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Thermal corrections in the Casimir interaction between a metal and dielectric

B. Geyer,* G. L. Klimchitskaya,† and V. M. Mostepanenko‡
Center of Theoretical Studies and Institute for Theoretical Physics, Leipzig University, Augustusplatz 10/11, D-04109 Leipzig, Germany

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The Casimir interaction between two thick parallel plates, one made of metal and the other of a dielectric, is investigated at nonzero temperature. It is shown that in some temperature intervals the Casimir pressure and the free energy of a fluctuating field are nonmonotonic functions of temperature and the corresponding Casimir entropy can be negative. A physical interpretation for these conclusions is given. At the same time we demonstrate that the entropy vanishes when the temperature goes to zero, i.e., in the Casimir interaction between a metal and a dielectric the Nernst heat theorem is satisfied. The investigation is performed both analytically, by using the model of an ideal metal and dilute dielectric or dielectric with a frequency-independent dielectric permittivity, and numerically for a real metal (Au) and dielectrics with different behaviors of the dielectric permittivity along the imaginary frequency axis (Si and α-Al_2O_3).

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I. INTRODUCTION

Recently, there has been a renewed interest in the investigation of the dispersion forces that originate from the existence of zero-point electromagnetic oscillations. The best-known physical phenomena of this kind are the van der Waals and Casimir forces (see, e.g., Refs. [1–4]). Both forces are of the same nature and are related to the nonrelativistic and relativistic situations, respectively. In the last few years many measurements of the Casimir force between metals were performed [5–14]. The increasing interest in experimental research on dispersion forces is motivated by their role in nanotechnology [15–17] and for constraining predictions of modern theories of fundamental interactions [13,14,18–20] (see also Ref. [21] covering both the experimental and theoretical developments of the subject). Quite recently both the van der Waals and Casimir forces were shown to be important [22–24] in experiments on quantum reflection and Bose-Einstein condensation.

The theoretical description of the Casimir force between real metals at nonzero temperature is problematic. The direct application [25–28] of the Lifshitz formula [29] in combination with the dielectric function of the Drude model leads [30] to a violation of Nernst's heat theorem in the case of perfect crystals, i.e., to a nonzero value of the Casimir entropy at zero temperature depending on the separation distance between the plates. As was shown in Refs. [28,31], for crystals with defects or impurities Nernst's heat theorem is satisfied, so that the entropy of a fluctuating field is equal to zero at zero temperature. This, however, does not solve the problem since a perfect crystal is a truly equilibrium system with a nondegenerate dynamic state of lowest energy. According to quantum statistical physics, the Nernst heat theorem is valid for any system possessing this property. It follows that any formalism applied to a perfect crystal must satisfy the Nernst heat theorem, whereas the approach of Refs. [25–28] does not. It is notable that this approach predicts *large* thermal corrections to the Casimir force at short separations as compared with the case of ideal metals.

Other theoretical descriptions of the thermal Casimir force between real metals are based on the use of the Lifshitz formula combined with the plasma model dielectric function [32–34] or with the Leontovich surface impedance [30,35]. In the framework of these approaches the Nernst heat theorem is satisfied for both perfect lattices as well as lattices with some small concentration of impurities. They predict *small* thermal corrections to the Casimir force at short separations in qualitative agreement with the case of ideal metals.

Recent experiments reveal the possibility of testing the validity of different theoretical approaches to the thermal Casimir force. Thus, the first modern measurement of the Casimir force by means of a torsion pendulum [5] was found [36,37] to be in disagreement with Refs. [25–28] and consistent with Refs. [30,32–35]. The most precise and accurate modern experiment by means of a micromechanical torsional oscillator [13,14] excludes the theoretical approach of Refs. [25–28] at 99% confidence and is consistent with the theoretical approaches of Refs. [30,32–35]. Notice that these conclusions concerning the comparison between theory and experiment are not universally accepted (see the discussion on this subject in Refs. [38,39]).

A distinguishing feature of the theoretical approach of Refs. [25–28] is the nonmonotonic dependence of the Casimir pressure and the free energy of a fluctuating field on the temperature. There are some temperature intervals where the magnitudes of the Casimir pressure and the free energy decrease with the increase of temperature. As a result, there are temperature intervals where the entropy of the fluctuating field takes negative values. The same is valid for hypothetical (nonexistent) dielectrics with a frequency-independent dielectric permittivity $\varepsilon=100$ or higher [27]. By contrast, in the approaches of Refs. [30,32–35] for metals and also for some real dielectrics considered in Ref. [30], the magnitudes

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*Email address: geyer@itp.uni-leipzig.de

†On leave from North-West Technical University, St. Petersburg, Russia. Email address: Galina.Klimchitskaya@itp.uni-leipzig.de

‡On leave from Noncommercial Partnership "Scientific Instruments," Moscow, Russia.
Email address: Vladimir.Mostepanenko@itp.uni-leipzig.de

of the Casimir free energy and pressure increase with temperature. There is, however, no general proof that the magnitudes of the Casimir free energy and pressure must increase with temperature for all real materials.

As can be seen from Ref. [22], the magnitudes of the Casimir-Polder free energy and force for the case of an atom interacting with a metal wall may decrease within some temperature region. The corresponding entropy of the fluctuating field becomes negative. When it is taken into account that the Lifshitz formula for an atom near a metal wall is obtained [22,40] from the case of a rarefied dielectric wall spaced parallel to a metal wall, it may be supposed that the primary reasons for the nonmonotonic behavior of the free energy are inherent in the Casimir interaction between the metal and dielectric.

In this paper we investigate the Casimir pressure, free energy, and entropy in a configuration of two thick parallel plates, one being metallic and the other dielectric. Previously, only two metallic or two dielectric plates were considered in the retarded regime (see also Refs. [41,42] where a so-called "unusual pair of plates" was treated—one of which is made of an ideal metal and the other is infinitely permeable). In the nonretarded limit of the van der Waals force the Hamaker constants for the configuration of a metallic and a dielectric plate were calculated in Ref. [43]. On the basis of the Lifshitz formula we find the analytic expressions for the Casimir pressure, free energy, and entropy in the configuration of a plate made of ideal metal and a parallel plate made of dilute dielectric separated by a gap of thickness a. It is shown that there are intervals on the temperature axis where the magnitude of the Casimir pressure decreases with the increase of temperature, whereas the magnitude of the free energy increases with the temperature and the Casimir entropy is positive. The asymptotic expressions of the obtained exact formulas at both low and high temperatures are presented and the validity of the Nernst heat theorem is demonstrated. The possibility of a decreasing Casimir energy and of negative values of the entropy of a fluctuating field within some temperature intervals is demonstrated for the cases of ideal and real (Au) metals near dielectrics with arbitrary constant ε (not dilute) and real dielectrics with the frequency-dependent dielectric permittivities (Si, α-$\mathrm{Al_2O_3}$). The physical interpretation of that negativeness of entropy is discussed. From this we give a positive answer to the above question of whether the entropy of a fluctuating field in the Casimir regime can be negative. This is important not only for the theory of the thermal Casimir force but also for experimental investigations of the Casimir interaction between metal and semiconductor [44].

