# Quantum and classical operational complementarity for single systems

Alfredo Luis*
*Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain*
(Received 8 April 2005; published 22 July 2005)

We investigate duality relations between conjugate observables after measurements performed on a single realization of the system. The application of standard inference methods implies the existence of duality relations for single systems when using classical as well as quantum physics.

DOI: 10.1103/PhysRevA.72.014106 PACS number(s): 03.65.Ta, 03.65.Ca, 42.50.Dv, 42.50.Xa

## I. INTRODUCTION

The concept of complementarity implies the existence of mutually exclusive observables. The wave-particle duality provides a classic illustration: If the path taken by the particle within a two-beam interferometer is observed, then the interference is wiped out. In spite of its long history, it seems that there are still many features of complementarity to be discovered. For example, some recent and striking results reveal that two observables can be complementary or not depending on the measure of fluctuations adopted, complementarity is not a symmetric relation, there are observables without complementary, and path measurements can increase visibility in multipath interferometers [1].

Complementarity can have different meanings. For example, we can distinguish between intrinsic and operational versions. Intrinsic complementarity refers to the mutual relationship between the probability distributions of conjugate observables, as suitably exemplified by standard position-momentum uncertainty relations. Intrinsic complementarity has no classical analog (classically there is no definite universal relationship between the statistics of different observables) and can only be observed by repeated measurements performed in ensembles of systems (since it involves statements about probability distributions).

On the other hand, operational complementarity emerges when we try to observe two conjugate observables simultaneously in each and every run of the experiment under the same experimental conditions. For example, for wave-particle duality it is always understood that we are monitoring the visibility of the interference at the same time that we try to determine the path followed by the particle within the interferometer. In principle, this idea differs from the intrinsic point of view since the determination of the intrinsic statistics of incompatible observable requires mutually exclusive measuring schemes. Nevertheless, care must be taken since one can be tempted to infer from the outputs of multiple measurements the quantum state of the system and then derive the intrinsic statistics [2]. An approach to operational complementarity after multiple measurements avoiding state reconstruction has been studied in Ref. [3], leading to operational duality relations more restrictive than the intrinsic ones.

*Electronic address: alluis@fis.ucm.es; URL: http://www.ucm.es/info/gioq

In this Brief Report we examine the case of measurements performed on single systems. We investigate whether there is actually any kind of operational complementarity between pairs of conjugate observables after measurements performed on an single realization of the system. In this case complementarity will involve our degree of knowledge about two conjugate properties rather than the dispersion of measured data. A natural arena for the inference of system properties is the Bayesian perspective, which can be naturally applied to individual events in the sense of degrees of belief [4]. For the sake of completeness we address this idea using both quantum and classical physics in Secs. II and III, respectively.

## II. QUANTUM CASE

The most general measurement is described by a positive operator measure $\Delta(w)$, where $w$ labels the outcomes [5]. Any inference about system properties must be derived exclusively from $w$ and $\Delta(w)$, which is the only information accessible to the experimenter. A very natural procedure consists in associating to every outcome $w$ a quantum state $\rho_w$ depending only on $w$ and $\Delta(w)$. This can be carried out in different ways [3,6,7]. Following a Bayesian perspective $\rho_w$ becomes a superposition of all possible density matrices $\rho_\Omega$ weighted by the conditional probability $p(\Omega|w)$ of inferring that the state of the system is $\rho_\Omega$ when the outcome is $w$,

$$\rho_w = \int d\Omega p(\Omega|w)\rho_\Omega = \frac{\mathcal{N}}{p(w)} \int d\Omega p(w|\Omega)\rho_\Omega, \qquad (1)$$

where $\Omega$ are parameters indexing the density matrices and

$$p(w|\Omega) = \mathrm{tr}[\rho_\Omega \Delta(w)], \quad p(w) = \int d\Omega p(w|\Omega) p_0(\Omega), \quad (2)$$

and we have used the Bayes' rule

$$p(\Omega|w) = \frac{p(w|\Omega)p_0(\Omega)}{p(w)}, \qquad (3)$$

assuming a uniform prior distribution $p_0(\Omega)=\mathcal{N}=(\int d\Omega)^{-1}$. The issue of uniform priors is not trivial, since it depends on the parameters $\Omega$ used [6]. Fortunately, this ambiguity does not affect the main conclusions of this work.

