}^4)$ where $\mathcal{A}$ refers to antisymmetrization. An arbitrary element $|\Psi\rangle$ of $\mathcal{H}$ has the form

$$|\Psi\rangle \equiv \sum_{\mu,\nu=0}^{3} P_{\mu\nu} c_\mu^\dagger c_\nu^\dagger |0\rangle \in \mathcal{H}, \tag{1}$$

where $c_\mu^\dagger$ and c_μ, $\mu=0,1,2,3$, are fermionic creation and annihilation operators satisfying the usual anticommutation relations

$$\{c_\mu, c_\nu^\dagger\} = \delta_{\mu\nu}, \quad \{c_\mu, c_\nu\} = 0, \quad \{c_\mu^\dagger, c_\nu^\dagger\} = 0, \tag{2}$$

and $|0\rangle$ is the fermionic vacuum. Due to anticommutation, the 4×4 matrix P with *complex* elements is an antisymmetric one; i.e., we have $P^T=-P$. Using these relations it can be shown that the normalization condition $\langle\Psi|\Psi\rangle=1$ implies

$$2\,\mathrm{Tr}\,PP^\dagger = 1. \tag{3}$$

It will be instructive in the following to stress the similarity with the structure of invariants characterizing fermionic entanglement and the ones arising from electrodynamics. Hence we parametrize our matrix P as

$$P_{\mu\nu} \equiv \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{4}$$

i.e., $P_{0j}=E_j$, $P_{jk}=-\epsilon_{jkl}B_l$, $j,k,l=1,2,3$. It is important to emphasize at this point that unlike in electrodynamics, here $\mathbf{E}$ and $\mathbf{B}$ are merely complex three-vectors—i.e., $\mathbf{E}, \mathbf{B}\in\mathbb{C}^3$.

As was demonstrated in [12] local unitary transformations $U\otimes U$ with $U\in\mathrm{U}(4)$ acting on $\mathbb{C}^4\otimes\mathbb{C}^4$ do not change the fermionic correlations we are intending to study. Under such transformations P transforms as

$$P \mapsto UPU^T. \tag{5}$$

A representation convenient for our purposes can be obtained by choosing the unitary matrix $U\in\mathrm{U}$ (4) as

$$U_{\mu\nu} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{6}$$

where i is the imaginary unit. The geometric meaning of the representation (hereafter to be called the "U representation") arising from using this unitary transformation will be explained later. Straightforward calculation shows that the transformed matrix $P'\equiv UPU^T$ has the form

$$P' = UPU^T = \frac{1}{2}(\varepsilon\otimes\varepsilon)(I\otimes\mathbf{a}\boldsymbol{\sigma} + \mathbf{b}\bar{\boldsymbol{\sigma}}\otimes I), \tag{7}$$

where

$$\mathbf{a} = \mathbf{E} + i\mathbf{B}, \quad \mathbf{b} \equiv \mathbf{E} - i\mathbf{B}, \quad \varepsilon \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{8}$$

$\mathbf{a}\boldsymbol{\sigma}\equiv a_1\sigma_1+a_2\sigma_2+a_3\sigma_3$ with σ_j, $j=1,2,3$ the standard Pauli matrices, I is the 2×2 unit matrix, and the overbar denotes complex conjugation. Notice also that with this notation we have at our disposal the important relations

$$\varepsilon\boldsymbol{\sigma}\varepsilon = \bar{\boldsymbol{\sigma}}, \quad \varepsilon^2 = -I, \tag{9}$$

and according to the normalization condition (3),

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \frac{1}{2}, \quad \text{with } \|\mathbf{a}\|^2 \equiv \bar{\mathbf{a}}\mathbf{a}, \quad \|\mathbf{b}\|^2 \equiv \bar{\mathbf{b}}\mathbf{b}. \tag{10}$$

Since the fermions are indistinguishable, the reduced one-particle density matrices are equal and have the form [16]

$$\rho = 2PP^{\dagger}. \tag{11}$$

However, now we cannot pretend that any of the one-particle density matrices describes the properties of precisely the first or second particle of the system. ρ of Eq. (11) describes the properties of a randomly chosen particle that cannot be better identified [11].