The paper is organized as follows. In Sec. II the main notation is introduced and the Casimir free energy, entropy, and pressure are found analytically in the framework of the exactly solvable model (one plate is made of ideal metal and the other of dilute dielectric). Section III contains the results of analytical and numerical computations performed for the case of an ideal-metal plate spaced parallel to the plate of dielectric with some frequency-independent ε (not dilute). The case of a Au plate and real dielectrics with frequency-dependent dielectric permittivities (Si, α-$\mathrm{Al_2O_3}$) is considered in Sec. IV. Section V contains our conclusions and discussion.

II. CASIMIR INTERACTION BETWEEN AN IDEAL METAL AND A DILUTE DIELECTRIC

We consider the configuration of two thick parallel plates, one made of a real metal and another of dielectric at a separation a and temperature T in thermal equilibrium. The Casimir free energy per unit area is given by the Lifshitz formula [29,40]. In terms of dimensionless variables it can be represented in the form

$$\mathcal{F}(a,T)=\frac{k_BT}{8\pi a^2}\sum_{l=0}^{\infty}\left(1-\frac{1}{2}\delta_{l0}\right)\times\int_{\zeta_l}^{\infty}y\,dy\{\ln[1-r_{\parallel}^{\mathrm{M}}(\zeta_l,y)r_{\parallel}^{\mathrm{D}}(\zeta_l,y)e^{-y}]+\ln[1-r_{\perp}^{\mathrm{M}}(\zeta_l,y)r_{\perp}^{\mathrm{D}}(\zeta_l,y)e^{-y}]\}. \tag{1}$$

Here, the dimensionless Matsubara frequencies are $\zeta_l=\xi_l/\xi_c$, where $\xi_l=2\pi k_BTl/\hbar$ are the usual Matsubara frequencies, and $\xi_c=c/(2a)$ is the characteristic frequency, k_B is the Boltzmann constant, and δ_{l0} is the Kronecker symbol. The reflection coefficients for two different polarizations of the electromagnetic field are expressed in terms of the dielectric permittivities $\varepsilon^{\mathrm{M,D}}(\omega)$ of a metal and a dielectric, respectively,

$$r_{\parallel}^{\mathrm{M,D}}(\zeta_l,y)=\frac{\varepsilon_l^{\mathrm{M,D}}y-\sqrt{y^2+\zeta_l^2(\varepsilon_l^{\mathrm{M,D}}-1)}}{\varepsilon_l^{\mathrm{M,D}}y+\sqrt{y^2+\zeta_l^2(\varepsilon_l^{\mathrm{M,D}}-1)}},$$

$$r_{\perp}^{\mathrm{M,D}}(\zeta_l,y)=\frac{\sqrt{y^2+\zeta_l^2(\varepsilon_l^{\mathrm{M,D}}-1)}-y}{\sqrt{y^2+\zeta_l^2(\varepsilon_l^{\mathrm{M,D}}-1)}+y}, \tag{2}$$

where $\varepsilon_l^{\mathrm{M,D}}=\varepsilon^{\mathrm{M,D}}(i\zeta_l\xi_c)$.

For later use we introduce the so-called effective temperature T_{eff} defined through $k_BT_{\mathrm{eff}}=\hbar\xi_c=\hbar c/(2a)$. In terms of the effective temperature the nondimensional Matsubara frequencies are expressed as $\zeta_l=\tau l$, where τ is a new parameter $\tau=2\pi T/T_{\mathrm{eff}}$.

For an ideal metal $r_{\parallel,\perp}^{\mathrm{M}}(\zeta_l,y)=1$ and Eq. (1) gives us the free energy of the Casimir interaction between an ideal metal and dielectric,

$$\mathcal{F}^{\mathrm{ID}}(a,T)=\frac{k_BT}{8\pi a^2}\sum_{l=0}^{\infty}\left(1-\frac{1}{2}\delta_{l0}\right)\int_{\tau l}^{\infty}y\,dy\times\left\{\ln[1-r_{\parallel}^{\mathrm{D}}(\zeta_l,y)e^{-y}]+\ln[1-r_{\perp}^{\mathrm{D}}(\zeta_l,y)e^{-y}]\right\}. \tag{3}$$

Let us assume that the dielectric is dilute, i.e., having dielectric permittivity $\varepsilon_l^{\mathrm{D}}=1+\eta$, where $\eta=\mathrm{const}\ll 1$. Expanding the reflection coefficients $r_{\parallel,\perp}^{\mathrm{D}}$ from Eq. (2) in powers of η, one obtains

$$\ln[1-r_{\parallel}^{\mathrm{D}}(\zeta_l,y)e^{-y}]=\eta\frac{e^{-y}}{4}\left(\frac{\tau^2l^2}{y^2}-2\right)-\eta^2\frac{e^{-y}}{32y^4}[(4+e^{-y})\tau^4l^4-4e^{-y}\tau^2l^2y^2+4(e^{-y}-2)y^4]+O(\eta^3),$$

$$\ln[1 - r_{\perp}^{\mathrm{D}}(\zeta_l, y)e^{-y}] = -\eta \frac{e^{-y}}{4}\frac{\tau^2 l^2}{y^2} - \eta^2 \frac{e^{-y}}{32y^4}(e^{-y} - 4)\tau^4 l^4 + O(\eta^3). \tag{4}$$

Substituting Eq. (4) in Eq. (3) and preserving only two powers in the small parameter η, we arrive at the following expression for the Casimir free energy:

$$\mathcal{F}^{\mathrm{ID}}(a,T) = \frac{\hbar c \tau}{32\pi^2 a^3}\sum_{l=0}^{\infty}\left(1 - \frac{1}{2}\delta_{l0}\right)\int_{\tau l}^{\infty} y\, dy \times \left(-\eta \frac{e^{-y}}{2} + \eta^2 \frac{e^{-y}}{16y^4} \times [4y^4 - e^{-y}(\tau^4 l^4 - 2\tau^2 l^2 y^2 + 2y^4)]\right). \tag{5}$$

