Decoherence time in self-induced decoherence

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A general method for obtaining the decoherence time in self-induced decoherence is presented. In particular, it is shown that such a time can be computed from the poles of the resolvent or of the initial conditions in the complex extension of the Hamiltonian's spectrum. Several decoherence times are estimated: $10^{-13}-10^{-15}$ s for microscopic systems, and $10^{-37}-10^{-39}$ s for macroscopic bodies. For the particular case of a thermal bath, the order of magnitude of our results agrees with that obtained by the einselection (environment-induced decoherence) approach.

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I. INTRODUCTION

The phenomenon of decoherence is usually considered as a relevant element for understanding how the classical world emerges from an underlying quantum realm. In a first period, decoherence was explained as the result of the destructive interference of the off-diagonal elements of a density matrix (see Refs. [1,2]); however, this line of research was abandoned due to technical difficulties derived from the formalism used to describe the process. As a consequence, decoherence began to be conceived as produced by the interaction between a system and its environment. This approach gave rise to the *environment-induced decoherence* (EID) program, based on the works of Zeh [3–5] and later developed by Zurek and co-workers [6–12]. Although many relevant results have been obtained by means of environment-induced decoherence, this approach still involves certain unsolved problems (see Refs. [13,14]):

(i) Einselection is based on the decomposition of the system into a relevant part, the *proper system,* and an irrelevant part, the *environment.* This decomposition is not always possible, as in the case of the universe. In fact, Zurek himself considers the criticism: "… the Universe as a whole is still a single entity with no 'outside' environment, and therefore any resolution involving its division is unacceptable" (Ref. [15], p. 181). The same problem appears in any closed system (and therefore with no interaction with an environment) that becomes classical. In fact, if "…the existence of emergent classically will be always accompanied by other manifestations of openness such a dissipation of energy into the environment" (Ref. [11], p. 6), the problem is to explain why many systems behave classically maintaining their energy constant.

(ii) The einselection approach does not provide a clear definition of the "cut" between the proper system and its environment. In fact, as Zurek himself admits, "In particular, one issue which has been often taken for granted is looming big, as a foundation of the whole decoherence problem. It is the question of what the 'system' is which plays such a crucial role in all the discussions of the emergent classicality. This question was raised earlier but the progress to date has been slow at best" (Ref. [16] p. 122).

(iii) In the einselection approach, the definition of the basis where the system becomes classical, i.e., the pointer basis, relies on the "predictability sieve" which would produce the set of the most stable states. But this definition seems very difficult to implement, at least in a generic case. In fact, the basis vectors are only good candidates for reasonable stable states [17].

As the result of these and other difficulties, a number of alternative accounts of decoherence have been proposed [18–24].

In a series of papers [13,25–38] we have returned to the initial idea of the destructive interference of the off-diagonal terms of the density matrix, but now on the basis of the van Hove formalism [39–43]. We have called this new approach "self-induced decoherence" (SID) because, from this viewpoint, decoherence is not produced by the interaction between a system and its environment, but results from the own dynamics of the whole quantum system governed by a Hamiltonian with continuous spectrum. The aim of this paper is to present a general method for obtaining the decoherence time in self-induced decoherence. In particular, we will show that such a time can be computed from the poles of the resolvent or of the initial conditions in the complex extension of the Hamiltonian's spectrum. The general formalism has been developed in Refs. [44,45], but in the context of a discussion about the nature and properties of Gamow vectors. Here we will adopt that formalism for the computation of the decoherence time.

Two comments are in order:

Comment 1. The time evolution of the decaying off-diagonal terms of the density matrix has three periods. (i) A short initial (nonexponential) Zeno period, where the time derivative is zero for $t=0$ [46]. (ii) Then, a long exponential decaying period with a factor e^{-t/t_D}. This is the only period

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studied in this paper because it allows us to define an exponential decoherence characteristic time t_D by means of the poles method. (iii) A final Khalfin period [47], where the evolution has a long powers series tail that essentially depends on the interaction.