A calculation using Eqs. (9) shows that

$$2U\rho U^{\dagger} = (I \otimes \mathbf{a}\bar{\boldsymbol{\sigma}} + \mathbf{b}\boldsymbol{\sigma} \otimes I)(I \otimes \overline{\mathbf{a}\boldsymbol{\sigma}} + \bar{\mathbf{b}}\boldsymbol{\sigma} \otimes I). \tag{12}$$

Using the relation $(\mathbf{u}\boldsymbol{\sigma})(\mathbf{v}\boldsymbol{\sigma}) = (\mathbf{u}\mathbf{v})I + i(\mathbf{u}\times\mathbf{v})\boldsymbol{\sigma}$ and the normalization condition (10), we obtain the result

$$U\rho U^{\dagger} = \frac{1}{4}(\mathbf{1} + \Lambda), \tag{13}$$

where

$$\Lambda = 2(I \otimes \mathbf{x}\bar{\boldsymbol{\sigma}} + \mathbf{y}\boldsymbol{\sigma} \otimes I + \mathbf{b}\boldsymbol{\sigma} \otimes \overline{\mathbf{a}\boldsymbol{\sigma}} + \bar{\mathbf{b}}\boldsymbol{\sigma} \otimes \mathbf{a}\bar{\boldsymbol{\sigma}}) \tag{14}$$

and

$$\mathbf{x} \equiv -i\mathbf{a} \times \bar{\mathbf{a}}, \quad \mathbf{y} \equiv i\mathbf{b} \times \bar{\mathbf{b}}, \quad \mathbf{1} \equiv I \otimes I. \tag{15}$$

Notice that the vectors $\mathbf{x}$ and $\mathbf{y}$ are *real* ones—i.e., elements of $\mathbb{R}^3$.

In order to obtain the eigenvalues of ρ we calculate Λ^2. The calculation is easily performed after noticing that due to the relations $\mathbf{x}\mathbf{a} = \mathbf{x}\bar{\mathbf{a}} = \mathbf{y}\mathbf{b} = \mathbf{y}\bar{\mathbf{b}} = 0$ the first two and the last two terms in Λ anticommute with each other. A straightforward calculation using the definitions (8), the normalization condition (10), and the relations $\|\mathbf{x}\|^2 = \|\mathbf{a}\|^4 - \mathbf{a}^2\bar{\mathbf{a}}^2$ and $\|\mathbf{y}\|^2 = \|\mathbf{b}\|^4 - \mathbf{b}^2\bar{\mathbf{b}}^2$ shows that

$$\Lambda^2 = (1 - 64|\mathbf{E}\mathbf{B}|^2)(I \otimes I) = (1 - \eta^2)\mathbf{1}. \tag{16}$$

The quantity

$$0 \leqslant \eta \equiv 8|P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12}| = 8|\mathbf{E}\mathbf{B}| \leqslant 1 \tag{17}$$

is the measure for fermionic correlations introduced in [12]. Since $\mathrm{Det}\, P = (\mathbf{E}\mathbf{B})^2$, we see that η is invariant under local unitary U(4) transformations of the form (5).

Now Eq. (16) implies that the eigenvalues of Λ are $\pm\sqrt{1-\eta^2}$, each of them doubly degenerate. Using this result in Eq. (13) we obtain for the eigenvalues of ρ

$$\lambda_{\pm} = \frac{1}{4}(1 \pm \sqrt{1 - \eta^2}), \tag{18}$$

each of them doubly degenerate.

In [12] a fermionic analog of the usual Schmidt decomposition for distinguishable particles was introduced. Adapted to our situation the theorem of [12] states that there exists a unitary matrix $\mathcal{U} \in \mathrm{U}(4)$ [not to be confused with our U of expression (6)] such that

$$Z = \mathcal{U}P\mathcal{U}^{T}, \quad \text{where } Z = \begin{pmatrix} 0 & z_1 & 0 & 0 \\ -z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2 \\ 0 & 0 & -z_2 & 0 \end{pmatrix}, \tag{19}$$

where z_1 and z_2 are *complex* numbers. When one of the complex numbers z_i $i=1, 2$ is zero we have *Slater rank 1*, for both z_i being nonzero *Slater rank-2* states. According to [12], a fermionic state is called *entangled* if and only if its Slater number is strictly greater than 1.

However, according to a theorem of Zumino [18] even more can be said.

