Constructing N-qubit entanglement monotones from antilinear operators

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We present a method to construct entanglement measures for pure states of multipartite qubit systems. The key element of our approach is an antilinear operator that we call "comb". For qubits (or spin 1/2) the combs are automatically invariant under $SL(2,\mathbb{C})$. This implies that the filters obtained from the combs are entanglement monotones by construction. We give alternative formulas for the concurrence and the three-tangle as expectation values of certain antilinear operators. As an application we discuss inequivalent types of genuine four-qubit entanglement.

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Entanglement is one of the most striking features of quantum mechanics, but it is also one of its most counterintuitive consequences of which we still have rather incomplete knowledge [1]. Although the concentrated effort during the past decade has produced impressive progress, there is no general qualitative and quantitative theory of entanglement.

A pure quantum-mechanical state of distinguishable particles is called disentangled with respect to a given partition $\mathcal{P}$ of the system if and only if it can be written as a tensor product of the parts of this partition. In the opposite case, the state must contain some finite amount of entanglement. The question then is to characterize and quantify this entanglement.

As to *measuring* the amount of entanglement in a given pure multipartite state, the first major step was made by Bennett *et al.* [2] who discovered that the partial entropy of a party in a bipartite quantum state is a measure of entanglement. It coincides (asymptotically) with the entanglement of formation (i.e., the number of Einstein-Podolsky-Rosen pairs required to prepare a given state). Subsequently, the entanglement of formation of a two-qubit state was related to the concurrence [3,4]. Interestingly, by exploiting the knowledge of the mixed-state concurrence, a measure for three-partite pure states could be derived, the so-called "three tangle" τ_3 [5]. In terms of the coefficients of the wave function $\{\psi_{000},\psi_{001},\ldots,\psi_{111}\}$ in the standard basis it reads

$$\tau_3 = |d_1 - 2d_2 + 4d_3|,$$

$$d_1 = \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2,$$

$$d_2 = \psi_{000}\psi_{111}\psi_{011}\psi_{100} + \psi_{000}\psi_{111}\psi_{101}\psi_{010} + \psi_{000}\psi_{111}\psi_{110}\psi_{001} + \psi_{011}\psi_{100}\psi_{101}\psi_{010} + \psi_{011}\psi_{100}\psi_{110}\psi_{001} + \psi_{101}\psi_{010}\psi_{110}\psi_{001},$$

$$d_3 = \psi_{000}\psi_{110}\psi_{101}\psi_{011} + \psi_{111}\psi_{001}\psi_{010}\psi_{100}.$$

This was a remarkable step since, loosely speaking, it opened the path to studying multipartite entanglement on solid grounds. Further, it was noticed by Uhlmann that antilinearity is an important property of operators that measure entanglement [6]. A particularly interesting consequence of the three-tangle formula was presented by Dür *et al.* who found that there are two inequivalent classes of states with three-party entanglement [7].

Another important aspect of the research on entanglement measures was the question regarding the requirements for a function that represents an entanglement monotone [8]. It turned out that the essential property to be satisfied is non-increasing behavior on average under stochastic local operations and classical communication (SLOCC) [7,9]. Later, Verstraete *et al.* demonstrated that, in general, an entanglement monotone can be obtained from any homogeneous positive function of pure-state density matrices that remain invariant under determinant-one SLOCC operations [10].

Despite the enormous effort, the only truly operational entanglement measure for arbitrary mixed states at hand, up to now, is the concurrence. For pure states we have a slightly farther view up to systems of two qutrits [11,12], and for three qubits, due to the three-tangle. Various multipartite entanglement measures for pure states have been proposed; but most of these measures do not yield zero for all possible product states (e.g., Refs. [13–16]).

This motivated the quest for an operational entanglement measure based on the requirement that it be zero for product states (not only for completely separable pure states). In particular, the goal has been to explore the idea that entanglement monotones are related to antilinear operators as pointed out for the concurrence by Uhlmann [6]. Here we show that it is possible to construct a filter, i.e., an operator that has zero expectation value for all product states. It will turn out that these filters are entanglement monotones by construction. Interestingly, the two-qubit concurrence and the three tangle have various equivalent filter representations (see below). In order to illustrate the application of the method to a nontrivial example, we will present filters for four-qubit states that are able to distinguish inequivalent types of genuine four-qubit entanglement. We use the term "genuine N-qubit entanglement" in a more restricted sense than, e.g., in Ref. [7]: a state with only genuine N-partite entanglement does not contain any genuine $(N-k)$-partite entanglement (or "subtangle") with $1 \leqslant k \leqslant N-2$. In this sense the only class of three-qubit states with genuine three-partite entanglement is represented by the GHZ state.