Upon integrating and summing in Eq. (5), we obtain

$$\begin{aligned}\mathcal{F}^{\mathrm{ID}}(a,T) = -\frac{\hbar c}{32\pi^2 a^3}\Bigg[& \eta\tau \frac{e^{2\tau} + 2\tau e^{\tau} - 1}{4(e^{\tau} - 1)^2} \\ & - \eta^2 \frac{\tau}{16}\Bigg(\frac{2(e^{2\tau} + 2\tau e^{\tau} - 1)}{(e^{\tau} - 1)^2} - \frac{e^{4\tau} + 4\tau e^{2\tau} - 1}{4(e^{2\tau} - 1)^2} \\ & + \tau^2 e^{2\tau}\frac{1 - e^{4\tau} + 2\tau(1 + 4e^{2\tau} + e^{4\tau})}{2(e^{2\tau} - 1)^4} \\ & + 2\tau^4 \sum_{l=1}^{\infty} l^4\, \mathrm{Ei}(-2\tau l) - 2\tau^2 \sum_{l=1}^{\infty} l^2\, \mathrm{Ei}(-2\tau l)\Bigg)\Bigg],\end{aligned} \tag{6}$$

where Ei(z) is the integral exponent.

The entropy of the Casimir interaction per unit area of the ideal metal and dilute dielectric is found as minus the first derivative of Eq. (6) with respect to the temperature,

$$\begin{aligned}S^{\mathrm{ID}}(a,T) &= -\frac{\partial \mathcal{F}^{\mathrm{ID}}(a,T)}{\partial T} \\ &= \frac{k_B}{8\pi a^2}\Bigg[\frac{\eta}{2}\left(\frac{e^{\tau} + 1}{2(e^{\tau} - 1)} - \tau e^{\tau}\frac{1 + \tau - (1 - \tau)e^{\tau}}{(e^{\tau} - 1)^3}\right) \\ &\quad - \frac{\eta^2}{16}\Bigg(\frac{7e^{\tau} + 9}{4(e^{\tau} - 1)} - 4\tau e^{\tau}\frac{1 + \tau - (1 - \tau)e^{\tau}}{(e^{\tau} - 1)^3} \\ &\quad - \frac{2\tau e^{2\tau} + e^{2\tau} - 1}{2(e^{2\tau} - 1)^2} + \tau^2 e^{2\tau} \\ &\quad \times \frac{3(1 - e^{4\tau}) + 10\tau(1 + 4e^{2\tau} + e^{4\tau})}{2(e^{2\tau} - 1)^4} \\ &\quad - \sum_{l=1}^{\infty}(6\tau^2 l^2 - 10\tau^4 l^4)\mathrm{Ei}(-2\tau l)\Bigg)\Bigg].\end{aligned} \tag{7}$$

Equation (6) permits one to find an analytic expression for the Casimir pressure between the ideal metal and dilute dielectric,

$$\begin{aligned}P^{\mathrm{ID}}(a,T) &= -\frac{\partial \mathcal{F}^{\mathrm{ID}}(a,T)}{\partial a} \\ &= -\frac{\hbar c \tau}{32\pi^2 a^4}\Bigg[\frac{\eta}{2}\left(\frac{e^{2\tau} + 2\tau e^{\tau} - 1}{(e^{\tau} - 1)^2} + \frac{\tau^2 e^{\tau}(e^{\tau} + 1)}{(e^{\tau} - 1)^3}\right) \\ &\quad - \frac{\eta^2}{4}\Bigg(\frac{e^{2\tau} + 2\tau e^{\tau} - 1}{(e^{\tau} - 1)^2} + \frac{\tau^2 e^{\tau}(e^{\tau} + 1)}{(e^{\tau} - 1)^3} \\ &\quad - \frac{e^{4\tau} + 4\tau e^{2\tau} - 1}{8(e^{2\tau} - 1)^2} - \frac{\tau^3 e^{2\tau}(e^{4\tau} + 4e^{2\tau} + 1)}{2(e^{2\tau} - 1)^4} \\ &\quad - \tau^4 \sum_{l=1}^{\infty} l^4\, \mathrm{Ei}(-2\tau l)\Bigg)\Bigg].\end{aligned} \tag{8}$$

Note that Eqs. (6)–(8) are perturbative in the small parameter η but for each perturbation order the thermal effects are taken into account exactly. This gives the possibility to find the high-temperature behavior ($\tau \gg 1$) of the Casimir free energy, entropy, and pressure. Thus, from Eq. (6) in the limit $\tau \gg 1$ ($T \gg T_{\mathrm{eff}}$) for the free energy it follows that

$$\mathcal{F}^{\mathrm{ID}}(a,T) = -\frac{k_B T}{32\pi a^2}\eta\left(1 - \frac{7}{16}\eta\right). \tag{9}$$

In a similar manner from Eqs. (7) and (8) one finds the behavior of the Casimir entropy and pressure at high temperature,

$$S^{\mathrm{ID}}(a,T) = \frac{k_B}{32\pi a^2}\eta\left(1 - \frac{7}{16}\eta\right),$$

$$P^{\mathrm{ID}}(a,T) = -\frac{k_B T}{16\pi a^3}\eta\left(1 - \frac{7}{16}\eta\right). \tag{10}$$

Needless to say, the entropy and pressure from Eq. (10) can be obtained also as minus the derivatives of Eq. (9) with respect to temperature and separation distance, respectively.