Theorem. If P is a complex $2K \times 2K$ skew symmetric matrix, then there exists a unitary transformation $\mathcal{V} \in U(2K)$ such that

$$R = \mathcal{V}P\mathcal{V}^{T}, \quad \text{where } R = \mathrm{diag}[R_1, R_2, \ldots, R_K], \tag{20}$$

with

$$R_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix}, \tag{21}$$

where r_i, $i = 1, 2, \ldots, K$, are *non-negative real* numbers. Notice that unlike the decomposition of [12] in terms of *complex* expansion coefficients the one presented in Eq. (20) is similar to the usual Schmidt decomposition where the expansion coefficents are *non-negative real numbers*. We will show in Sec. IV that these non-negative real numbers can be related to the geodesic distance between certain states with respect to a suitably chosen Riemannian metric on the space of fermionic states.

Now let us use the theorem above to see that the canonical form of our P in Eq. (1) is given by Eq. (20) with $K=2$ and

$$r_1 = \sqrt{\frac{\lambda_+}{2}}, \quad r_2 = \sqrt{\frac{\lambda_-}{2}}. \tag{22}$$

With this notation,

$$|\Psi\rangle = \sqrt{2\lambda_+}C_0^{\dagger}C_1^{\dagger}|0\rangle + \sqrt{2\lambda_-}C_2^{\dagger}C_3^{\dagger}|0\rangle, \tag{23}$$

where

$$c_{\mu}^{\dagger} = \sum_{\nu=0}^{3} \mathcal{V}_{\nu\mu}C_{\nu}^{\dagger}. \tag{24}$$

It is clear that for the calculation of the Slater states (the analogs of the Schmidt states) appearing in Eq. (23) we have to determine the unitary $\mathcal{V}$ along the lines as presented in [18]. Notice that in our case this $\mathcal{V}$ as a function of the complex numbers $\mathbf{E}$ and $\mathbf{B}$ can be obtained explicitly. In order to give some hints notice that according to Eq. (16) the 4×4 matrices

$$\Pi_{\pm} \equiv \frac{1}{2}\left(\mathbf{1} \pm \frac{1}{r}\Lambda\right), \quad r \equiv \sqrt{1 - \eta^2} \tag{25}$$

are orthogonal projectors of rank 2; i.e., they satisfy $\Pi_{\pm}^2 = \Pi_{\pm}$ and $\Pi_{\pm}\Pi_{\mp} = 0$. Let us define the vectors

$$v_0 = N_0\Pi_+ e_0, \quad v_1 = N_1\Pi_+ e_1, \tag{26}$$

$$v_2 = N_2\Pi_- e_2, \quad v_3 = N_3\Pi_- e_3, \tag{27}$$

where e_{μ}, $\mu = 0,1,2,3$, are unit vectors corresponding to the columns of the unitary in Eq. (6) and N_{μ} are normalization constants. Then the v_{μ} are normalized eigenvectors of ρ.

With the help of these eigenvectors we can build up the unitary diagonalizing the density matrix with the dependence on **E** and **B** explicitly displayed, the first step needed for the determination of $\mathcal{V}$ [18].

III. ENTROPY

Having the eigenvalues and the canonical form at our disposal we can now write down the explicit form of entropies used in quantum information theory. These are the von Neumann and the quantum counterpart of Rényi's α ($\alpha=2,3,\ldots$) entropies [19] defined as

$$S_1 \equiv -\operatorname{Tr}\rho\log_2\rho, \quad S_\alpha \equiv \frac{1}{1-\alpha}\log_2 \operatorname{Tr}\rho^\alpha, \quad \alpha>1. \tag{28}$$

Notice that for convenience we have chosen 2 for the base of the logarithm, and the von Neumann entropy can be regarded as the α tends to 1 decreasing limit of S_α.

Using Eq. (18) we obtain the explicit formula

$$S_1(\eta) = 1 - x\log_2 x - (1-x)\log_2(1-x), \tag{29}$$

$$S_\alpha(\eta) = 1 + \frac{1}{1-\alpha}\log_2[x^\alpha + (1-x)^\alpha], \quad \alpha>1, \tag{30}$$

