Entanglement and the quantum-to-classical transition

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We analyze the quantum-to-classical transition (QCT) for coupled bipartite quantum systems for which the position of one of the two subsystems is continuously monitored. We obtain the surprising result that the QCT can emerge concomitantly with the presence of highly entangled states in the bipartite system. Furthermore, the changing degree of entanglement is associated with the backaction of the measurement on the system and is itself an indicator of the QCT. Our analysis elucidates the role of entanglement in von Neumann's paradigm of quantum measurements comprised of a system and a monitored measurement apparatus.

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The emergence of classical dynamics in a world fundamentally described by quantum mechanics remains a profound issue in the foundations of physics. Recently, it has been shown how such a quantum-to-classical transition (QCT) arises through continuous weak measurement of an evolving observable such that the dynamical measurement record is well predicted by the classical equations of motion [1]. This operational approach has significant relevance given the proliferation of experiments in quantum control based on continuous monitoring of individual and ensembles of quantum systems [2]. Of particular interest are systems with coupled degrees of freedom whose classical dynamics can exhibit chaos [3,4], something not seen in closed quantum systems. At the quantum level this coupling leads to entanglement, which is typically responsible for the most nonclassical phenomena. Here we study the entanglement associated with the quantum states of continuously observed bipartite systems whose dynamical measurement record follows classical trajectories. We find the surprising result that the dynamical QCT emerges even if the entanglement between degrees of freedom grows. This highlights the fact that not all nonclassical features of a quantum system are concomitant.

The QCT is achieved via continuous measurement through a balancing of strong localization and weak measurement backaction. According to Ehrenfest's theorem, expectation values of quantum observables follow the classical equations of motion if all correlation functions factorize. Corrections to the classical dynamics arise through higher-order cumulants of the relevant observables. Their effect can be neglected when their extent is small compared to the size of the phase space explored by the dynamics. Bhattacharya and co-workers showed that in the limit of large actions ($\hbar \to 0$) continuous measurement can sufficiently limit the growth of these cumulants [1]. In previous work [3,4], we extended their analysis to bipartite systems in which only one degree of freedom is observed. We found that the QCT emerged when both subsystems were made sufficiently macroscopic so that neither experienced strong backaction. Backaction is distributed to the unobserved subsystem through the entanglement generated between the degrees of freedom. It thus appears that the QCT requires weak entanglement between the subsystems. Indeed, by Enhrenfest's theorem, separable pure states imply factorizable correlation functions, a sufficient condition for classical dynamics. We show here that that this is not necessary. The QCT in fact is concomitant with *increased entanglement* between the subsystems. By relating the weak backaction condition to a change in entanglement with time, we provide an alternative condition for the QCT to the one derived by Bhattacharya *et al.*, useful for multipartite systems.

Consider a bipartite system, $\mathcal{A}+\mathcal{B}$, in which a continuous weak measurement is performed on $\mathcal{A}$, treated, without loss of generality, as a position degree of freedom q. The measurement record is described by a quantum trajectory [5] including the irreducible quantum backaction noise commensurate with the information gain-disturbance tradeoff. For ideal measurements (which acquire all information that leaves $\mathcal{A}+\mathcal{B}$), the state $\rho=|\psi\rangle\langle\psi|$ of the joint system is always pure, with the specific pure state differing according to the corresponding measurement record. The degree of bipartite entanglement can thus be obtained equivalently as the entropy $S(\rho_{\mathcal{A}})$ or $S(\rho_{\mathcal{B}})$ for $\rho_{\mathcal{A}}=\mathrm{Tr}_{\mathcal{B}}\rho$ and $\rho_{\mathcal{B}}=\mathrm{Tr}_{\mathcal{A}}\rho$. We work with the linear entropy $S=1-\mathrm{Tr}(\rho_{\mathcal{A}}^2)=1-\mathrm{Tr}(\rho_{\mathcal{B}}^2)$, which is convenient to calculate and can be employed as an estimator for other required entropy measures [6].