Now let us consider the low-temperature behavior of the Casimir free energy, entropy, and pressure. For this purpose the direct application of Eqs. (6)–(8) is rather cumbersome. The same results can be obtained more simply by using the Abel-Plana formula (see, e.g., Refs. [3,21]),

$$\sum_{l=0}^{\infty}\left(1 - \frac{1}{2}\delta_{l0}\right)F(l) = \int_0^{\infty} F(t)dt + i\int_0^{\infty}\frac{dt}{e^{2\pi t} - 1} \times [F(it) - F(-it)], \tag{11}$$

where $F(z)$ is an analytic function on the right half plane. Before the application of Eq. (11), we return to Eq. (5) and perform the integration with respect to y (but not the summation over l). The result is

$$\mathcal{F}^{\mathrm{ID}}(a,T) = -\frac{\hbar c \tau}{32\pi^2 a^3}\sum_{l=0}^{\infty}\left(1 - \frac{1}{2}\delta_{l0}\right)F(l), \tag{12}$$

where

$$F(l)=\frac{\eta}{2}(1+\tau l)e^{-\tau l}-\frac{\eta^2}{16}\Bigg(4(1+\tau l)e^{-\tau l} -\frac{1}{2}(1+2\tau l+\tau^2 l^2-2\tau^3 l^3)e^{-2\tau l} -2\tau^2 l^2(1-\tau^2 l^2)\mathrm{Ei}(-2\tau l)\Bigg). \tag{13}$$

The sum in Eq. (12) can be calculated using the Abel-Plana formula (11). In doing so, we note that the first term on the right-hand side of Eq. (11), where $F(t)$ is given by Eq. (13) with a substitution of t for l, is proportional to the Casimir energy at zero temperature, whereas the second gives the thermal correction. Direct integration results in

$$\int_0^\infty F(t)dt=\frac{\eta}{\tau}\left(1-\frac{457}{960}\eta\right). \tag{14}$$

The second integral on the right-hand side of Eq. (11) is calculated as a perturbative expansion in powers of a small parameter τ with the aim of obtaining the low-temperature asymptotics. Preserving all terms up to τ^3 inclusive, we obtain from Eq. (13)

$$F(it)-F(-it)=-\frac{i\eta}{3}[\tau^3t^3+O(\tau^5)]+\frac{i\eta^2}{8}\times[-\pi\tau^2t^2+6\tau^3t^3+O(\tau^4)]. \tag{15}$$

Using this equation, the integral can be calculated:

$$i\int_0^\infty \frac{dt}{e^{2\pi t}-1}[F(it)-F(-it)]=\frac{\eta\tau^3}{720}-\frac{\eta^2\tau^2}{32}\left(\frac{\tau}{10}-\frac{\zeta(3)}{\pi^2}\right), \tag{16}$$

where $\zeta(z)$ is the Riemann zeta function.

Substituting Eqs. (11), (14), and (16) in Eq. (12), the low-temperature behavior of the free energy is obtained:

$$\mathcal{F}^{\mathrm{ID}}(a,T)=-\frac{\hbar c}{32\pi^2a^3}\eta\left[1+\frac{\tau^4}{720}-\frac{\eta}{32}\left(\frac{457}{30}-\frac{\zeta(3)\tau^3}{\pi^2}+\frac{\tau^4}{10}\right)\right] \tag{17}$$

[here the terms of order $O(\eta\tau^6)$ and $O(\eta^2\tau^5)$ are omitted].

The low-temperature asymptotics of the Casimir entropy and pressure can be obtained by the calculation of minus the derivatives of Eq. (12) (with respect to temperature and separation, respectively) and by subsequent application of the Abel-Plana formula with the resulting functions $F(l)$. The same asymptotic expressions are obtained also by direct differentiation of Eq. (17). They are as follows:

$$S^{\mathrm{ID}}(a,T)=\frac{k_B}{32\pi a^2}\eta\tau^2\left[\frac{\tau}{45}+\frac{\eta}{4}\left(\frac{3\zeta(3)}{2\pi^2}-\frac{\tau}{5}\right)\right],$$

$$P^{\mathrm{ID}}(a,T)=-\frac{\hbar c}{32\pi^2a^4}\eta\left(3-\frac{\tau^4}{720}-\frac{\eta}{320}(457-\tau^4)\right). \tag{18}$$

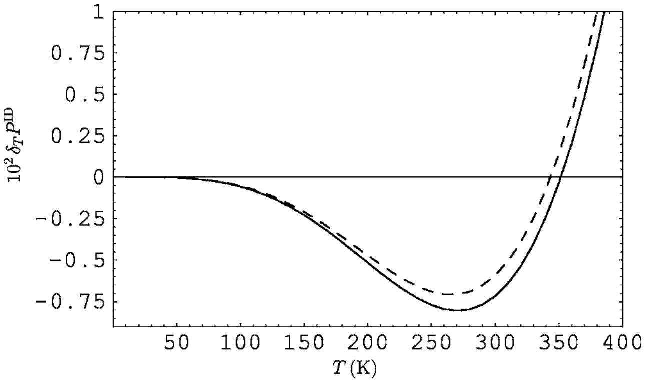

FIG. 1. Relative thermal correction to the Casimir pressure as a function of temperature at a separation $a=2$ μm for two plates, one made of an ideal metal and the other of a dilute dielectric with $\varepsilon_0^{\mathrm{D}}=1.001$ (solid line) and 1.1 (dashed line).

It is significant that $S^{\mathrm{ID}}\to 0$ when $T\to 0$ (recall that $\tau\sim T$ by definition), so that the Nernst heat theorem is satisfied. For dilute dielectrics from Eqs. (7) and (18) it follows also that $S^{\mathrm{ID}}\geqslant 0$. At the same time, there are temperature intervals where the Casimir pressure is nonmonotonic. To demonstrate this, in Fig. 1 the relative thermal correction to the Casimir pressure, defined as

$$\delta_T P=\frac{P(a,T)-P(a,0)}{P(a,0)}, \tag{19}$$

is plotted as a function of temperature for $\eta=0.001$ (solid line) and 0.1 (dashed line) at a separation distance $a=2\mu$m [here we use Eq. (8) and $P(a,T)=P^{\mathrm{ID}}(a,T)$]. As is seen from Fig. 1, the region where the thermal correction is negative is in accordance with the results of Ref. [22] for the Casimir-Polder energy and force between an atom and a metal wall.

From Fig. 1 it is seen that for $\eta=0.1$ it holds that $\delta_T P^{\mathrm{ID}}<0$ at temperatures $T\leqslant 343$ K. In this case the relative thermal correction has the minimum value $\delta_T P^{\mathrm{ID}}\approx -0.007$ at $T=270$ K. For comparison, at $T=400$ K the thermal correction has the positive value $\delta_T P^{\mathrm{ID}}\approx 0.018$. Note that there is some analogy between the behavior of the Casimir pressure and thermal correction as a function of temperature, obtained here on the one hand, and discussed in Refs. [25–28] on the other hand, in the case of real metals described by the Drude dielectric function. We will return to this analogy in more detail in Sec. V after the extension of the above results for more realistic plate materials which will be obtained in the following sections.