where

$$x \equiv \frac{1}{2}(1+\sqrt{1-\eta^2}), \tag{31}$$

with η given by Eq. (17). All of our entropies satisfy the inequalities

$$1 \leqslant S_\alpha(\eta) \leqslant 2, \quad \alpha \geqslant 1. \tag{32}$$

Any one of our entropies measures the lack of information about the single-particle subsystem. For indistinguishable particles, however, we must add another type of uncertainty to the list: namely, the one arising from the uncertainty concerning the state to be assigned to each of our two identical subsystems. Indeed one can only say that one of the particles is characterized by the state described by the Eq. (11) density matrix without identifying which. This is another way of saying that the nonzero value occurring on the left-hand-side inequality of Eq. (32) is a manifestation of the exchange property's two-particle fermionic state $|\Psi\rangle$. This statement, however, is a special case of a more general result obtained from the so-called Pauli principle for density matrices, related to the N-representability problem of ρ. For N-particle fermionic systems the following question is of physical relevance. Given a one-particle density matrix ρ, does there exist an N-particle density matrix ρ_N with the usual properties and which is antisymmetric with respect to the exchange of particles, satisfying the relation $\rho=\mathrm{Tr}_{2,\ldots,N}\rho_N$? The operation $\mathrm{Tr}_{2,\ldots,N}$ is a partial trace of the particle indices $2,\ldots,N$. Clearly, a physical reduced density matrix should satisfy this requirement; in this case, it is called N representable. If, furthermore, $\rho_N=|\Psi\rangle\langle\Psi|$, ρ is called pure state N representable.

It is a result of Coleman [20] that a necessary and sufficient condition for N representability can be formulated using the eigenvalues $\lambda_0,\ldots,\lambda_{M-1}$ of ρ (here M stands for the dimension of the basis set of the one-particle Hilbert space). The reduced density operator ρ is N representable iff

$$0 \leqslant \lambda_\mu \leqslant 1/N \quad \text{for any } \mu=0,\ldots,M-1. \tag{33}$$

The above condition is known as the generalized Pauli principle in the literature and is obviously satisfied by the eigenvalues (18) with $N=2$, $M=2K=4$.

Considering now the entropy expressions (28) Jensen's inequality results in the standard relations

$$0 \leqslant S_1 \leqslant \log_2 M, \quad 0 \leqslant S_2 \leqslant \log_2 M. \tag{34}$$

Moreover, it can be shown [21] that

$$S_2 \leqslant S_1 \tag{35}$$

also holds. Applying (33)

$$-S_2 = \log_2 \sum_{\mu=0}^{M-1} \lambda_\mu^2 \leqslant \log_2 \sum_{\mu=0}^{M-1} \lambda_\mu \frac{1}{N} = -\log_2 N \tag{36}$$

and using (35) finally leads to

$$\log_2 N \leqslant S_1 \leqslant \log_2 M, \quad \log_2 N \leqslant S_2 \leqslant \log_2 M, \tag{37}$$

which is clearly a generalization of (32) for an arbitrary particle number N with $\alpha=1,2$.

Notice that $S_\alpha=1$ iff $\eta=0$. These are the states having Slater rank 1; i.e., $|\Psi\rangle$ in this case can be transformed via local unitaries $U\otimes U$ with $U\in \mathrm{U}(4)$ to a single Slater determinant. Mathematically this means that $P_{\mu\nu}$ of (1) is a *separable* bivector; i.e., there exist four-vectors u_μ and v_ν, μ, $\nu=0$, 1, 2, 3, such that $P_{\mu\nu}=u_\mu v_\nu - u_\nu v_\mu$.

In order to study in our formalism the $S_\alpha=2(\eta=1)$ case corresponding to Slater rank-2 states satisfying an additional requirement we introduce some terminology. Let us define the matrix

$$g \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{38}$$

Then a short calculation using Eq. (6) shows that

$$UgU^T = \varepsilon \otimes \varepsilon. \tag{39}$$

We use g to raise and lower indices in the usual way; hence, for example, we have $P^{\mu\nu} \equiv g^{\mu\kappa}g^{\nu\varrho}P_{\kappa\varrho}$. In short a quantity like gPg corresponds to P with both indices raised.

Now let us define the *dual* ${}^*P$ of P as

$${}^*P_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\kappa\varrho}P^{\kappa\varrho}. \tag{40}$$

Here $\epsilon_{\mu\nu\kappa\varrho}$ is the fourth-order totally antisymmetric tensor defined by the condition $\epsilon_{0123}=1$. Then we see that ${}^*\mathbf{E}=-\mathbf{B}$ and ${}^*\mathbf{B}=\mathbf{E}$. Moreover, for the U(4) invariant η of Eq. (17) occurring in our formulas for the entropy we have

$$\eta = 2|{}^{*}P_{\mu\nu}P^{\mu\nu}| \equiv 2|\mathrm{Tr}({}^{*}PgPg)|. \tag{41}$$