Consider how $|\psi\rangle$ behaves under mappings that obey the QCT. The stochastic evolution of the pure state for $\mathcal{A}+\mathcal{B}$ is given by

$$d|\psi\rangle = \left[-\frac{i}{\hbar}H - k(q-\langle q\rangle)^2\right]dt|\psi\rangle + \sqrt{2k}(q-\langle q\rangle)dW|\psi\rangle \tag{1}$$

with measurement record $dX=\langle q\rangle dt+(8k)^{-1/2}dW$ on $\mathcal{A}$ [7], with k the strength (resolution) of the position measurement and dW the Wiener noise. From Eq. (1), the evolution of the reduced density operator for $\mathcal{A}$ is

$$d\rho_{\mathcal{A}} = -\frac{i}{\hbar}\mathrm{Tr}_{\mathcal{B}}([H,\rho])dt + k(2q\rho_{\mathcal{A}}q - q^2\rho_{\mathcal{A}} - \rho_{\mathcal{A}}q^2)dt + \sqrt{2k}(q\rho_{\mathcal{A}} + \rho_{\mathcal{A}}q - 2\langle q\rangle\rho_{\mathcal{A}})dW. \tag{2}$$

The evolution of the marginal linear entropy obeys

$dS=-2\mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}}d\rho_{\mathcal{A}})-\mathrm{Tr}_{\mathcal{A}}[(d\rho_{\mathcal{A}})^2]$. From Eq. (2) [and only retaining terms to $O(dt)$], the evolution of the entanglement for a given measurement strength k is

$$dS_k = dS_0 - 8k\mathrm{Tr}_{\mathcal{A}}\{[\rho_{\mathcal{A}}(q-\langle q\rangle)]^2\}dt - 4\sqrt{2k}\mathrm{Tr}_{\mathcal{A}}[\rho_{\mathcal{A}}^2(q-\langle q\rangle)]dW \quad (3)$$

with dS_0 the term corresponding to measurement-free ($k=0$) evolution.

To consider the QCT, we study the evolution of the moments of position and momentum of the measured subsystem $\mathcal{A}$. These equations of motion are

$$d\langle q\rangle = (\langle p\rangle/m)dt + \sqrt{8k}C_{qq}dW,$$

$$d\langle p\rangle = -\langle \partial_q V\rangle dt + \sqrt{8k}C_{qp}dW \quad (4)$$

with centroid coordinates $\langle q\rangle$ and $\langle p\rangle$ and covariances $C_{ab} = (\langle ab\rangle + \langle ba\rangle)/2 - \langle a\rangle\langle b\rangle$. The force $-\partial_q V$ acting on $\mathcal{A}$ is derived from the potential V. Provided that the noise terms (due to backaction) are sufficiently small and the measurement is sufficiently strong to localize the state, these equations of motion will closely follow the dynamics of the corresponding classical system: the strong localization and weak noise conditions require that the cumulants remain small compared to the phase space explored by the classical equations of motion [1].

In the regime of the QCT, the wave function remains close to Gaussian [1]. In this limit (covariances are small), the measurement terms in Eq. (3), which can be written in terms of the covariances, become negligible, and the degree of change in entanglement,

$$\Delta S_k = \sup_t\left(1 - \frac{S_k(t)}{S_0(t)}\right), \quad (5)$$

caused by observation of $\mathcal{A}$, approaches zero. Since ΔS_k is negligible only if the covariances, which are proportional to the backaction noise terms in Eq. (4) become sufficiently small, it quantifies the degree of backaction resulting from the measurement of $\mathcal{A}$ with a given measurement strength k. Hence $\Delta S_k \to 0$, coupled with the localization condition discussed above, are sufficient conditions to ensure the QCT for this bipartite system.