III. THERMAL CORRECTIONS TO THE CASIMIR INTERACTION BETWEEN AN IDEAL METAL AND A DIELECTRIC WITH FREQUENCY-INDEPENDENT DIELECTRIC PERMITTIVITY

In this section we consider one plate made of dielectric with arbitrary constant ε^{D} (not dilute) and the other plate made of ideal metal as before. The application regions of the asymptotic expressions at low and high temperatures, obtained in Sec. II, will be our initial concern. To gain an im-

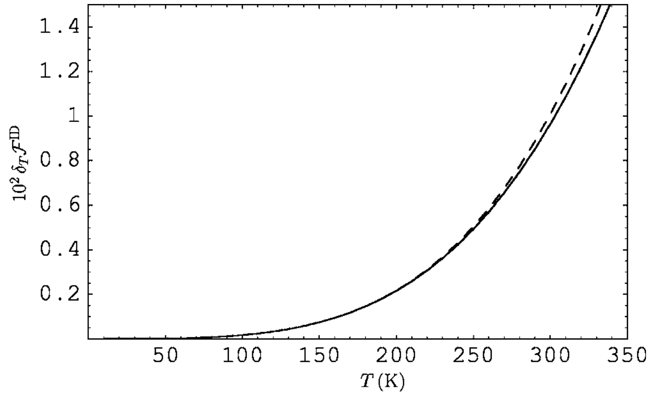

FIG. 2. Relative thermal correction to the Casimir energy at a separation $a=1$ μm as a function of temperature for two plates, one made of an ideal metal and the other of a dilute dielectric with $\varepsilon_0^{\rm D}=1.1$, computed numerically (solid line) and by the asymptotics of low temperatures (dashed line).

pression of how accurate the asymptotic of Eq. (17) is, let us calculate the relative thermal correction to the Casimir energy,

$$\delta_T\mathcal{F}=\frac{\mathcal{F}(a,T)-\mathcal{F}(a,0)}{\mathcal{F}(a,0)}, \tag{20}$$

as a function of temperature both numerically [by the direct use of the Lifshitz formula (3)] and analytically [using Eq. (17)]. In Fig. 2 the numerical results are presented by the solid line and the analytic ones by the dashed line for a dielectric with $\varepsilon_0^{\rm D}=1.1$ $(\eta=0.1)$ at a separation $a=1\,\mu$m. As is seen from this figure, at $T<220$ K there is practically exact coincidence of numerical and analytic results. What this means is that at $\tau\leqslant 1.2$ (the effective temperature here is equal to $T_{\rm eff}=1145$ K) one can use the asymptotic expression of Eq. (17) in order to get accurate results.

The asymptotic regime of high temperatures is given by Eq. (9). This regime is achieved at $T=1500$ K where the values of $\delta_T\mathcal{F}^{\rm ID}$, calculated by Eqs. (3) and (9), differ for only 0.2%. For higher temperatures $(\tau\geqslant 8.2)$ Eq. (9) is applicable for accurate calculations.

We now turn to the dielectrics with larger $\varepsilon_0^{\rm D}$. For the sake of convenience in the comparison of experiment and theory, we will present the results of the calculation as a function of separation rather than temperature. The point is that in all the previous experiments of Refs. [5–14] the temperature was constant ($T=300$ K) and the separation distance was variable (there is only one proposed but not yet completed experiment [45,46] exploiting the effect of two different temperatures). Theoretically, the two asymptotic regimes of low $(T\ll T_{\rm eff})$ and high $(T\gg T_{\rm eff})$ temperatures are equivalent to the limits of small and large separations, respectively. This becomes evident if it is recalled that $T_{\rm eff}\sim a^{-1}$. We consider the separation region $100\leqslant a\leqslant 1400$ nm where the nonmonotonic behavior of the thermal correction to the Casimir energy is observed for dielectrics with larger ε (not dilute). In fact, even in that separation region effects which are due to the nonideality of a metal and the absorption bands in a dielectric (see, e.g., Refs. [47,48]) become important. At shorter separations $(a<100$ nm) these effects lead to the result that the model of an ideal metal and a dielectric with constant dielectric permittivity becomes inapplicable (see Sec. IV where the case of a real metal and dielectric is considered).

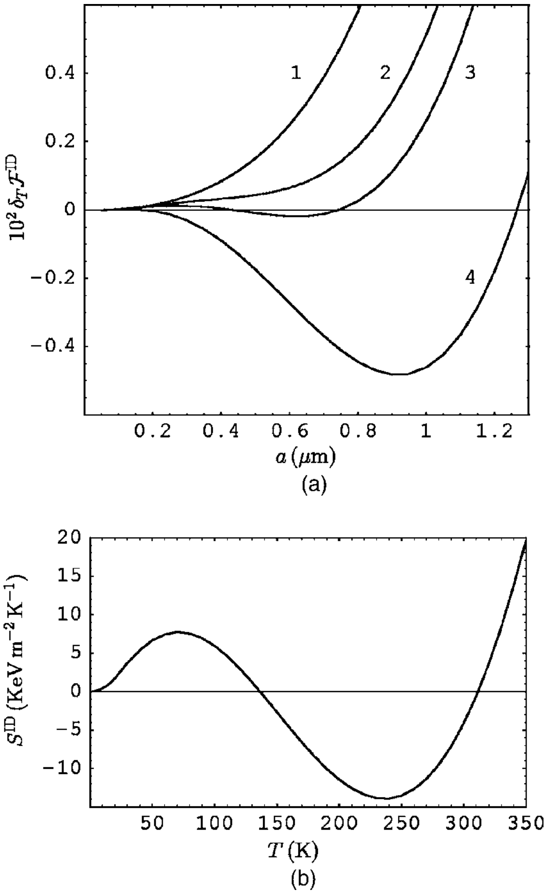

FIG. 3. (a) Relative thermal correction to the Casimir energy at $T=300$ K as a function of separation for two plates, one made of an ideal metal and the other of dielectrics with $\varepsilon_0^{\rm D}=3$, 6, 7, and 10 (lines 1, 2, 3, and 4, respectively); (b) Casimir entropy as a function of temperature for the dielectric plate with $\varepsilon_0^{\rm D}=7$ at a separation $a=600$ nm from a metal plate.