These three periods are studied in Refs. [27,48–50] with the same methods used in the present paper (see also the coincidence of our results with those of Ref. [51]). These three periods are ubiquitous in decaying processes.[1]

Comment 2. The models studied by EID and SID are quite different. In the typical models treated by EID, the proper system interacts with an environment, the evolution spectrum is usually discrete, only decoherence in the proper system is considered since the environment is traced away, and normally there is dissipation of energy from the system to the environment; this means that EID is a dissipative decoherence theory. On the contrary, SID models are closed systems and therefore there is no dissipation, the evolution spectrum is continuous, and the decoherence of the whole system is considered: SID is a nondissipative decoherence theory. Unfortunately, the comparative study between EID and SID is only in a first stage in Ref. [53] and, in certain sense, in the present paper. More generally, there is no comparative study between dissipative theories (like Refs. [3–5],) and nondissipative ones [54–57].[2] Therefore the comparison between the decoherence times obtained by EID and SID is a comparison of data resulting from different models. As a consequence, we will try to find obviously not an exact coincidence, but only a certain similarity in orders of magnitude. Nevertheless, to the extent that EID is an accepted theory (albeit the problems listed above), this comparison is interesting and necessary, and it is the best we can do while we wait for more model research and more experimental data.

The paper is organized as follows. In Sec. II we briefly review the formal basis of self-induced decoherence. In Sec. III we show that the decoherence time can be computed as the characteristic decaying time of the expectation value equation, and that this time can be obtained in terms of the poles of the involved functions. In Sec. IV we find the origin of these poles in the resolvent of the Hamiltonian or in the initial conditions. In Secs. V and VI we estimate the decoherence time in microscopic systems and in macroscopic bodies, respectively. Section VII is devoted to study the case of a thermal bath, where the decoherence time is estimated and compared with the corresponding results obtained by the einselection approach. After the conclusions, we include two appendixes: in Appendix A we introduce a necessary mathematical remark, and in Appendix B we sketch a model with two characteristic times: decoherence and relaxation.

II. SELF-INDUCED DECOHERENCE

In this section we will present the formalism of self-induced decoherence by means of a very simple case. We refer the reader to our previous papers; in particular, for more complex cases, see Ref. [30], and for a conceptual discussion about the physical meaning of self-induced decoherence, see Ref. [13].

Let us consider a quantum system endowed with a Hamiltonian with continuous spectrum,

$$H = \int_0^\infty \omega |\omega\rangle\langle\omega| d\omega. \tag{1}$$

A generic observable reads

$$O = \int_0^\infty \int_0^\infty \tilde{O}(\omega,\omega') |\omega\rangle\langle\omega'| d\omega d\omega', \tag{2}$$

where $O(\omega,\omega')$ is any kernel or distribution. Since the algebra of these observables is too large for our purposes, we only consider the van Hove operators such that

$$\tilde{O}(\omega,\omega') = O(\omega)\delta(\omega-\omega') + O(\omega,\omega'), \tag{3}$$

where $O(\omega,\omega')$ is a regular function. Then,

$$O = \int_0^\infty O(\omega)|\omega\rangle\langle\omega| d\omega + \int_0^\infty \int_0^\infty O(\omega,\omega')|\omega\rangle\langle\omega'| d\omega d\omega', \tag{4}$$

where the first term of the right-hand side (rhs) is the *singular term* O_S, and the second term is the *regular term* O_R. These observables belong to an algebra $\hat{\mathcal{A}}$, that we will call *van Hove algebra*: they are the self-adjoint operators of $\hat{\mathcal{A}}$ and belong to a space of operators $\hat{\mathcal{O}}$. The basis of $\hat{\mathcal{A}}$ is $\{|\omega), |\omega,\omega')\}$, where $|\omega) = |\omega\rangle\langle\omega|$ and $|\omega,\omega') = |\omega\rangle\langle\omega'|$.

Let us now consider the space $\hat{\mathcal{O}}'$, that is, the dual of space $\hat{\mathcal{O}}$ with basis $\{(\omega|, (\omega,\omega'|\}$. A generic state belonging to $\hat{\mathcal{O}}'$ reads

$$\rho = \int_0^\infty \rho(\omega)(\omega| d\omega + \int_0^\infty \int_0^\infty \rho(\omega,\omega')(\omega,\omega'| d\omega d\omega' \tag{5}$$

and must satisfy the usual constraints: $\rho(\omega)$ must be real and positive and $\int_0^\infty \rho(\omega) d\omega = 1$. We also introduce the require-

[1] In some cases where an approximate analytical solution can be obtained by means of the EID approach, these periods can be easily identified. For instance, in Ref. [52], when the distribution of the interaction couplings is Lorentzian, the time evolution is exponential, but when these couplings are constant, an approximately Gaussian evolution is obtained. In the Gaussian regime, the initial period can be conceived as a Zeno period where the time derivative is zero for $t=0$; after the first inflection point, the evolution can be approximated by an exponential; and at infinity, the Gaussian evolution and all its derivatives vanish, so we find a final Khalfin constat zero tail (see also Fig. 29 of Ref. [48] as an illustration).