In the QCT, ΔS_k must remain small, but no such restriction applies to $S_k(t)$ itself. For example, a quantum chaotic system of $\mathcal{A}+\mathcal{B}$ may enhance the bipartite entanglement between $\mathcal{A}$ and $\mathcal{B}$ more rapidly than the measurement process can diminish the entanglement, and therefore large bipartite entanglement may be compatible with the QCT. The key point is that the measured system will approximately follow classical trajectories when the cumulants are a small fraction of the total classical phase space measured by some characteristic action. In contrast, entanglement depends on the growth of these cumulants with respect to the *absolute* scale of action, $\hbar$. Thus as the action increases and one moves into the classical domain where a macroscopic phase space is explored, the relative size of the cumulants decreases while the entanglement increases. In such a regime of large entanglement, the QCT can be recovered if the condition of small change in entanglement ΔS_k is fulfilled along with the strong localization condition. Thus entanglement can be used to quantitatively identify the QCT in coupled systems.

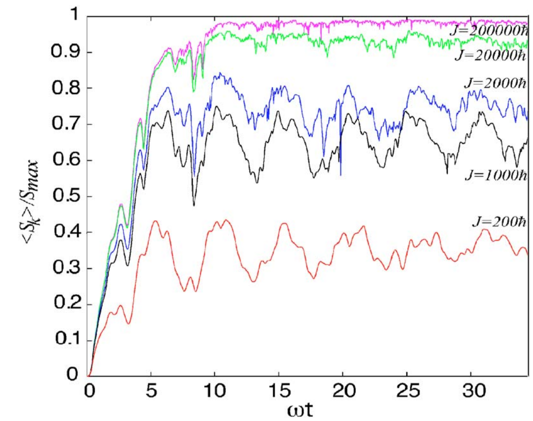

FIG. 1. (Color online) The normalized average linear entropy increases as J is increased, keeping I_0/J constant.

The compatibility of large entanglement and the QCT for the dynamics is illustrated by the following example. The bipartite system $\mathcal{A}+\mathcal{B}$ we consider consists of a particle of mass m in a harmonic trap of angular frequency ω. The particle is coupled via an internal magnetic moment to a gradient magnetic field along its axis of motion z and a constant transverse field along x. The corresponding Hamiltonian governing the dynamics is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 z^2 + bzJ_z + cJ_x, \quad (6)$$

which applies to various phenomena including the simplest Jahn-Teller model [8], the Jaynes-Cummings model, or the Tavis-Cummings model without the rotating wave approximation [9], and the motion of ultracold atoms in a magneto-optical trap [10]. The classical Hamiltonian has the same form as Eq. (6) with the z and p operators replaced by classical variables and the spin replaced by a classical magnetic moment. The transverse magnetic field along x causes the Hamiltonian to become nonintegrable and leads to chaotic dynamics [11]. Previous studies showed that a continuous position measurement resulted in quantum trajectories that exhibit classical chaos when the actions associated with the spin and harmonic motion are both large relative to $\hbar$ [3,4]. Here we analyze the behavior of entanglement in this limit.

We introduce initial states that are products of Gaussian and spin coherent states $|\psi(0)\rangle = |\alpha\rangle|\theta,\phi\rangle$. We start with a spin of $J=200\hbar$, which puts us in the semiclassical regime. We let $c=0.5\omega$ and $b\Delta z=2.5\omega$ with $\Delta z=45z_g$ where $z_g = \sqrt{\hbar/2m\omega}$. This results in a characteristic external action $I_0 = m\omega\Delta z^2 = 1000\hbar$. Classical trajectories are recovered for a measurement strength of $k=\omega/20z_g^2$ given these parameter choices [4].