In Fig. 3(a) we present the values of the relative thermal correction (20) to the Casimir energy at $T=300$ K as a function of separation computed by the use of the Lifshitz formula (3) for ideal metal and dielectrics with different dielectric permittivities $\varepsilon_0^{\rm D}=3$, 6, 7, and 10 (lines 1, 2, 3, and 4, respectively). As is seen from Fig. 3(a), lines 3 and 4 demonstrate the nonmonotonic behavior of the thermal correction to the free energy (recall that in the case of an ideal metal and dilute dielectric only the Casimir pressure is a nonmonotonic function of the temperature; see Fig. 1). Curiously, line 3 possesses both a minimum and a maximum value within the separation interval under consideration.

For line 4 $(\varepsilon_0^{\rm D}=10)$ the zero value of the thermal correction is achieved at $a_1=0.2$ μm and at $a_2=1.25$ μm. Within the separation region $a_1<a<a_2$ the relative thermal correction is negative. Using the corresponding effective temperatures ($T_{\rm eff}^{(1)}=5724$ K at $a=a_1$ and $T_{\rm eff}^{(2)}=916$ K at $a=a_2$) we find that in terms of the dimensionless parameter τ, intro-

duced in Sec. II, the thermal correction is negative within the interval $0.33 \leqslant \tau \leqslant 2.06$. The minimum on line 4 is achieved at $a=0.9$ μm ($\tau=1.48$). What this means is that the entropy of a fluctuating field takes negative values within the interval $0.33 \leqslant \tau \leqslant 1.48$ (in the case of a plate made of dilute dielectric the entropy is positive). In terms of the usual temperatures for the plates at a separation $a=0.2$ μm, the Casimir entropy is negative within the interval $300 \leqslant T \leqslant 1350$ K (if, as assumed in this section, ε^{D} is independent of the temperature). Note that for the dielectric with $\varepsilon_0^{\mathrm{D}}=10$ the thermal correction is almost 0.5% at $a=0.9$ μm, i.e., much greater than in the case of a dilute dielectric. It remains, however, still too small to be observed experimentally.

To illustrate the behavior of the Casimir entropy as a function of temperature in the configuration of a plate made of ideal metal and another plate made of dielectric with a frequency-independent dielectric permittivity, we plot it in Fig. 3(b) for $\varepsilon_0^{\mathrm{D}}=7$ and separation distance $a=600$ nm. As is seen from Fig. 3(b), the entropy is negative within the interval from 137 to 311 K (i.e., from 0.45 to 1.02 in terms of τ). The minimum value of the entropy equal to -14 keV m^{-2} K^{-1} is achieved at a temperature $T=238$ K ($\tau=0.78$). When this result is compared with the above case of $\varepsilon_0^{\mathrm{D}}=10$, it is apparent that the region where the entropy is negative is narrowed with a decrease of the dielectric permittivity. Thus, for $\varepsilon_0^{\mathrm{D}}<6$ the Casimir entropy is already positive at any temperature.

The negativeness of the Casimir entropy within some temperature (separation) intervals should not become of concern. It is self-evident that the entropy of a closed system, which includes the space occupied by the dielectric plate, is positive (the second plate is made of an ideal metal with the Dirichlet boundary condition on it; for this reason it does not contribute to the entropy of the system under consideration). It should be noted also that the Nernst heat theorem is satisfied perfectly well. To prove this fact analytically, we apply the Abel-Plana formula (11) in Eq. (3) where the role of $F(l)$ is played by the integral in the right-hand side of Eq. (3). Preserving only the lowest expansion order in the parameter τl, we arrive at

$$F(l)=\tau^2 l^2 \frac{\varepsilon_0^{\mathrm{D}}-1}{\varepsilon_0^{\mathrm{D}}+1}\int_{\tau l}^{\infty} dy \times\left(\frac{\varepsilon_0^{\mathrm{D}}}{(\varepsilon_0^{\mathrm{D}}+1)-(\varepsilon_0^{\mathrm{D}}-1)e^{-y}}-\frac{\varepsilon_0^{\mathrm{D}}+1}{4}\right)\frac{e^{-y}}{y}+O(\tau^3 l^3). \tag{21}$$

From this after integration it follows that

$$F(it)-F(-it)=i\pi\tau^2 t^2\frac{(\varepsilon_0^{\mathrm{D}}-1)^2}{4(\varepsilon_0^{\mathrm{D}}+1)}. \tag{22}$$

Performing the integration with respect to t in Eq. (11), we obtain from Eq. (3) the asymptotic behavior of the Casimir free energy,

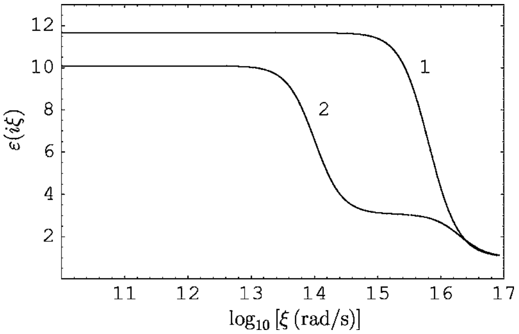

FIG. 4. Dielectric permittivity of Si (line 1) and α-Al$_2$O$_3$ (line 2) along the imaginary frequency axis as a function of the logarithm of frequency.

$$\mathcal{F}^{\mathrm{ID}}(a,T)=\mathcal{F}^{\mathrm{ID}}(a,T=0)-\frac{\hbar c\zeta(3)\tau^3}{512\pi^4 a^3}\frac{(\varepsilon_0^{\mathrm{D}}-1)^2}{\varepsilon_0^{\mathrm{D}}+1}. \tag{23}$$

Note that for a dilute dielectric this coincides with the term of Eq. (17) of order τ^3, as it should.

From Eq. (23) we finally arrive at the low-temperature asymptotics of the entropy,

$$S^{\mathrm{ID}}(a,T)=\frac{3k_B\zeta(3)\tau^2}{128\pi^3 a^2}\frac{(\varepsilon_0^{\mathrm{D}}-1)^2}{\varepsilon_0^{\mathrm{D}}+1}, \tag{24}$$

which is also in perfect agreement with the second-order term in Eq. (18) obtained for dilute dielectrics. From Eq. (24) it follows that the Casimir entropy goes to zero as the second power of the temperature, i.e., the Nernst heat theorem is satisfied. The comparison with the numerical computations shows that the asymptotic expressions (23) and (24) work well for all $\tau \leqslant 0.1$.