Combs and filters. The basic concept is that of a comb. The name is with reference to the hairy-ball theorem: if an odd-dimensional sphere is covered with hair, there is no continuous way of combing the hair so that it lies flat at every point. We define a comb of first order as an antilinear operator A with zero expectation value for all states of a certain Hilbert space $\mathcal{H}$. That is,

$$\langle\psi|A|\psi\rangle = \langle\psi|LC|\psi\rangle = \langle\psi|L|\psi^*\rangle \equiv 0 \tag{1}$$

for all $|\psi\rangle \in \mathcal{H}$, where L is a linear operator and C is the complex conjugation. Here A necessarily has to be antilinear (a linear operator with this property is zero itself). For simplicity we abbreviate

$$\langle\psi|LC|\psi\rangle =: \langle L\rangle_C. \tag{2}$$

Note that the complex conjugation is *included* in the definition of the expectation value $\langle\cdots\rangle_C$ in Eq. (2).

We will use the comb operators[1] in order to construct the desired filters which are antilinear operators whose expectation values vanish for all product states. While a comb is a local, i.e., a single-qubit operator, a filter is a nonlocal operator that acts on the whole multiqubit state. It is worth mentioning already at this point that such a filter is invariant under $\mathcal{P}$-local unitary transformations if the combs have this property. Even more, it is invariant under the complex extension of the corresponding unitary group which is isomorphic to the special linear group. Since the latter represents the SLOCC operations for qubits [7,9], the filters will be entanglement monotones by construction.

In this work, we restrict our focus to multipartite systems of qubits (i.e., spin 1/2). The local Hilbert space is $\mathcal{H}_j = \mathbb{C}^2 =: \mathfrak{h}$ for all j. We need the Pauli matrices $\sigma_0 := \mathbb{1}$, $\sigma_1 := \sigma_x$, $\sigma_2 := \sigma_y$, and $\sigma_3 := \sigma_z$. It is straightforward to verify that the only single-qubit comb of first order is the operator σ_y:

$$\langle\psi|\sigma_y C|\psi\rangle = \langle\sigma_y\rangle_C \equiv 0.$$

Notice that any tensor product $f(\{\sigma_\mu\}) := \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n}$ with an odd number N_y of σ_y is a n-qubit comb. This can be seen immediately from $\langle f(\{\sigma_\mu\})\rangle_C \equiv \langle\psi|f(\{\sigma_\mu\})|\psi^*\rangle = (\langle\psi^*|f(\{\sigma_\mu^*\})|\psi\rangle)^* = (-1)^{N_y}\langle\psi|f(\{\sigma_\mu\})^\dagger|\psi^*\rangle = (-1)^{N_y}\langle f(\{\sigma_\mu\})\rangle_C$. Since its expectation value is a bi(antilinear) expression in the coefficients of the state we denote it a comb of order 1. In general we will call a comb to be of order n if its expectation value is $2n$ linear in the coefficients of the state. There is one independent single-qubit comb which is of second order. One can verify that for an arbitrary single-qubit state

$$0 = \langle\sigma_\mu\rangle_C\langle\sigma^\mu\rangle_C := \sum_{\mu,\nu=0}^{3} \langle\sigma_\mu\rangle_C g^{\mu,\nu}\langle\sigma_\nu\rangle_C, \tag{3}$$

with $g^{\mu,\nu} = \mathrm{diag}\{-1,1,0,1\}$ being very similar to the Minkowski metric.

[1]If the antilinear operator $A = LC$ is a comb (with the complex conjugation C), for the sake of brevity we will also call the linear operator L a comb.