[2] It is interesting to note that, since in SID the whole system (proper system plus environment) reaches an equilibrium decohered state, all the parts of the whole system (in particular, the proper system) also reach a final equilibrium state. Therefore the proper system has a final ρ of equilibrium whose eigenbasis can be obtained: the proper system decoheres in that basis. This means that, when the conditions for the application of SID are satisfied in the whole system (see Ref. [30]), the SID approach should explain the decoherence of the proper system as treated by the EID approach. Of course, this is only a preliminar suggestion that we will develop elsewhere.

ment that $\rho(\omega,\omega')$ be a regular function. Again, the first term of the rhs of Eq. (5) is the *singular term* ρ_S, and the second term is the *regular term* ρ_R.

The expectation value of the observable O in the state ρ results from the action of the functional ρ on the operator O, $(\rho|O)$,

$$\langle O\rangle_\rho = \int_0^\infty \overline{\rho(\omega)}O(\omega)d\omega + \int_0^\infty\int_0^\infty \overline{\rho(\omega,\omega')}O(\omega,\omega')d\omega d\omega', \tag{6}$$

where functions $\overline{\rho(\omega)}$ and $O(\omega)$ are such that the first integral is well defined. The time evolution of this last equation reads

$$\langle O\rangle_{\rho(t)} = \int_0^\infty \overline{\rho(\omega)}O(\omega)d\omega + \int_0^\infty\int_0^\infty \overline{\rho(\omega,\omega')}O(\omega,\omega')e^{i[(\omega-\omega')/\hbar]t}d\omega d\omega'. \tag{7}$$

Since the first term of the rhs is time-constant and the second is a function of time, we will call them *constant term* and *fluctuating term*, respectively.

The Riemann-Lebesgue theorem, which mathematically expresses the phenomenon of destructive interference, states that, if $f(\nu)\in\mathbb{L}_1$,

$$\lim_{t\to\infty}\int d\nu f(\nu)e^{i\nu t} = 0. \tag{8}$$

Therefore, if the function $\overline{\rho(\omega,\omega')}O(\omega,\omega')$ of Eq. (7) is $\mathbb{L}_1$ in variable $\nu=\omega-\omega'$, we can apply the Riemann-Lebesgue theorem,

$$\lim_{t\to\infty}(\rho_R(t)|O_R) = \lim_{t\to\infty}\int_0^\infty\int_0^\infty \overline{\rho(\omega,\omega')}O(\omega,\omega')e^{i[(\omega-\omega')/\hbar]t} \times d\omega d\omega' = 0. \tag{9}$$

As a consequence,

$$\lim_{t\to\infty}\langle O\rangle_{\rho(t)} = \lim_{t\to\infty}(\rho(t)|O) = \int_0^\infty \overline{\rho(\omega)}O(\omega)d\omega. \tag{10}$$

This last equation can be expressed as a weak limit,

$$W-\lim_{t\to\infty}(\rho(t)| = (\rho_*| = \int_0^\infty \overline{\rho(\omega)}(\omega|d\omega, \tag{11}$$

where $(\rho_*|$ has only singular-diagonal terms.[3]

It is not difficult to see that this approach to decoherence avoids the problems of the einselection program. In fact, self-induced decoherence can be applied to closed systems as the universe [34], the problem of the "cut" between the proper system and its environment is absent [13], and the pointer basis is perfectly defined (e.g., the energy eigenbasis in the above example) [28]. Nevertheless, three points need to be emphasized:

(i) In this approach, the coarse graining necessary to turn the usual unitary time evolutions of quantum mechanics into nonunitary time evolutions is not explicit in the formalism. However, the choice of a particular algebra $\hat{\mathcal{A}}$, the van Hove algebra, among all possible algebras and the systematic use of expectation values $\langle O\rangle_{\rho(t)}=(\rho(t)|O)$ play the role of a coarse graining. In fact, we can define the projector $\Pi=|O)(\rho_0|$, with $|O)\in\hat{\mathcal{A}}$ and $(\rho_0|O)=1$, that projects $(\rho(t)|$ as $(\rho(t)|\Pi=\langle O\rangle_{\rho(t)}(\hat{\rho}_0|$ and allows us to translate our results in the language of projectors: in this case, from Eq. (10) we will obtain $\lim_{t\to\infty}(\rho(t)|\Pi=(\rho_*|\Pi$, where $(\rho_*|$ is diagonal. This projection is what breaks the unitarity of the primitive evolution. A detailed discussion on this point can be found in Refs. [13,38].