For definiteness we focus on the most widely studied example of complementarity: path-visibility complementarity in two-beam interferometers. For two-dimensional systems the most general $\Delta(w)$ is

 

$$\Delta(w) = \mu_w + \boldsymbol{\mu}_w \cdot \boldsymbol{\sigma}, \tag{4}$$

where $\sigma$ are the three Pauli matrices and $\mu_w$ and $\boldsymbol{\mu}_w$ are real coefficients satisfying

$$\sum_w \mu_w = 1, \quad \sum_w \boldsymbol{\mu}_w = 0, \quad \mu_w \geqslant |\boldsymbol{\mu}_w|, \tag{5}$$

and we have assumed without loss of generality a discrete set of outcomes $w$. On the other hand, the most general density matrix $\rho_\Omega$ can be expressed as

$$\rho_\Omega = \frac{1}{2}(1 + \boldsymbol{\Omega} \cdot \boldsymbol{\sigma}), \tag{6}$$

being

$$\boldsymbol{\Omega} = \begin{pmatrix} r \sin\theta \cos\phi \\ r \sin\theta \sin\phi \\ r \cos\theta \end{pmatrix}, \quad d\Omega = r^2 \sin\theta dr d\theta d\phi, \tag{7}$$

with $1 \geqslant r \geqslant 0$, $\pi \geqslant \theta \geqslant 0$, and $2\pi \geqslant \phi \geqslant 0$. The application of Eqs. (1)–(7) leads to

$$\rho_w = \frac{1}{2}\left(1 + \frac{1}{5\mu_w}\boldsymbol{\mu}_w \cdot \boldsymbol{\sigma}\right) = \frac{2}{5}\left(1 + \frac{\Delta(w)}{2\mathrm{tr}\Delta(w)}\right). \tag{8}$$

Next we examine the mutual relationship between the inferred uncertainties of path and visibility. The path observable can be represented by the operator $\sigma_z$, while the phase-difference observable $\phi$ is represented by the operator $\sigma_y$ [8,9]. A suitable assessment of fluctuations for finite-dimensional systems is provided by the *certainties*

$$\mathcal{C}_z = |\langle \sigma_z \rangle|, \quad \mathcal{C}_y = |\langle \sigma_y \rangle|, \tag{9}$$

representing the degree of certainty one can have concerning the value of the corresponding observable [9]. They are bounded, $1 \geqslant \mathcal{C}_{y,z} \geqslant 0$, and satisfy the intrinsic duality relations [9]

$$\mathcal{C}_z^2 + \mathcal{C}_y^2 \leqslant 1, \quad \mathcal{C}_z\mathcal{C}_y \leqslant \frac{1}{2}, \tag{10}$$

so that $\mathcal{C}_{y,z}$ cannot simultaneously reach their maximum values.

When we apply this measure the quantum state $\rho_w$ we get

$$\mathcal{C}_j = \frac{1}{5}\left|\frac{\mu_{w,j}}{\mu_w}\right|, \tag{11}$$

for $j = y, z$. The corresponding operational duality relations are

$$\mathcal{C}_z^2 + \mathcal{C}_y^2 \leqslant \frac{1}{25}, \quad \mathcal{C}_z\mathcal{C}_y \leqslant \frac{1}{50}. \tag{12}$$

We can appreciate that the operational bounds are lower than the intrinsic ones (10). This is natural since we are inferring system properties from a single noisy simultaneous measurement of incompatible observables. Compared to the case of duality relations for multiple measurements [3], the bound for the sum of certainties is essentially the same while for the product the case of multiple measurements is more restrictive.

In the next section we address this same issue from the classical perspective. In order to simplify the comparison between the quantum and the classical analyses we formulate the quantum results in terms of Wigner functions.