Comparing this with Eq. (3) we see that $\eta=1$ iff

$$P = e^{i\theta}g(\overline{{}^{*}P})g, \tag{42}$$

where $e^{i\theta}$ is an arbitrary complex phase factor. In terms of **E** and **B** this means that $\eta\equiv 1$ iff

$$\mathbf{E} = e^{i\theta}\bar{\mathbf{B}}. \tag{43}$$

Transforming this equation with the unitary (6) we get

$$P' = e^{i\theta}(\varepsilon\otimes\varepsilon)\overline{{}^{*}P'}(\varepsilon\otimes\varepsilon). \tag{44}$$

With an abuse of notation we can omit the prime and we can say that in the U representation of Eq. (7), $\eta=1$ if and only if

$${}^{*}P = e^{i\theta}\tilde{P}, \tag{45}$$

where we have introduced the spin-flip operation of Wootters [9] playing a crucial role in the definition of the entanglement of formation for two-qubit systems (recall that $i\sigma_2=\varepsilon$):

$$\tilde{P} \equiv \sigma_2\otimes\sigma_2\bar{P}\sigma_2\otimes\sigma_2. \tag{46}$$

Hence in the U representation for states with maximal fermionic entanglement their duals are equal to their spin-flipped transforms (up to a phase). This result has to be compared with a similar one obtained in [15]. Here we also uncovered the instructive connection of dualization and its connection with the spin-flip operation (i.e., time reversal) of quantum information theory. Notice also that in the original, Eq. (42), representation taking the spin-flip transform amounts to complex conjugation followed by raising both indices with the matrix g known from special relativity. The roots of this correspondence will be revealed in the next section.

IV. GEOMETRY OF FERMIONIC ENTANGLEMENT

In this section we clarify the geometric meaning of our U representation and the residual entropy $S_{\min}=1$. To begin with, notice that our U representation is a variant of the method of expressing quantities instead of the computational base in the so-called "magic base" of Hill and Wootters [13]. The use of this base has its roots in the group-theoretical correspondences $[\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})]/\mathbb{Z}_2\simeq\mathrm{SO}(4,\mathbb{C})$ and $[\mathrm{SU}(2)\times\mathrm{SU}(2)]/\mathbb{Z}_2\simeq\mathrm{SO}(4)$. These correspondences have been used succesfully for establishing exact results for the behavior of the entanglement of formation [9,22]. Here we would like to provide a different insight into the effectiveness of this base provided by geometry.

Let us consider the quantities (Infeld–van der Waerden symbols)

$$\sigma^{0}_{AB'} = \sigma_{0}^{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{47}$$

$$\sigma^{1}_{AB'} = \sigma_{1}^{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{48}$$

$$\sigma^{2}_{AB'} = -\sigma_{2}^{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \tag{49}$$

$$\sigma^{3}_{AB'} = \sigma_{3}^{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{50}$$

Here A, $B'=0$, 1 are the matrix (spinor) indices of the Pauli matrices. The quantities $\sigma_{\mu}^{AB'}$ and $\sigma^{\mu}_{AB'}$, $\mu=0$, 1, 2, 3, can be used to convert vector and spinor indices back and forth. For example, for a four-vector a_μ we can form the four-component spinorial object $a_{AB'}=\sigma^{\mu}_{AB'}a_\mu$ where summation over μ is understood. Writing out this relation explicitly we have

$$\begin{pmatrix}a_{00'}\\ a_{01'}\\ a_{10'}\\ a_{11'}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & -i & 0\\ 0 & 1 & i & 0\\ 1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\end{pmatrix}. \tag{51}$$

Comparing this with Eq. (6) we see that our use of the U representation amounts to reverting to the spinorial analog of our tensorial quantities. In particular the transformation of Eq. (5) in this formalism takes the form

$$P_{\mu\nu}\mapsto P_{AA'BB'} = \sigma^{\mu}_{AA'}\sigma^{\nu}_{BB'}P_{\mu\nu}. \tag{52}$$

Moreover, Eq. (39) becomes one of the basic identities of the spinorial formalism

$$\varepsilon_{AB}\varepsilon_{A'B'} = \sigma^{\mu}_{AA'}\sigma^{\nu}_{BB'}g_{\mu\nu}. \tag{53}$$

Our decomposition (7) in this picture corresponds to the well-known one in the spinor formalism of Penrose and Rindler [23]:

$$P_{AA'BB'} = \varepsilon_{AB}\psi_{A'B'} + \varphi_{AB}\varepsilon_{A'B'}. \tag{54}$$