(ii) As a consequence of the Riemann-Lebesgue theorem, full decoherence occurs at $t\to\infty$. However, as in any exponential decaying process, there is a characteristic decaying time that can be considered as the time at which, in practice, the decaying is approximately completed. In the next sections we will compute the decoherence time of a self-induced decoherence process as the characteristic decaying time of the fluctuating term in the expression of $\langle O\rangle_{\rho(t)}$ of Eq. (7).

(iii) In the model presented here, full decoherence is obtained for $t\to\infty$. However, as explained in Ref. [30], if the total Hamiltonian has more than one discrete eigenvalue nonoverlapping with the continuous spectrum, there is no decoherence in the cross terms of the discrete eigenvalues.

III. POLES IN THE EXPECTATION VALUE EQUATION

In order to study and compute the decoherence time, we will use the standard theory of analytical continuation in scattering quantum theory (see, e.g., Refs. [58,59]) and its extension to the Liouville–von Neumann space (see, e.g., Refs. [26,44]). By means of this theory, we can compute the decoherence time in terms of the poles corresponding to the functions involved in the fluctuating term of $\langle O\rangle_{\rho(t)}$ [see Eq. (7)]:

$$(\rho_R(t)|O_R) = \int_0^\infty\int_0^\infty \overline{\rho(\omega,\omega')}O(\omega,\omega')e^{i[(\omega-\omega')/\hbar]t}d\omega d\omega'. \tag{12}$$

In this equation, we can introduce the following change of variables (see Appendix A):

$$\lambda = \frac{1}{2}(\omega+\omega'), \quad \nu = \omega-\omega', \quad d\omega d\omega' = Jd\lambda d\nu = d\lambda d\nu. \tag{13}$$

Then,

[3]Here we are working in the Schrödinger picture. However, analog results can be obtained in the Heisenberg picture:

$$\lim_{t\to\infty}\langle O(t)\rangle_\rho = \lim_{t\to\infty}(\rho|O(t)) = \int_0^\infty \overline{\rho(\omega)}O(\omega)d\omega = (\rho|O_*).$$

$$(\rho_R(t)|O_R) = \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu \overline{\rho'(\nu,\lambda)} O'(\nu,\lambda) e^{i(\nu/\hbar)t}, \quad (14)$$

where $\overline{\rho'(\nu,\lambda)} = \overline{\rho(\omega,\omega')}$, $O'(\nu,\lambda) = O(\omega,\omega')$ and the new limits of the integrals are due to the fact that ω, $\omega' \geqslant 0$. Now we promote the real variable ν to a complex variable Z; if the function $\overline{\rho'(Z,\lambda)} O'(Z,\lambda)$ has *no poles* in the upper Z half plane, we obtain

$$(\rho_R(t)|O_R) = \int_0^\infty d\lambda \int_{C(-2\lambda,2\lambda)} dz \overline{\rho'(Z,\lambda)} O'(Z,\lambda) e^{i(Z/\hbar)t}, \quad (15)$$

where $C(-2\lambda, 2\lambda)$ is any curve that goes from -2λ to 2λ by the upper complex half plane. If the function $\rho'(Z,\lambda)O'(Z,\lambda)$ has, say, a pole at $Z_0 = \tilde{\omega} + i\gamma$ in the upper half plane, we can, as usual, decompose $C(-2\lambda, 2\lambda) = \Gamma(-2\lambda, 2\lambda) \cup C_{Z_0}$, where C_{Z_0} is a residue circle around the pole Z_0 and $\Gamma(-2\lambda, 2\lambda)$ is the remaining "background" curve. If, as usual, we neglect the background, only the factor $e^{i(Z/\hbar)t}$ becomes relevant at the pole Z_0; this factor reads

$$e^{i(Z_0/\hbar)t} = e^{i[(\tilde{\omega}+i\gamma)/\hbar]t} = e^{i(\tilde{\omega}/\hbar)t} e^{-(\gamma/\hbar)t}, \quad (16)$$

where $e^{-(\gamma/\hbar)t}$ is a dumping factor appearing in the regular fluctuating term of $\langle O \rangle_{\rho(t)} = (\rho(t)|O)$. Therefore the decoherence time can be computed as the characteristic decaying time of the process as

$$t_D = \frac{\hbar}{\gamma}. \quad (17)$$

Let us note that, up to this point, we have worked in the eigenbasis of the complete Hamiltonian H, that we will call $\{|\omega\rangle^+\}$.[4]

Once the decoherence time has been computed, a single question remains: what is the origin of the pole in the product $\overline{\rho'(Z,\lambda)} O'(Z,\lambda)$? We will address this problem in the next section.