It will prove useful to introduce the embedding

$$\mathcal{E}_n: \begin{array}{c} \mathcal{H} \hookrightarrow \mathfrak{H}_n = \mathcal{H}^{\otimes n}, \\ |\psi\rangle \to \mathcal{E}_n|\psi\rangle = |\psi\rangle^{\otimes n}. \end{array} \tag{4}$$

Further define the product $\bullet$ for operators $O, P: \mathcal{H} \to \mathcal{H}$ such that

$$O \bullet P: \begin{array}{c} \mathfrak{H}_2 \to \mathfrak{H}_2, \\ O \bullet P\mathcal{E}_2(|\psi\rangle) = O|\psi\rangle \otimes P|\psi\rangle. \end{array} \tag{5}$$

Then we have the single-site ($\mathcal{H} = \mathbb{C}^2$) comb σ_y for $\mathfrak{H}_1 = \mathcal{H}$ and $\sigma_\mu \bullet \sigma^\mu$ for $\mathfrak{H}_2$. We will discuss in more detail below how one can see that both σ_y and $\sigma_\mu \bullet \sigma^\mu$ are invariant under SLOCC.

With these two one-site combs we are now equipped to construct filters for multipartite qubit systems. For n-qubit filters we will use the symbol $\mathcal{F}^{(n)}$. For two qubits the filters are

$$\mathcal{F}_1^{(2)} = \sigma_y \otimes \sigma_y, \tag{6}$$

$$\mathcal{F}_2^{(2)} = \frac{1}{3}(\sigma_\mu \otimes \sigma_\nu) \bullet (\sigma^\mu \otimes \sigma^\nu). \tag{7}$$

Both forms are explicitly permutation invariant, and they are filters since, if the state were a product, the combs would annihilate its expectation value. From the filters we obtain the pure-state concurrence in two different equivalent forms

$$C = |\langle\langle\mathcal{F}_1^{(2)}\rangle\rangle_C|,$$

$$C^2 = |\langle\langle\mathcal{F}_2^{(2)}\rangle\rangle_C| \equiv \frac{1}{3}|\langle\sigma_\mu \otimes \sigma_\nu\rangle_C\langle\sigma^\mu \otimes \sigma^\nu\rangle_C|. \tag{8}$$

While the first form in Eq. (8) has the well-known convex-roof extension of the pure-state concurrence via the matrix [3,4,6]

$$R = \sqrt{\rho}\sigma_y \otimes \sigma_y\rho^*\sigma_y \otimes \sigma_y\sqrt{\rho} \tag{9}$$

it can be shown that the second form in Eq. (8) leads to

$$R^2 = \sqrt{\rho}\sigma_\mu \otimes \sigma_\nu\rho^*\sigma_\kappa \otimes \sigma_\lambda\rho\, \sigma^\mu \otimes \sigma^\nu\rho^*\sigma^\kappa \otimes \sigma^\lambda\sqrt{\rho}. \tag{10}$$

Now let us consider the three-tangle [5]. For states of three qubits we find, e.g.,

$$\mathcal{F}_1^{(3)} = (\sigma_\mu \otimes \sigma_y \otimes \sigma_y) \bullet (\sigma^\mu \otimes \sigma_y \otimes \sigma_y), \tag{11}$$

$$\mathcal{F}_2^{(3)} = \frac{1}{3}(\sigma_\mu \otimes \sigma_\nu \otimes \sigma_\lambda) \bullet (\sigma^\mu \otimes \sigma^\nu \otimes \sigma^\lambda). \tag{12}$$

Both $\mathcal{F}_1^{(3)}$ and $\mathcal{F}_2^{(3)}$ are filters and the latter is explicitly permutation invariant. From these operators the pure-state three-tangle is obtained in the following way:

$$\tau_3 = |\langle\langle\mathcal{F}_1^{(3)}\rangle\rangle_C| = |\langle\langle\mathcal{F}_2^{(3)}\rangle\rangle_C| \tag{13}$$

Interestingly, *all* three-qubit filters reproduce the three-tangle as entanglement measure. We mention, however, that there is no immediate extension to mixed states as in the case of the "alternative" two-qubit concurrence, Eq. (10).

Invariance of filters under SLOCC. Up to here we have shown that the concepts of combs and filters reproduce the well-known pure-state entanglement measures of concurrence and three-tangle. Now we will briefly explain that the expectation values of N-qubit filter operators are invariant under SLOCC operations.

It has been demonstrated in Refs. [7,9,10] that invariance under SLOCC operations reduces to invariance with respect to the group $\mathrm{SL}(2,\mathbb{C})^{\otimes N}$. As to the single-qubit combs, it is easily verified that $V\sigma_y V^T=\sigma_y$ for any local transformation $V\in \mathrm{SL}(2,\mathbb{C})$. Further, there is an operator identity $(V\otimes V)\times(\sigma_\mu\otimes\sigma^\mu)(V^T\otimes V^T)=\sigma_\mu\otimes\sigma^\mu$ which expresses directly the $\mathrm{SL}(2,\mathbb{C})$ invariance of $\sigma_\mu\bullet\sigma^\mu$.