IV. THERMAL CORRECTIONS TO THE CASIMIR INTERACTION BETWEEN A REAL METAL AND A DIELECTRIC

In this section we consider one of the plates made of real metal (Au) and the other plate made of a real dielectric (Si or α-Al$_2$O$_3$). Both these chosen dielectrics possess relatively large values of the static dielectric permittivity and quite different behavior of $\varepsilon^{\mathrm{D}}(i\xi)$ around the characteristic frequencies for the separations under consideration. The Casimir free energy is found by the use of the complete Lifshitz formula (1) describing the case of two parallel plates made of real materials. The dielectric permittivity of real materials along the imaginary frequency axis can be obtained through the dispersion relation

$$\varepsilon^{\mathrm{M,D}}(i\xi)=1+\frac{2}{\pi}\int_0^{\infty} d\omega \frac{\omega\, \mathrm{Im}\, \varepsilon^{\mathrm{M,D}}(\omega)}{\omega^2+\xi^2}. \tag{25}$$

Here the imaginary part of the dielectric permittivity is calculated as $2n_1n_2$ where n_1 and n_2 are the real and imaginary parts of the complex refraction index tabulated, e.g., in Ref. [49]. For Au the available tabulated data are extended for lower frequencies using the usual procedure (see, e.g., Refs. [47,48]). The resulting behavior of $\varepsilon^{\mathrm{Au}}$ as a function of ξ can be found in Refs. [21,24,47,48]. The same procedure, applied in the case of Si (here tabulated data for lower frequencies than for Au are available so that no additional extension of data is needed), leads to the results shown by line 1 in Fig. 4.

There are also good analytic formulas describing the behavior of the dielectric permittivities of different materials along the imaginary frequency axis. As an example, the dielectric permittivity of α-Al_2O_3 is well described [43] in the Ninham-Parsegian representation [1]

$$\varepsilon^{D}(i\xi) = 1 + \frac{C_{\rm ir}}{1 + \xi^2/\omega_{\rm ir}^2} + \frac{C_{\rm uv}}{1 + \xi^2/\omega_{\rm uv}^2}, \qquad (26)$$

where $\omega_{\rm ir}=1\times 10^{14}$ rad/s and $\omega_{\rm uv}=2\times 10^{16}$ rad/s are the characteristic absorption frequencies, and $C_{\rm ir}=7.03$ and $C_{\rm uv}=2.072$ are the corresponding absorption strengths in the infrared and ultraviolet ranges, respectively. The dielectric permittivity of α-Al_2O_3 as a function of ξ is plotted in Fig. 4, line 2. As is seen from Fig. 4, the dielectric permittivities of Si and α-Al_2O_3 are qualitatively different in the region of characteristic frequencies $\xi_c \sim 10^{15}$ rad/s. In fact, for α-Al_2O_3 in the region around ξ_c the values of ε^{D} correspond to the second step of line 2 and are several times less than $\varepsilon_0^{D}=10.1$, whereas for Si the static value of the dielectric permittivity $\varepsilon_0^{D}=11.66$ is preserved up to the region of characteristic frequencies.

It should be stressed that the computations below are unaffected by the controversies concerning the contribution of the zero-frequency term of the Lifshitz formula in the case of real metals (this contribution is different in the approaches using the Drude dielectric function and the Leontovich surface impedance; see the Introduction). The reason is that in our case only one plate is made of a real metal, whereas the other one is made of a dielectric. According to Eq. (2), for dielectrics with finite $\varepsilon^{D}(0)=\varepsilon_0^{D}$ it follows that

$$r_{\parallel}^{D}(0,y) = \frac{\varepsilon_0^{D} - 1}{\varepsilon_0^{D} + 1}, \quad r_{\perp}^{D}(0,y) = 0. \qquad (27)$$

As a result, if one plate is made of a dielectric, the transverse electric mode at zero frequency does not contribute to the Casimir free energy regardless of the approach used to describe the metal of the other plate [i.e., regardless of the value of the transverse reflection coefficient $r_{\perp}^{M}(0,y)$].

A further distinctive feature of the different approaches to the thermal Casimir force in the case of a real metal, which might play a part in determining the contribution to the free energy at nonzero Matsubara frequencies, is the form of the reflection coefficients. In the framework of the impedance approach [30,35], in place of the usual reflection coefficients (2) expressed in terms of the dielectric permittivity, one should use the coefficients expressed in terms of the Leontovich surface impedance. This, however, does not present a problem because, as was demonstrated in Ref. [14], at all nonzero Matsubara frequencies the contributions from both types of reflection coefficients are practically the same.

In Fig. 5 we present the results of the calculation for the relative thermal correction to the Casimir energy between Au and Si plates obtained by Eqs. (1) and (2) using the procedure described above at $T=300$ K (solid line). For comparison, in the same figure the dashed line shows the results obtained using the approach of Sec. III (i.e., for an ideal metal and a dielectric with a frequency-independent dielectric permittivity $\varepsilon_0^{D}=11.66$ equal to the static permittivity of Si). As is seen from Fig. 5 (solid line), there is a wide separation interval $0.2 \leqslant a \leqslant 1.3$ μm where the relative thermal correction to the Casimir energy in the case of real materials is negative (in terms of the dimensionless variable this holds for $0.33 \leqslant \tau \leqslant 2.14$). The minimum value of the thermal correction $\delta_T\mathcal{F}=-0.006$ is achieved at $a=0.95$ μm ($\tau=1.56$). What this means is that the Casimir entropy in the case of real materials is negative within the separation region $0.2 \leqslant a \leqslant 0.95$ μm (or, in terms of τ, for $0.33 \leqslant \tau \leqslant 1.56$). The comparison with the dashed line shows that for Si the simple model, used in Sec. III, leads to the same qualitative results with only minor differences in the minimum values of $\delta_T\mathcal{F}$ and the width of the intervals where the thermal correction and the Casimir entropy are negative.

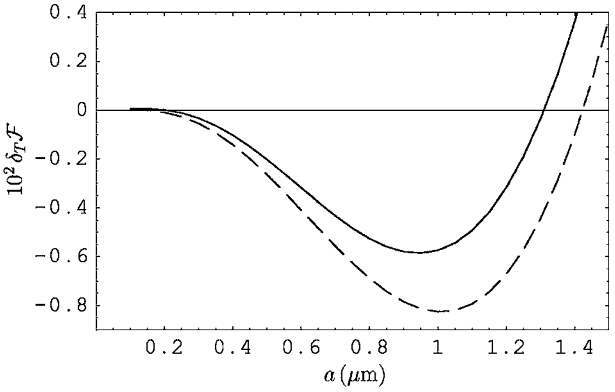

FIG. 5. Relative thermal correction to the Casimir energy at $T=300$ K as a function of separation for two plates, one made of real metal (Au) and the other of dielectric (Si) (solid line). The same is shown by the dashed line for an ideal metal and dielectric with $\varepsilon_0^{D}=11.67$.