IV. ORIGIN OF THE POLES

In the physical evolutions we are interested on, there are "free" periods with no decoherence (e.g., very long ones like the "in" and "out" periods used to modelize a scattering process, or short periods but long enough to fix the initial conditions for an "interaction" period) and interaction periods where decoherence occurs. On this basis, we will first consider in detail the free period governed by a free Hamiltonian H_0 (case 1), and then the interaction period governed by a "perturbed" Hamiltonian $H = H_0 + V$ (case 2).

Case 1. Let us consider the free case with a free Hamiltonian H_0, and call $\{|E\rangle\}$ the eigenbasis of H_0, where $0 \leqslant E < \infty$. Decoherence is due to the vanishing of the fluctuating term of $\langle O \rangle_{\rho(t)}$ [see Eq. (12)],

$$\lim_{t\to\infty}(\rho_R(t)|O_R) = \lim_{t\to\infty} \int_0^\infty \int_0^\infty \overline{\rho(E,E')} O(E,E') e^{i[(E-E')/\hbar]t} \times dE dE' = 0, \quad (18)$$

where $\rho(E,E') = \langle E|\rho|E'\rangle$ are the coordinates of the state, and $O(E,E') = \langle E|O|E'\rangle$ are the coordinates of the considered observable, both in the basis $\{|E\rangle\}$. Calling, as above,

$$\lambda_0 = \frac{1}{2}(E + E'), \quad \nu_0 = E - E'$$

the limit reads

$$\lim_{t\to\infty}(\rho_R(t)|O_R) = \int_0^\infty d\lambda_0 \lim_{t\to\infty} \int_{-2\lambda_0}^{2\lambda_0} d\nu_0 \overline{\rho(\lambda_0 + \nu_0/2, \lambda_0 - \nu_0/2)} \times O(\lambda_0 + \nu_0/2, \lambda_0 - \nu_0/2) e^{i(\nu_0/\hbar)t}. \quad (19)$$

Proceeding in the same way as in the previous section, we arrive at an expression similar to Eq. (15),

$$\lim_{t\to\infty}(\rho_R(t)|O_R) = \int_0^\infty d\lambda_0 \times \lim_{t\to\infty} \int_{C(-2\lambda_0, 2\lambda_0)} dZ \overline{\rho(\lambda_0 + Z/2, \lambda_0 - Z/2)} \times O(\lambda_0 + Z/2, \lambda_0 - Z/2) e^{i(Z/\hbar)t}, \quad (20)$$

where some poles could be found (see details in Refs. [26,44]). However, since this is a free period, there should be no decoherence, i.e., the decoherence time should be infinite. This means that we have to adjust our theory to this physical fact by asking the following conditions for the complex continuation of the coordinates $\rho(E,E') = \langle E|\rho|E'\rangle$, $O(E,E') = \langle E|O|E'\rangle$:

(i) Condition 1: In the free eigenbasis $\{|E\rangle\}$, the coordinates of the observable O, $O(\lambda_0 + Z/2, \lambda_0 - Z/2)$, have no poles in the upper half plane. This is a natural requirement because, if not, the observable O would introduce by itself a finite decoherence time *for any state* and *any evolution Hamiltonian*, which is certainly not a physical situation.

(ii) Condition 2: In the free eigenbasis $\{|E\rangle\}$, the coordinates of the state ρ, $\rho(\lambda_0 + Z/2, \lambda_0 - Z/2)$, have no poles in the upper half plane.[5]

If these two conditions are satisfied, the function $\overline{\rho(\lambda_0 + Z/2, \lambda_0 - Z/2)} O(\lambda_0 + Z/2, \lambda_0 - Z/2)$ will have no poles in the upper half plane. In this case, the decoherence time is *infinite* and decoherence is only nominal, as one would have expected in a free evolving situation.

Case 2. Once we have "calibrated" our theory in order to satisfy the physical condition according to which a free

[4] Strictly speaking, we will find a double basis $\{|\omega\rangle^\pm\}$ corresponding to a decaying process and a growing process, respectively. But since we are only interested in the former one, we will use only $\{|\omega\rangle^+\}$ and work in the lower half plane, trying to find poles in the second sheet. In Sec. II, we have simply called $|\omega\rangle$ the eigenvectors $|\omega\rangle^+$ [see Eq. (1)].