Notice that the *symmetric spinors* ψ and φ correspond to $(1/2)\varepsilon(\mathbf{a}\boldsymbol{\sigma})$ and $(1/2)\varepsilon(\mathbf{b}\bar{\boldsymbol{\sigma}})$, respectively. It is straightforward to check that ${}^{*}\mathbf{a}=i\mathbf{a}$ and ${}^{*}\mathbf{b}=-i\mathbf{b}$. Tensors satisfying ${}^{*}P=\pm iP$ are called [23] *self-dual* and *anti-self-dual*, respectively. Hence our decomposition (7) is in terms of the self-dual and anti-self-dual parts of our matrix P. As is well known spinorial methods proved to be of basic importance for a Petrov type of classification of curvature tensors in general relativity [23]. It is interesting to note that these methods proved to be of relevance for the classification of three-qubit (and possibly n-qubit) entanglement, too [24]. The basic idea behind this approach to n-qubit entanglement is to convert spinorial indices reflecting transformation properties under the group of n-fold tensor products of $\mathrm{SL}(2,\mathbb{C})$ representing stochastic local operations and classical communication of the entangled parties to vectorial ones (or vice versa) and then use the techniques as developed in twistor theory [23].

Finally, let us discuss the geometric meaning of $S_{\min}=1$ characterizing *nonentangled* fermionic states. From Eq. (1) it is clear that an unnormalized fermionic state $|\Psi\rangle$ can be characterized by *six complex* numbers—i.e., elements of $\mathbb{C}^6$. However, the space of states of a quantum system is the space of rays $\mathcal{P}$, a space obtained by identifying states $|\Psi\rangle$ and $|\Phi\rangle$ if they are related by $|\Psi\rangle=c|\Phi\rangle$ where $0\neq c\in\mathbb{C}$. In our case $\mathcal{P}$ is the five-dimensional complex projective

space—i.e., $\mathcal{P}\simeq\mathbb{CP}^5$. Alternatively, one can consider the space of normalized states which is the 11-dimensional sphere $S^{11}\subset\mathbb{C}^6\simeq\mathbb{R}^{12}$. In this case $\mathbb{CP}^5$ can also be regarded as the space of equivalence classes of normalized states defined up to a complex phase. (S^{11} has 11 real dimensions, a complex phase of unit magnitude is the circle S^1 which has one real dimension, and $\mathbb{CP}^5$ has $11-1=10$ real dimensions.)

Let us now consider the constraint $\eta=0$ which is a sufficient and necessary condition for our fermionic system to be nonentangled and, according to Eqs. (29)–(31), also for having $S_{\min}=1$ for all of our entropies. This condition is

$$P_{01}P_{23}-P_{02}P_{13}+P_{03}P_{12}=0, \tag{55}$$

which is the Plücker relation among the six complex coordinates characterizing separable bivectors. As is well known, a bivector (an antisymmetric 4×4 matrix) is separable if and only if condition (55) holds [23].

Using the coordinates

$$(z_0,z_1,z_2,z_3,z_4,z_5)\equiv(a_1,a_2,a_3,ib_1,ib_2,ib_3), \tag{56}$$

where the components of **a** and **b** are related to the Plücker coordinates $P_{\mu\nu}$ by Eqs. (4) and (8), the Plücker relations can be written as

$$z_0^2+z_1^2+z_2^2+z_3^2+z_4^2+z_5^2=0. \tag{57}$$

This equation is homogeneous of degree 2 and defines a quadric surface $Q_4(\mathbb{C})$ (the so-called Klein quadric) in $\mathbb{CP}^5$ [25]. The Klein quadric is the eight-real- (four-complex-) dimensional manifold characterizing nonentangled fermionic states. $Q_4(\mathbb{C})$ is a submanifold of $\mathbb{CP}^5$. States of $\mathbb{CP}^5$ not lying on $Q_4(\mathbb{C})$ are exhibiting nontrivial fermionic correlations; hence, they are entangled.

In order to gain more insight into the nature of fermionic entanglement let us compare these results with the corresponding ones known for two distinguishable qubits. Two unnormalized qubits are characterized by *four complex* numbers; hence, the relevant space of rays in this case is the three-dimensional complex projective space $\mathbb{CP}^3$. Nonentangled states are the ones for which the concurrence $\mathcal{C}$ is zero. It can be shown [13] (again by using the magic base) that four complex coordinates w_0, w_1, w_2, and w_3 can be introduced such that for vanishing $\mathcal{C}$ they satisfy the relation

$$w_0^2+w_1^2+w_2^2+w_3^2=0. \tag{58}$$

This equation defines the four-real-dimensional quadric $Q_2(\mathbb{C})$ in $\mathbb{CP}^3$. Now as in the fermionic case nonentangled states are parametrized by the points of $Q_2(\mathbb{C})$ and entangled ones belong to its complement in $\mathbb{CP}^3$.