We now turn to the Casimir interaction of a Au plate with a plate made of α-Al_2O_3. As was discussed above, the be-

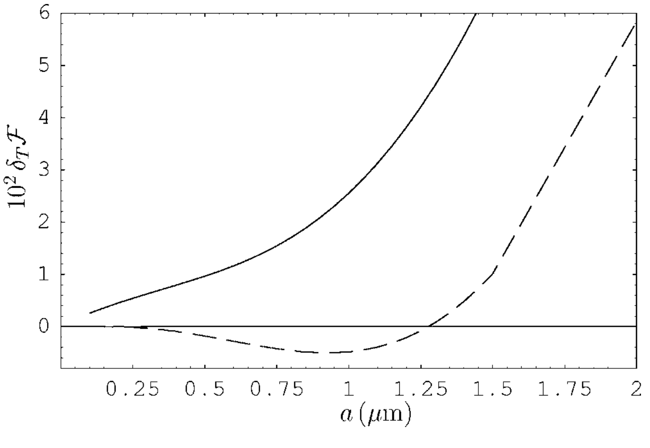

FIG. 6. Relative thermal correction to the Casimir energy at $T=300$ K as a function of separation for two plates, one made of a real metal (Au) and the other of a dielectric (α-Al_2O_3) (solid line). The same is shown by the dashed line for an ideal metal and dielectric with $\varepsilon_0^{D}=10.1$.

havior of the dielectric permittivity of α-Al_2O_3 along the imaginary frequency axis is different from that of Si. The results of the calculation for the relative thermal correction to the Casimir energy as a function of separation at $T=300$ K, obtained by Eqs. (1), (2), and (26), are shown in Fig. 6 by the solid line. The dashed line is calculated for an ideal metal and a dielectric with the frequency-independent dielectric permittivity $\varepsilon_0^{\mathrm{D}}=10.1$ equal to the static dielectric permittivity of α-Al_2O_3. As is seen from Fig. 6, in this case the solid line presents the monotonically increasing positive function of the separation distance. The corresponding Casimir entropy is also nonnegative within the separation region reflected in the figure. The application of the simplified model of Sec. III to α-Al_2O_3 leads to qualitatively different results shown by the dashed line in Fig. 6. This line demonstrates the negative thermal correction within the separation region from $a_1=0.25$ μm to $a_2=1.27$ μm and the negative Casimir entropy within the separations from 0.25 to 0.9 μm. Thus, the use of realistic data for the dielectric permittivities of the plates is essential for the final results.

V. CONCLUSIONS AND DISCUSSION

In the foregoing we have investigated the thermal corrections to the Casimir interaction between metallic and dielectric plates. This was done both analytically (using the idealized model of an ideal metal and a dilute dielectric) and numerically (for the ideal metal and a dielectric with a frequency-independent dielectric permittivity, and for a real metal and two different dielectrics with dissimilar behavior of their dielectric permittivities along the imaginary frequency axis). The main conclusion obtained above is that the pressure and the free energy of the Casimir interaction between metal and dielectric plates may be nonmonotonic functions of the temperature within some definite regions. This leads to the possibility of negative relative thermal corrections and negative values of entropy of the fluctuating field (the latter holds only for dielectrics with sufficiently large dielectric permittivity). Using the proximity force theorem, one can conclude that the relative thermal correction to the Casimir force between a plane metal plate and a spherical dielectric lens (the configuration used in many experiments) also can be negative.

The physical interpretation of the obtained results is based on the fact that both the free energy and entropy of the closed system under consideration consist of contributions from the plates and from their interaction (in the previous sections the latter were denoted as $\mathcal{F}^{\mathrm{ID}},S^{\mathrm{ID}}$ or as $\mathcal{F},S$ for real materials). The above conclusions about the possibility of a nonmonotonic behavior of the free energy and of the negativeness of the entropy are not relevant for the closed system but are due to the interaction between its parts. In the case of two plates made of an ideal metal with the Dirichlet boundary conditions on their surface, there is no penetration of the fluctuating field inside the plates. In this case the characteristics of the closed system coincide with those obtained for the interaction between the plates. As a result, for ideal metals the free energy of a fluctuating field is a monotonic function and the entropy is positive. For two dielectric plates [27,50] or for one dielectric and one metal plate this is not necessarily so.

It is important to keep in mind that only the interaction parts of the free energy and entropy depend on the separation distance: $\mathcal{F}=\mathcal{F}(a,T)$, $S=S(a,T)$. This leads to two conclusions of considerable significance. The first is that the thermal correction to the Casimir force (which is minus the derivative of the free energy with respect to the separation) can be negative. The second is that the Nernst heat theorem must be valid separately for the contribution to the entropy from the interaction between the plates, so that $S(a,0)=0$, and for the entropy of the plates. If this were not the case, i.e., if the equation $S(a,0)=f(a)\neq 0$ were valid (as in Refs. [25–28] for a perfect crystal with no impurities), then the Nernst heat theorem for the closed system would be violated as the entropies of the plates do not depend on a. Both these conclusions were illustrated above by the example of the Casimir interaction between metal and dielectric.

To conclude, the nonmonotonic dependence of the Casimir free energy on temperature and the negative values of the relative thermal correction (as, for instance, was predicted for real metals in the approach of Refs. [25–28] and for Si in Ref. [50]) are not in themselves excluded thermodynamically. Such behavior for real metals is, however, unlikely because there is only a small penetration of the electromagnetic fluctuations at the characteristic frequencies in the interior of a metal (recall that in the approach of Refs. [30,35], where this property of real metals is taken into account, the free energy is monotonic, and the corresponding thermal correction is positive). The decisive theoretical argument to give preference to any approach is, thus, the satisfaction of the Nernst heat theorem for the Casimir entropy of a fluctuating field for both perfect crystals and crystals with impurities. In particular, if one interacting body is made of a metal and the other of a dielectric, the entropy of the fluctuating field vanishes when the temperature goes to zero.

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