Maybe the simplest approach to a phase-space formulation of quantum mechanics for a two-dimensional system is provided by a Wigner function defined on a discrete and finite phase space made of $2 \times 2$ points $(m,s)$, with $m,s = \pm 1$, formed by the product of the spectra of $\sigma_z$ and $\sigma_y$ [8,10]. The Wigner-Weyl correspondence between operators $A$ and functions $W_A(m,s)$ on this phase space is given by [10]

$$W_A(m,s) = \mathrm{tr}[A\Lambda(m,s)], \quad A = 2\sum_{m,s} W_A(m,s)\Lambda(m,s), \tag{13}$$

with

$$\Lambda(m,s) = \frac{1}{4}(1 + \sqrt{3}\boldsymbol{\alpha}_{m,s} \cdot \boldsymbol{\sigma}), \tag{14}$$

where $\boldsymbol{\alpha}_{m,s}$ are the four unit vectors,

$$\boldsymbol{\alpha}_{m,s} = \frac{1}{\sqrt{3}}(ms, s, m), \tag{15}$$

so that mean values can be computed as phase-space averages:

$$\mathrm{tr}\, A = \sum_{m,s} W_A(m,s), \quad \mathrm{tr}(AB) = 2\sum_{m,s} W_A(m,s) W_B(m,s). \tag{16}$$

Incidentally, we note that the vectors $\boldsymbol{\alpha}_{m,s}$ define a positive operator measure for minimal qubit tomography [11].

Denoting by $W_w^{(q)}(m,s)$ the Wigner function associated with the quantum state (8) we get

$$W_w^{(q)}(m,s) = \frac{1}{5}\left(1 + \frac{W_{\Delta(w)}(m,s)}{\sum_{m',s'} W_{\Delta(w)}(m,s)}\right), \tag{17}$$

where $W_{\Delta(w)}(m,s)$ is the distribution associated to the operator $\Delta(w)$:

$$W_{\Delta(w)}(m,s) = \frac{1}{2}(\mu_w + \sqrt{3}\boldsymbol{\alpha}_{m,s} \cdot \boldsymbol{\mu}_w). \tag{18}$$

It is worth noting that $W_w^{(q)}(m,s)$ never takes negative values. As can be seen in Eqs. (17) and (18) the minimum value would be reached at a given point $(m_0,s_0)$ when $\boldsymbol{\mu}_w/\mu_w = -\boldsymbol{\alpha}_{m_0,s_0}$, leading to $W_w^{(q)}(m_0,s_0) = 0.163 > 0$. Therefore $W_w^{(q)}(m,s) > 0$, so it can be regarded as a true probability distribution in phase space.

## III. CLASSICAL CASE

In classical physics the state of the system is specified by a probability distribution on the phase space. The question of a proper phase space for the classical analog of two-dimen-

sional systems is not straightforward. For simplicity we consider the same phase space of the Wigner-Weyl formulation of quantum physics for a two-dimensional system employed above [10]. This choice of phase space might appear to be somewhat exotic, but it allows a workable parametrization of the classical states that would be rather impracticable with a different phase space. This simplifies the comparison with the quantum case without altering the essence of the conclusions.

The classical inference is governed by exactly the same equations (1)–(3), simply replacing density matrices by phase-space distributions $\rho_j \to W_j(m,s)$ for $j=w,\Omega$. Furthermore, the conditional statistics $p(w|\Omega)$ can be always expressed as a suitable phase-space average in terms of some function $\mathcal{W}_w(m,s)$:

$$p(w|\Omega) = \sum_{m,s} W_\Omega(m,s)\mathcal{W}_w(m,s). \tag{19}$$

Therefore, the classical counterparts of Eqs. (1)–(3) lead to, taking into account Eq. (19),

$$W_w(m,s) = \frac{\mathcal{N}}{p(w)} \sum_{m',s'} \mathcal{W}_w(m',s') \int d\Omega W_\Omega(m',s') W_\Omega(m,s). \tag{20}$$

In order to proceed we must address the parametrization of the classical states, which must be different from the quantum one since the spaces of classical and quantum distributions do not coincide. The requirements that classical distributions must satisfy are reality, normalization, and non-negativeness. Normalization imposes that $W_\Omega(+,+) + W_\Omega(+,-) + W_\Omega(-,+) + W_\Omega(-,-) = 1$, so that reality, positivity, and normalization can be satisfied simultaneously in the form

$$\sqrt{W_\Omega(+,+)} = \cos\theta \sin\phi, \quad \sqrt{W_\Omega(-,-)} = \cos\theta\cos\phi,$$
$$\sqrt{W_\Omega(+,-)} = \sin\theta \sin\varphi, \quad \sqrt{W_\Omega(-,+)} = \sin\theta\cos\varphi, \tag{21}$$

with $\pi/2 \geq \theta, \phi, \varphi \geq 0$, and $d\Omega = \cos\theta \sin\theta d\theta d\phi d\varphi$.