We can, therefore, conclude that tensor products of single-site combs are invariant with respect to $\mathrm{SL}(2,\mathbb{C})$ operations at each site. Hence, as long as a N-qubit filter operator is built from tensor products of single-site combs it will be invariant under SLOCC operations.

Note that the SLOCC invariance of a filter expectation value with respect to some state (which vanishes if there is a way of writing the state as a tensor product) means that this expectation value represents an entanglement monotone [8,9]. This imposes the question of what kind of entanglement is being measured by these quantities. Recall the case of three qubits [7] where there are two kinds of three-partite entanglement: GHZ-type (or genuine) entanglement which is detected by the three-tangle and W-type entanglement with zero three-tangle. As to the N-qubit case, we do not have a complete answer to the question above at this moment (although it is straightforward to write down N-qubit filter operators). There are, however, indications that the filters measure the maximally entangled states discussed, e.g., in Refs. [10,17]. In order to see this it is instructive to consider the four-qubit case.

Filters for four-qubit states. Classifications of four-qubit states with respect to their entanglement properties have been studied, e.g., in Refs. [18–20]. Here we introduce several four-qubit filter operators and study the classes of entangled states they are measuring.

A four-qubit filter has the property that its expectation value for a given state is zero if the state is separable, i.e., if there is a one-qubit or a two-qubit part which can be factored out (note that for a three-qubit filter it is enough to extract one-qubit parts only). An expression that obeys this requirement for any single qubit and any combination of qubit pairs is given by

$$\mathcal{F}_1^{(4)}=(\sigma_\mu\sigma_\nu\sigma_y\sigma_y)\bullet(\sigma^\mu\sigma_y\sigma_\lambda\sigma_y)\bullet(\sigma_y\sigma^\nu\sigma^\lambda\sigma_y). \tag{14}$$

Recall that any combination of the type $\sigma_\mu\sigma_y$ $(\mu\neq 2)$ represents a two-qubit comb. Note that the expectation value of an nth-order four-qubit filter has to be taken with respect to the corresponding $\mathfrak{H}_n$, see Eq. (4). It is straightforward to check that for a four-qubit GHZ state

$$|\Phi_1\rangle=\frac{1}{\sqrt{2}}(|0000\rangle+|1111\rangle) \tag{15}$$

we have $\langle\Phi_1|\mathcal{F}_1^{(4)}|\Phi_1^*\rangle=1$. However, there is another state for which $\langle\mathcal{F}_1^{(4)}\rangle_C$ does not vanish. For

$$|\Phi_2\rangle=\frac{1}{\sqrt{6}}(\sqrt{2}|1111\rangle+|1000\rangle+|0100\rangle+|0010\rangle+|0001\rangle) \tag{16}$$

we find $\langle\Phi_2|\mathcal{F}_1^{(4)}|\Phi_2^*\rangle=8/9$.

In addition to the third-order filter $\mathcal{F}_1^{(4)}$ there exist also filters of fourth order and of sixth order. Examples are

$$\mathcal{F}_2^{(4)}=(\sigma_\mu\sigma_\nu\sigma_y\sigma_y)\cdot(\sigma^\mu\sigma_y\sigma_\lambda\sigma_y)\cdot(\sigma_y\sigma^\nu\sigma_y\sigma_\tau)\cdot(\sigma_y\sigma_y\sigma^\lambda\sigma^\tau),$$

$$\mathcal{F}_3^{(4)}=\frac{1}{2}(\sigma_\mu\sigma_\nu\sigma_y\sigma_y)\cdot(\sigma^\mu\sigma^\nu\sigma_y\sigma_y)\cdot(\sigma_\rho\sigma_y\sigma_\tau\sigma_y)\cdot(\sigma^\rho\sigma_y\sigma^\tau\sigma_y)\cdot(\sigma_y\sigma_\rho\sigma_\tau\sigma_y)\cdot(\sigma_y\sigma^\rho\sigma^\tau\sigma_y). \tag{17}$$