[5] Nevertheless, in Sec. VII we will see that, in some cases, there are poles in the free period, e.g., in the case of the evolution of a system with a thermal bath.

evolving system does not decohere, we will consider the process in the interaction period. The total Hamiltonian now reads

$$H = H_0 + V = \int_0^\infty \omega|\omega\rangle\langle\omega| d\omega + \int_0^\infty \int_0^\infty V(\omega,\omega)|\omega\rangle \times\langle\omega'| d\omega d\omega', \tag{21}$$

where $\{|\omega\rangle\}$ is the eigenbasis of H_0 (here we have replaced E with ω to emphasize that now this is an interaction period). Let us consider the resolvent, namely, the *complex value operator* (see Ref. [60])

$$R(z) = (z - H)^{-1}, \tag{22}$$

i.e., the analytical continuation to the lower second sheet of

$$R(\omega) = (\omega + i0 - H)^{-1}. \tag{23}$$

The poles of $R(z)$ are known as the poles of the resolvent, and they coincide with those of the S matrix. In fact, for the Hamiltonian (21), the S-matrix coefficients read (see Ref. [61])

$$S(\omega) = 1 - 2\pi i\langle\omega|V|\omega\rangle - 2\pi i\langle\omega|V\frac{1}{\omega + i0 - H}V|\omega\rangle. \tag{24}$$

Then, if V is well behaved, i.e., the analytical continuation of $V|\omega\rangle$ is a vector value analytical function [60], functions $R(\omega)$ and $S(\omega)$ have the same poles.[6]

As before, for the sake of simplicity we will assume that $R(z)$ has just one pole z_0,[7] and we will only consider pure states. On this basis we will show that, if the continuation to the lower second sheet of $\langle\varphi|\omega\rangle$ has no poles, the continuation to the lower second sheet of $\langle\varphi|\omega\rangle^+$ gets the pole z_0 of function $R(z)$. In fact, from the Lippmann-Schwinger equation we know that there are two eigenbases for H (see Ref. [61]),

$$|\omega\rangle^\pm = |\omega\rangle + \frac{1}{\omega \pm i0 - H}V|\omega\rangle, \tag{25}$$

where $\pm i0$ symbolizes the analytical continuation to the lower (upper) half plane in the second sheet. As explained, we will consider only the decaying case and therefore we will only use $\{|\omega\rangle^+\}$. Then,

$$\langle\varphi|\omega\rangle^+ = \langle\varphi|\omega\rangle + \langle\varphi|\frac{1}{\omega + i0 - H}V|\omega\rangle \tag{26}$$

and we can make the analytical continuation of this equation to the lower second sheet,

$$\langle\varphi|z\rangle^+ = \langle\varphi|z\rangle + \langle\varphi|\frac{1}{z - H}V|z\rangle. \tag{27}$$

If we suppose, as before, that V is a well behaved vector value function, then even if $\langle\varphi|\omega\rangle$ has no poles (as required in case 1), $\langle\varphi|\omega\rangle^+$ has a pole at z_0, coming from the resolvent term. The same applies to ${}^+\langle z|\psi\rangle$: even if $\langle z|\psi\rangle$ has no poles, ${}^+\langle z|\psi\rangle$ gets the poles of the resolvent. The argument can be extended to states and observables: even if $\langle z|\rho|z'\rangle$ has no poles, ${}^+\langle z|\rho|z'\rangle^+$ has poles, and even if $\langle z|O|z'\rangle$ has no poles, ${}^+\langle z|O|z'\rangle^+$ has poles.

As a consequence, if the initial condition of the interaction period is given by the states and operators of a previous free period which, as shown in case 1, have no poles in the eigenbasis of H_0 [precisely, the continuations of $\rho(E,E') = \langle E|\rho|E'\rangle$ and $O(E,E') = \langle E|O|E'\rangle$ have no poles], then $\rho(z,z') = \langle z|\rho|z'\rangle$ and $O(z,z') = \langle z|O|z'\rangle$ have no poles but ${}^+\langle z|\rho|z'\rangle^+$ and ${}^+\langle z|O|z'\rangle^+$ *do have poles* that produce the dumping factor of Eq. (16).[8] These results are presented in great detail in Ref. [45], where the dumping factor appears in Eq. (70), precisely,

$$\begin{aligned}(\rho(t)|O) = &\int_0^\infty d\omega(\rho_0|\Phi_\omega)(\widetilde{\Phi_\omega}|O) + e^{i(\bar{z}_0 - z_0)t}(\rho_0|\Phi_{00})(\widetilde{\Phi_{00}}|O) \\ &+ \int_\Gamma dz' e^{i(\bar{z}_0 - z')t}(\rho_0|\Phi_{0z'})(\widetilde{\Phi_{0z'}}|O) \\ &+ \int_{\bar{\Gamma}} dz e^{i(z - z_0)t}(\rho_0|\Phi_{z0})(\widetilde{\Phi_{z0}}|O) \\ &+ \int_\Gamma dz' \int_{\bar{\Gamma}} dz e^{i(z - z')t}(\rho_0|\Phi_{zz'})(\widetilde{\Phi_{zz'}}|O).\end{aligned} \tag{28}$$