It was shown in [26] that for distinguishable qubits a measure of entanglement can be defined as follows. The space of rays $\mathbb{CP}^3$ can be equipped with the Fubini-Study metric [27], which is induced by the standard Hermitian scalar product on $\mathbb{C}^4$. Let us fix an entangled state off the (58) quadric. Then the measure of entanglement for this state is related to the length of the *shortest* arc of the geodesic with respect to the Fubini-Study metric, connecting the state in question with the (58) quadric. More precisely we have

$$\cos^2\frac{s}{2}=\frac{1}{2}(1+\sqrt{1-\mathcal{C}^2}), \tag{59}$$

where $0\leqslant\mathcal{C}\leqslant1$ is the concurrence and s is the geodesic distance. The separable states corresponding to the *two* points of intersection of this geodesic with $Q_2(\mathbb{C})$ are just the ones occurring in the Schmidt decomposition [28]. The proof of this theorem in [26] can be trivially generalized for an arbitrary quadric $Q_{n-1}(\mathbb{C})$ in $\mathbb{CP}^n$. In particular for $n=5$ we get the result

$$\cos^2\frac{s}{2}=\frac{1}{2}(1+\sqrt{1-\eta^2}). \tag{60}$$

Hence the measure of nontrivial fermionic correlations is related to the geodesic distance between the fermionic state in question and the Klein quadric of nonentangled states by Eq. (60). Notice also that the Slater decomposition of Eq. (23) can be reexpressed as

$$|\Psi\rangle=\cos\frac{s}{2}C_0^\dagger C_1^\dagger|0\rangle+\sin\frac{s}{2}C_2^\dagger C_3^\dagger|0\rangle, \tag{61}$$

which for variable s describes a family of entangled states lying on a *horizontal* [28] geodesic. The normalized separable Slater states $C_0^\dagger C_1^\dagger|0\rangle$ and $C_2^\dagger C_3^\dagger|0\rangle$ are on the quadric $Q_4(\mathbb{C})$. They can also be calculated by a Lagrange multiplier technique as in [26], giving a geometric meaning to the Slater decomposition of a fermionic state having four single-particle states.

Now the question arises, is there any basic difference between the quadrics $Q_2(\mathbb{C})$ and $Q_4(\mathbb{C})$ that can account for the different physical situations as reflected by the different minimum values of their respective entropies? The answer to this question is surprisingly yes. It is a theorem in differential geometry that the quadrics $Q_{n-1}(\mathbb{C})$ in $\mathbb{CP}^n$ parametrized by homogeneous coordinates Z_0, $Z_1,\ldots,Z_n$ satisfying the additional constraint $\Sigma_{j=0}^n Z_j^2=0$ are symmetric spaces that can be represented in the form [27]

$$Q_{n-1}(\mathbb{C})\simeq \mathrm{SO}(n+1)/\mathrm{SO}(2)\times\mathrm{SO}(n-1). \tag{62}$$

For the very special value $n=3$, $\mathrm{SO}(4)\sim\mathrm{SU}(2)\times\mathrm{SU}(2)$; i.e., this group exhibits a product structure. Since $\mathrm{SO}(2)\simeq\mathrm{U}(1)$ and $\mathrm{SU}(2)/\mathrm{U}(1)\simeq S^2$, one can show that $Q_2(\mathbb{C})\simeq S^2\times S^2$—i.e., the direct product of two-spheres. These spheres are just the Bloch spheres corresponding to the distinguishable qubits in a separable state. The embedding of $Q_2(\mathbb{C})$ in the form $S^2\times S^2\hookrightarrow\mathbb{CP}^3$ is a special case of the so-called Segré embedding, having already been used in geometric descriptions of separability for distinguishable particles [29].

For $n\geqslant4$ the corresponding symmetric spaces are irreducible [27]; hence, they cannot be represented in a product form of two manifolds. Hence we can conclude that the manifold of nonentangled states representing quantum systems of distinguishable or indistinguishable constituents exhibits different topological structure. For nonentangled distinguishable particles ($S_{\min}=0$) we have a product structure of the state space $Q_2(\mathbb{C})$ which conforms with our expectations coming from classical mechanics. However, for nonentangled indistinguishable fermions ($S_{\min}=1$) no product

structure of the state space $Q_4(\mathbb{C})$ can be observed due to correlations reflecting the exchange properties of the fermions. These correlations are intrinsically quantum in nature. However, they are not to be confused with the correlations that can be regarded as true manifestations of entanglement. Representative states of this kind belonging to the complement of $Q_4(\mathbb{C})$ can be used to implement quantum information processing tasks.