Figure 1 shows the evolution of the average normalized linear entropy $\langle S_k\rangle/S_{\max}$ of 100 trajectories with $S_{\max}=1-1/(2J+1)$ and energy $E=0.58E_0$ ($E_0=m\omega^2\Delta z^2$) for increasing values of J, keeping the measurement strength k constant. We scale Δz up appropriately relative to z_g in order to

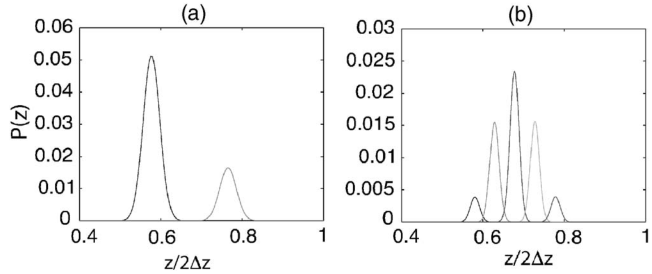

FIG. 2. (a) In the spin-$\frac{1}{2}$ system, the measurement is strong enough to resolve the two spinor components of the wave function and thus remove the entanglement. (b) For larger actions even though the wave-packet components are spatially distinguishable so that the state is highly entangled, the same measurement is too weak to resolve all the different wave packets and remove all the entanglement.

keep the ratio I_0/J constant in the classical limit. As both J and I_0 are increased and the measured quantum trajectories approach the classically predicted trajectories [4], the average entanglement increases. This indicates that in a regime where classical trajectories emerge from the measured quantum system, the underlying states are highly entangled.

This behavior can be understood by examining the measured state more closely. If we write the state $|\phi\rangle$ in terms of its spin components $|m\rangle$ in some basis as

$$|\phi\rangle = \sum_{m=-J}^{J} \alpha_m |\phi_m\rangle |m\rangle, \tag{7}$$

then we can relate the entanglement between spin and motion to the overlap between the spinor components in the different spin states:

$$S_k = 1 - \sum_{m,n} |\alpha_m^* \alpha_n \langle \phi_m | \phi_n \rangle|^2. \tag{8}$$

If there is zero overlap between different spinor components, then the only contributions to the sum are for $m=n$. In this case, when in addition the α_m are all equal, the state is maximally entangled.

In order to understand the behavior in the classical limit, we first consider the spin-$\frac{1}{2}$ case studied in Ref. [3] with $c=0$. The spinor components in the diabatic ($|m_z\rangle$) basis of an initial spatially localized state move along two different harmonic wells centered at $\pm b$ so that their overlap reduces almost to zero [Fig. 2(a)]. At this point, the entanglement increases to its maximum value, but falls back to zero when the measurement eventually projects the state into one of the two spin states. Thus in this example maximum entanglement (zero overlap between wave packets) corresponds to a measurement that perfectly distinguishes the spinor components, resulting in a projective measurement with maximum measurement backaction. The entanglement acts as a measure of the noise on the spin due to the position measurement.

In the large action chaotic limit, the increase in S_k with the actions can be understood in a similar manner. The overall extent of the initial state in position and momentum spreads as the $2J+1$ spinor components move along different diabatic potentials and are coupled by the transverse magnetic field. As the spinor components spatially separate, their overlap decreases, leading to an increase in entanglement. At the same time, the measurement acts to localize the state, thereby damping out the tails of the spatial distribution and preventing further spatial separation between the spinor components. Whereas in the spin-$\frac{1}{2}$ case, the measurement was strong enough to eventually resolve the two spinor components, for larger spin J, the same measurement strength cannot distinguish between all $2J+1$ components [Fig. 2(b)]. Thus the measurement does not project the state into a single spin state and hence the entanglement does not decrease back to zero. Instead, a quasisteady state is reached where some nonoverlapping spinor components remain that lead to a nonzero steady-state entanglement, but are indistinguishable by the measurement. For a constant measurement strength k, as the actions increase, more nonoverlapping spinor components fit within the width of the measurement resolution, and hence this steady-state entanglement increases as seen in Fig. 1.