Independently of the precise meaning of each symbol (which can be found in Ref. [45]), it is quite clear that the term $e^{i(\bar{z}_0 - z_0)t}(\rho_0|\Phi_{00})(\widetilde{\Phi}_{00}|O) = e^{iZ_0 t}(\rho_0|\Phi_{00})(\widetilde{\Phi}_{00}|O)$ represents the main contribution to decoherence, that is, the pole contribution. The first term is the "constant term" and the last three terms are background terms.[9] Furthermore, it can be proved that a sufficient condition for the coefficient $(\rho_0|\Phi_{00})(\widetilde{\Phi}_{00}|O)$ be well defined is that $\rho(z,z') = \langle z|\rho|z'\rangle$ and $O(z,z') = \langle z|O|z'\rangle$ have no poles.

[6] If V is not well behaved, some new poles may appear.

[7] z_0 is the pole of the $|\psi(t)\rangle$ evolution, $Z_0 = \bar{z}_0 - z_0$ is the pole of the $\rho(t)$ evolution. The minus sign in the $-z_0$ produces the change from the lower half plane for the decaying processes in the $|\psi(t)\rangle$ evolution to the upper half plane for these processes in the $\rho(t)$ evolution.

[8] Moreover, if conditions 1 or 2 of case 1 are not satisfied, $\rho(z,z') = \langle z|\rho|z'\rangle$ and $O(z,z') = \langle z|O|z'\rangle$ may have poles, as it will be shown in Sec. VII. Nevertheless, in a certain sense more poles are welcomed because what we are essentially trying to prove is that decoherence time is very small.

[9] All this computation can also be made by using the Laplace transform (see Ref. [62])

$$\exp(-iLt) = \frac{1}{2\pi i}\int_C \exp(-izt)\frac{1}{L - z}dz$$

with the same result. In this case, all the conditions required in this section are also needed.

V. DECOHERENCE TIME FOR MICROSCOPIC SYSTEMS

In these three final sections we will present some estimates of decoherence time, in order to show that self-induced decoherence can account for already known results and opens the way to more detailed models.

As already explained, if γ is the imaginary part of the pole (or of the pole closer to the real axis), the decoherence time is

$$t_D = \frac{\hbar}{\gamma} \tag{29}$$

because, as we have said, the *characteristic decaying time of the fluctuating term* of $\langle O\rangle_{\rho(t)} = (\rho(t)|O)$ is the *decoherence time* t_D.

The decoherence time can be estimated in particular cases like, e.g., the Friedrich model studied in Refs. [26,44], where we obtain

$$t_{D_1} = \frac{\hbar}{2\pi|V_\Omega|^2} \tag{30}$$

being V_Ω the interaction function. It is clear that, if the interaction vanishes, $t_D \to \infty$. In turn, if the characteristic energy $2\pi|V_\Omega|^2 \sim V$ is, say, 1 eV (a natural energy scale for quantum atomic interactions, see, e.g., Ref. [63]), the decoherence time is $\sim 10^{-15}$ s.[10]

This means that, when the theory is calibrated in such a way that the free period does not lead to decoherence, a generic system does decohere (and, in general, *very fast*) in the interaction period.

It is interesting to remark that the method of Ref. [45] was compared with the usual methods of nuclear physics ([64–66]) in the case of a $^{208}P_b(2d_{5/2})$ proton state in a Woods-Saxon potential, including spin-orbit interaction with parameters as in Ref. [67], with an excellent agreement (see Fig. 3 of Ref. [45]). In this case, $\gamma = 10^{-1}$ MeV and $t_D \sim 10^{-20}$ s.