Using Eq. (21) in Eq. (20) and performing the $\theta$, $\phi$, $\varphi$ integration we get

$$W_w(m,s) = \frac{1}{6}\left(1 + 2\frac{\mathcal{W}_w(m,s)}{\sum_{m',s'} \mathcal{W}_w(m',s')}\right). \tag{22}$$

This classical result closely resembles the quantum one in Eq. (17) since $\mathcal{W}_w(m,s)$ is the classical analog of $W_{\Delta(w)}(m,s)$. In particular, this implies the existence of classical duality relations for single measurements. We can asses the fluctuations by using the certainties

$$\mathcal{C}_j = |\langle \sigma_j \rangle^{(c)}| = \left| \sum_{m,s} W_w(m,s) W_{\sigma_j}(m,s) \right|, \tag{23}$$

for $j=y,z$, being

$$W_{\sigma_z}(m,s) = \delta_{m,+} - \delta_{m,-}, \quad W_{\sigma_y}(m,s) = \delta_{s,+} - \delta_{s,-}. \tag{24}$$

This leads to

$$\mathcal{C}_z = \frac{1}{3}|\tilde{W}(+,+) + \tilde{W}(+,-) - \tilde{W}(-,+) - \tilde{W}(-,-)|,$$
$$\mathcal{C}_y = \frac{1}{3}|\tilde{W}(+,+) - \tilde{W}(+,-) + \tilde{W}(-,+) - \tilde{W}(-,-)|, \tag{25}$$

where the distribution

$$\tilde{\mathcal{W}}_w(m,s) = \frac{\mathcal{W}_w(m,s)}{\sum_{m',s'} \mathcal{W}_w(m',s')}, \tag{26}$$

is normalized and positive because $\mathcal{W}_w(m,s)$ is positive. Therefore, the parametrization (21) can be used for $\tilde{\mathcal{W}}(m,s)$ in Eq. (25), leading to

$$\mathcal{C}_z^2 + \mathcal{C}_y^2 \leq \frac{2}{9}, \quad \mathcal{C}_z\mathcal{C}_y \leq \frac{1}{9}. \tag{27}$$

The equalities are reached when $\theta, \phi, \varphi = 0, \pi/2$—i.e., when the measurement is of the form

$$\mathcal{W}_w(m,s) \propto \delta_{m,m_0}\delta_{s,s_0}, \tag{28}$$

which is an exact noiseless joint measurement of path and interference (not allowed in the quantum domain).

Relations (27) are duality relations since they express the way the certainty is distributed between two conjugate observables in such a way that they exclude maximum certainty for both observables simultaneously, exactly in the same way as occurs in quantum mechanics. Actually it can be can appreciated that the bounds (27) are compatible with the quantum intrinsic certainty relations (10) [even in the case in relation (28)], being less restrictive than the quantum operational ones in relations (12). In other words, the classical duality relations can violate the operational duality relations (12) but not the intrinsic ones (10).

In this regard, let us show that the classically inferred state is always compatible with quantum mechanics. With the help of the Wigner-Weyl correspondence (13) outlined above we can determine the operator $\rho_w^{(c)}$ associated with the classical state $W_w(m,s)$ in Eq. (22):

$$\rho_w^{(c)} = \frac{1}{3}\left(1 + \frac{\Delta^{(c)}(w)}{\mathrm{tr}\Delta^{(c)}(w)}\right), \tag{29}$$