V. CONCLUSIONS

In this paper we studied the nature of quantum correlations for fermionic systems having four single-particle states. Though these are the simplest systems among the fermionic ones exhibiting such correlations, they clearly show some of the basic differences between entanglement properties of quantum systems with distinguishable and indistinguishable constituents.

Following Ref. [11] we connected the notion of entanglement for quantum systems composed of two identical components with four single-particle states to the impossibility of assigning a complete set of properties to the subsystems. This definition implies precise constraints on the mathematical form of the state vector, state space, single-particle density matrices, and entropies representing the entanglement properties of the composite system. In order to study these constraints we employed a comfortable, so-called U representation for the 4×4 antisymmetric matrix P containing the six complex amplitudes representing our fermionic system. This representation enabled an explicit construction of the reduced density matrix and its eigenvalues and eigenstates. We have shown that these eigenvalues can be expressed in terms of the invariant η of [12] via an elementary formula analogous to the one well known for distinguishable qubits. In this way we managed to represent our entangled state in a canonical form (the so-called Slater decomposition) with *real non-negative* expansion coefficents. Using this decomposition a geometric interpretation of the Slater coefficents in terms of lengths of suitably defined geodesic segments was given.

Using these results we have computed the von Neumann and Rényi entropies that can also be used to characterize fermionic correlations. These entropies satisfy the bound $1 \leqslant S_\alpha(\eta)$, $\alpha=1, 2,\ldots$. This inequality is to be contrasted with the corresponding one $0\leqslant S_\alpha(\mathcal{C})$ known for distinguishable qubits, where $\mathcal{C}$ is the concurrence. We have shown that the difference in the bounds can be traced back to the fact that the so-called Pauli principle for density matrices has to hold. We related the special values for the entropies satisfying the lower or upper bounds to the algebraic properties of the matrix P.

We have also clarified the geometric meaning of the U representation. An interesting and useful connection with the spinor formalism of Penrose and Rindler hitherto used merely within the rather exotic realm of twistor theory was pointed out. Next we initiated a study of quadrics $Q_{n-1}(\mathbb{C})$ embedded in the space of rays $\mathbb{CP}^n$ for revealing the geometric aspects of entanglement. The cases $n=3$ and $n=5$ correspond to the ones of entanglement for systems with distinguishable and indistinguishable fermionic constituents. We have proved that the different physical situations showing up as the occurrence of different minimum values for the entanglement entropies are also reflected in the different topological properties of the quadrics $Q_2(\mathbb{C})$ and $Q_4(\mathbb{C})$. $Q_2(\mathbb{C})$ exhibits a product structure $S^2\times S^2$ of two Bloch spheres which conforms with our expectations coming from classical physics. However, for $Q_4(\mathbb{C})$ as the manifold of nonentangled states no product representation is available. In this case no classical picture giving rise to the possibility of assigning particle labels to some submanifold of $Q_4(\mathbb{C})$ is arising. The lack of information in identifying such submanifolds is reflected in the nonzero value of $S_{\min}$. Of course, in spite of this restriction purely of quantum nature, we are still capable of attributing a complete set of properties to at least one of the particles no matter that we do not know which is which. Hence nonentangled states represented by the points of $Q_4(\mathbb{C})$ give rise merely to correlations that cannot be used to implement a teleportation process or to violate Bell's inequality. On the other hand, the main concern of quantum information science could be the use of states off the Klein quadric. These are the fermionic states that can be used to implement quantum information processing tasks using, for example, quantum dot techniques as described in [15,17].

We would like to remark in closing that as far as we know until now fermionic entanglement has merely been used as a resource for generating entanglement for *distinguishable* particles (see, e.g., the quantum dot techniques of [15,17]). It would be interesting to find physically interesting situations where fermionic entanglement can be used directly. In this respect we mention the rapidly evolving research field of entanglement and quantum phase transitions. Here, where the connection between quantum phase transitions and entanglement can be nontrivial (see [30] and references therein), the use of alternative fermionic measures in understanding the underlying physics might be crucial.

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