Whereas the weak measurement does not remove all the entanglement between spin and motion, it is *sufficient* to localize the state and damp the higher-order cumulants that lead to "nonclassical" dynamics [3,4]. The connection between entanglement and the cumulants becomes clear in the large action (classical) limit, where the measurement causes the reduced state of the motional subsystem to remains approximately Gaussian. In that case the linear entropy can be written solely in terms of the variances and covariance as $S_{\text{Gauss}}=1-\hbar/2A$. The quantity $A=\sqrt{C_{zz}C_{pp}-C_{zp}^2}$ represents the effective area of the "uncertainty bubble" of the Gaussian distribution and its ratio to $\hbar/2$ measures the number of minimum uncertainty wave packets which fit within this area and thus the dimension of the Hilbert space required to describe the marginal state. This ratio therefore determines the effective rank of the reduced density operator, or Schmidt number of the entangled state. From this equation it is clear that even if the variances and covariances remain small relative to the *total phase space* of the dynamics, they may still be large compared to $\hbar$, and the entanglement can be close to maximal ($S\approx 1$). In this way, one can simultaneously satisfy the QCT conditions (covariance matrix remains bounded), and obtain an evolution that results in a highly entangled quantum state. This arises because the various J_z components are *in principle* distinguishable by a strong enough position measurement alone. However, the QCT emerges precisely because *in practice* the measurement is weak and hence cannot induce strong quantum backaction.

The measurement backaction can be quantified by ΔS_k. Consider first the extreme quantum limit, $J=\frac{1}{2}$. In that case, any measurement strong enough to localize the wave packet in position will necessarily resolve the two spinor components and cause *maximum backaction*, projecting the maximally entangled state on to one of the two spin states. This is accompanied by a maximal change in degree of entanglement ΔS_k from its maximum value (corresponding to a "Schrödinger cat" state) to zero (a product state). In contrast, in the large action limit, the measurement is only weakly

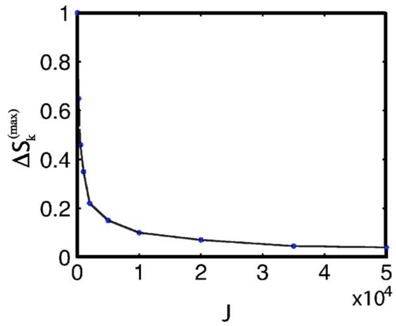

FIG. 3. (Color online) $\Delta S_k^{(\mathrm{max})}$, which is the upper bound on ΔS_k, decreases as $1/\sqrt{J}$ for a fixed measurement strength k.

projective (small backaction) on the spin system and correspondingly, does not change the entanglement as much. If we replace S_0 by $S_{\max}=1-1/(2J+1)$ in Eq. (5), we obtain an upper bound on ΔS_k. Figure 3 shows the steady-state behavior of this upper bound as a function of the size of the spin system. As the system is made more classical by increasing J and I_0, keeping k fixed, the maximum value of ΔS_k decreases rapidly, indicating that the backaction due to the measurement decreases, as is expected in the classical limit. On the other hand, keeping the actions fixed, as k increases, ΔS_k increases, reflecting the larger backaction on the spin system caused by a stronger measurement of the position. ΔS_k is thus a good quantitative measure of the small backaction condition required for the QCT and provides an alternative to the covariance matrix conditions obtained in Ref. [4].

This treatment of the QCT for bipartite systems sheds new light on the von Neumann paradigm of quantum measurements [12], comprising a system $\mathcal{B}$ and apparatus $\mathcal{A}$, followed by observation of the apparatus. The standard paradigm involves *strong* projective measurement, leading to strong backaction of the system when there is large entanglement between the two parts. Tracing over the system, quantum interference effects between the different states of the apparatus are removed, at which point they can be considered as classical alternatives. The treatment here extends this paradigm to the regime of weak measurement on the system—a general POVM (Positive Operator-Valued Measure). Our results show that although our goal is to achieve *weak backaction* on the system (the opposite regime of the standard von Neumann paradigm), we still require large entanglement between the system and probe in order to achieve a measurement record that evolves according to classical equations of motion. These results underscore the subtle relations between dynamics and states in defining the QCT [13].

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