where $\Delta^{(c)}(w)$ is the operator associated with $\mathcal{W}_w(m,s)$ via the same relations (13). The similarity with Eq. (8) is manifest. The operator $\rho_w^{(c)}$ represents a legitimate quantum state (i.e., it is a density matrix) since $\rho_w^{(c)}$ is Hermitian with unit trace [granted by the Wigner-Weyl correspondence since $W_w(m,s)$ is real] and positive $\rho_w^{(c)} > 0$. To demonstrate positivity we note that for $2\times 2$ matrices $\rho_w^{(c)} > 0$ if and only if $\mathrm{tr}[(\rho_w^{(c)})^2] \leq 1$ [12]. In our case the maximum $\mathrm{tr}[(\rho_w^{(c)})^2]$ is

obtained when $\mathcal{W}_w(m,s) \propto \delta_{m,m_0}\delta_{s,s_0}$, leading to $\mathrm{tr}[(\rho_w^{(c)})^2] = 2/3$. Therefore, $\rho_w^{(c)} > 0$.

## IV. DISCUSSION

The Bayesian approach allows us to develop the idea of operational complementarity for single systems suitably expressed by the operational duality relations (12) and (27). In this case complementarity involves our amount of knowledge about two conjugate system properties, rather than the dispersion of recorded data.

Maybe surprisingly, within the Bayesian approach there are no essential differences between the quantum and classical analysis. The key point for this classical-quantum similarity is that the logic of the Bayesian inference is the same in quantum and classical physics. The only difference is that the state spaces are different (there are classical distributions that cannot be the Wigner function of any legitimate quantum state [12,13] and quantum Wigner functions that are not classically valid distributions).

The strategy of inference adopted is crucial in the classical case. For example, classical complementarity fades away if we adopt a maximum-likelihood strategy, which stipulates that the best estimate is the state that maximizes $p(w|\Omega)$—i.e., the phase-space point for which $\mathcal{W}_w(m,s)$ takes the maximum value. Phase-space points carry full certainty for path and phase difference simultaneously $\mathcal{C}_z = \mathcal{C}_y = 1$ so that there is no room for duality relations.

We think that these results are interesting since they may open promising routes to analyze the relationship between classical and quantum physics.

## ACKNOWLEDGMENT

This work has been supported by Project No. FIS2004-01814 of the Spanish Dirección General de Investigación del Ministerio de Educación y Ciencia.

[1] M. Mei and M. Weitz, Phys. Rev. Lett. **86**, 559 (2001); A. Luis, J. Phys. A **34**, 8597 (2001); Phys. Rev. Lett. **88**, 230401 (2002).
[2] M. Freyberger, Phys. Lett. A **242**, 193 (1998).
[3] A. Luis, Phys. Rev. A **70**, 062107 (2004).
[4] Ch. A. Fuchs and R. Schack, in *Quantum State Estimation*, Vol. 649 of Springer Lecture Notes in Physics, edited by M. G. A. Paris and J. Rehacek (Springer, Berlin, 2004), p. 147; D. M. Appleby, e-print quant-ph/0408058.
[5] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976); A. Peres, Found. Phys. **20**, 1441 (1990); *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1993); W. M. de Muynck, *Foundations of Quantum Mechanics, An Empiricist Approach* (Kluwer Academic, Dordrecht, 2002).
[6] K. R. W. Jones, Ann. Phys. (N.Y.) **207**, 140 (1991); Phys. Rev. A **50**, 3682 (1994).
[7] J. Audretsch, L. Diósi, and Th. Konrad, Phys. Rev. A **68**, 034302 (2003); e-print quant-ph/0210205; H. F. Hofmann, Phys. Rev. A **67**, 022106 (2003); M. J. W. Hall, *ibid.* **69**, 052113 (2004).
[8] A. Luis and L. L. Sánchez-Soto, Phys. Rev. A **48**, 4702 (1993); A. Luis and J. Peřina, *ibid.* **54**, 4564 (1996); A. Luis and L. L. Sánchez-Soto, in *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 2000), Vol. 41, p. 421.
[9] A. Luis, Phys. Rev. A **64**, 012103 (2001); **67**, 032108 (2003).
[10] A. Luis and J. Peřina, J. Phys. A **31**, 1423 (1998).
[11] J. Řeháček, B.-G. Englert, and D. Kaszlikowski, Phys. Rev. A **70**, 052321 (2004).
[12] A. Luis and L. M. Johansen (unpublished).
[13] A. Luis, Phys. Rev. A **69**, 052112 (2004).