While $\mathcal{F}_2^{(4)}$ measures only GHZ-type entanglement $(\langle\Phi_1|\mathcal{F}_2^{(4)}|\Phi_1^*\rangle=1)$ the sixth-order filter $\mathcal{F}_3^{(4)}$ has the nonzero expectation values $1/2$ for the GHZ state and 1 for yet another state,

$$|\Phi_3\rangle=\frac{1}{2}(|1111\rangle+|1100\rangle+|0010\rangle+|0001\rangle). \tag{18}$$

$\mathcal{F}_1^{(4)}$ and $\mathcal{F}_2^{(4)}$ have a zero expectation value for this state. Finally, it is not difficult to convince oneself that all four-qubit filters $\mathcal{F}_j^{(4)}$ $(j=1,2,3)$ have a zero expectation value for the four-qubit W state $1/2(|0111\rangle+|1011\rangle+|1101\rangle+|1110\rangle)$.

By analyzing the states $|\Phi_j\rangle$ we find that they are peculiar in the sense that the local density operators for each qubit are given by $\frac{1}{2}\mathbb{1}$. As also all other reduced density operators do not have any k tangle $(k\in\{2,3\})$ we conclude that the $|\Phi_j\rangle$ represent classes of genuine four-qubit entanglement. They are maximally entangled in the sense of Refs. [10,17]. Note that they cannot be transformed into one another by SLOCC operations: A state with a finite expectation value for one filter cannot be transformed by means of SLOCC operations into a state with zero expectation value for the same filter. For example, $\mathcal{F}_2^{(4)}$ detects the GHZ state $|\Phi_1\rangle$ but gives zero for the other two states. Therefore, the four-qubit entanglement in those states must be different from that of the GHZ state.

Hence, there are at least three inequivalent types of genuine entanglement for four qubits.[2] We mention that the three maximally entangled states $|\Phi_j\rangle$ are not distinguished by the classification for pure four-qubit states of Ref. [18]. This can be seen by computing the expectation values of the four-qubit filters and the reduced one-qubit density matrices for each of the nine class representatives of Ref. [18]. Only the classes 1-4 and 6 have non-vanishing "four tangle." The corresponding local density matrices can be completely mixed *only* for class 1. Therefore, all three states $|\Phi_j\rangle$ must belong to that class.

Conclusions. We have presented an efficient way of generating entanglement monotones. It is based on operators

[2]In fact, there are *exactly* three maximally entangled states for four qubits. This will be discussed in a forthcoming publication.

which we called "filters." The expectation values of these operators are zero for all possible product states, not only for the completely factoring case. The building blocks of the filters (denoted "combs") guarantee invariance under $SL(2,\mathbb{C})^{\otimes N}$ for qubits. As a consequence, all filters are automatically entanglement monotones. They are measures of genuine multipartite entanglement. This circumvents the difficult task to construct entanglement monotones from the essentially known (linear) local unitary invariants.

Further advantages of our approach lie in the feasibility of constructing specific monotones that vanish for certain separable (pure) states and in the applicability of this concept to partitions into subsystems other than qubits (i.e., qutrits, …). Although only filters for four qubits are given explicitly in this work, it is possible to build filters for any $N>4$ from the two presented single-qubit combs in a straightforward manner.

As an immediate result of our method the concurrence for pure two-qubit states is reproduced. Moreover, we have found an alternative expression for the concurrence with the corresponding convex roof extension. The application of the method to pure three-qubit states yields several operator-based expressions for the three tangle, including an explicitly permutation-invariant form. Finally we have given explicit expressions for four-qubit entanglement measures that detect three different types of genuine four-qubit entanglement. As we have found, these types of genuine four-qubit entanglement are not distinguished by the classification of four-qubit states in Ref. [18].

As to N-qubit systems, there remain various interesting questions. Clearly, it would be desirable to have a recipe how to build invariant combs for more complicated systems (e.g., higher spin). It would also be interesting to know what characterizes a complete set of filters for a given N. While it is not obvious how the convex roof construction for two qubits can be generalized, we believe that the operator form of the N tangles in terms of filters makes it easier to solve this problem. The question is whether there is a systematic way to obtain a convex-roof construction for a given filter with general multilinearity. Returning to the case of two qubits, one may conclude that the crucial quality of $\sigma_y \otimes \sigma_y$ (together with the complex conjugation) in Wootters' concurrence formula is that it is a filter constructed with the comb $\sigma_y C$, rather than the time-reversal property of this operator.

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