VI. DECOHERENCE TIME OF MACROSCOPIC BODIES

A pure state of a macroscopic object $|\psi\rangle$ can be considered as the tensor product of N states $|\psi_i\rangle$ of microscopic particles, belonging to Hilbert spaces $\mathcal{H}_i$, respectively,

$$|\psi\rangle = \overset{N}{\underset{i=1}{\otimes}} |\psi_i\rangle \in \overset{N}{\underset{i=1}{\otimes}} \mathcal{H}_i \tag{31}$$

or, more generally, if the microscopic states are mixed, $\rho_i \in \mathcal{L}_i = \mathcal{H}_i \otimes \mathcal{H}_i$, then the state of the macroscopic object will be

$$\rho = \overset{N}{\underset{i=1}{\otimes}} \rho_i \in \overset{N}{\underset{i=1}{\otimes}} \mathcal{L}_i. \tag{32}$$

Therefore the generic total Hamiltonian reads

$$H = \sum_{i=1}^{N} H_i + \sum_{i=1}^{N} V_i + \sum_{i,j=1}^{N} V_{ij} + \sum_{i,j,k=1}^{N} V_{ijk} + \cdots, \tag{33}$$

where the H_i are the free Hamiltonians of the microscopic particles, the V_i are the averages of the interactions between each particle and the remaining particles, the V_{ij} are the two particle interactions, and so forth. The eigenvectors of H are $|\omega, x_1, \ldots, x_{N-1}\rangle$, such that

$$H|\omega, x_1, \ldots, x_{N-1}\rangle = \omega|\omega, x_1, \ldots, x_{N-1}\rangle, \tag{34}$$

where ω is the total energy and $x_1, \ldots, x_{N-1}$ are the remaining labels necessary to define the eigenstate. Since decoherence occurs in variable ω, we can ignore the rest of the labels for our argument. Disregarding for simplicity the $V_{ij}, V_{ijk}, \ldots$, and considering all the V_i equal, we obtain

$$V = \sum_{i=1}^{N} V_i = NV_i. \tag{35}$$

Then, the characteristic energy in Eq. (30) is now

$$t_D = \frac{\hbar}{NV_i} = \frac{t_{D_1}}{N}. \tag{36}$$

Let us suppose again that all the V_i are of the order of 1 eV. If we consider a macroscopic object of 1 mol, where $N = 10^{24}$, the decoherence time results,

$$t_D = 10^{-39} \text{ s},$$

a very tiny time indeed. This decoherence time is so close to Plank time 10^{-43} s that it should be considered more as an illustration than as a physical result. Nevertheless, it shows that decoherence is fantastically fast in macroscopic bodies.

Let us compare Eq. (36) with the low temperature t_{dec} of Ref. [11], p. 51,

$$t_{dec} = \gamma_0^{-1}\left(\frac{\Delta x}{2L_0}\right)^2$$

where we have also a interaction factor term $\gamma_0^{-1} \sim t_{D_1}$ and a macroscopic coefficient $(\Delta x/2L_0)^2 \sim 1/N$. We can see a coincidence between the orders of magnitude of our result and of the standard theory's result, in the spirit of the Comment 2 of the Introduction.

VII. INITIAL THERMAL BATH

Finally, we will consider the case of a system with a thermal bath, following the formalism of Ref. [27], Sec. IV.B. For the model considered in that section, an oscillator in a thermal bath, the fluctuating term of Eq. (7) is

$$\int O_{\mathbf{pp}'} \bar{\rho}_{\mathbf{pp}'} e^{i[(\omega_p - \omega_{p'})t/\hbar]} d\mathbf{p} d\mathbf{p}', \tag{37}$$

where $\bar{\rho}_{\mathbf{pp}'} = (\rho|A_{\mathbf{p}}^{\dagger} A_{\mathbf{p}'})$, given in Eq. (37) of Ref. [27], is the initial condition of the oscillator and thermal bath,

[10] This is, of course, a general case: γ is usually of the order of magnitude of the characteristic interaction energy V.

$$(\rho|A_{\mathbf{p}}^{\dagger}A_{\mathbf{p}'}) = \rho(\mathbf{p})\,\delta^3(\mathbf{p}-\mathbf{p}') + \frac{\langle n\rangle_0 V_p V_{p'}}{\eta_+(\omega_p)\,\eta_-(\omega_{p'})} + \frac{V_p V_{p'}\rho(\mathbf{p})}{\eta_-(\omega_{p'})(\omega_{p'}-\omega_p-i0)} + \frac{V_p V_{p'}\rho(\mathbf{p})}{\eta_+(\omega_p)(\omega_p-\omega_{p'}+i0)} + \frac{V_p V_{p'}}{\eta_+(\omega_p)\,\eta_-(\omega_{p'})} \times \int \frac{d\mathbf{k}V_k^2\rho(\mathbf{k})}{\eta_-(\omega_{p'})(\omega_{p'}-\omega_p-i0)(\omega_p-\omega_{p'}+i0)}, \tag{38}$$

where $\rho(\mathbf{p})$ is given in Eq. (40) of that paper,

$$\rho(\mathbf{p}) = \frac{1}{e^{\beta\omega_p}-1} \tag{39}$$

and where $\beta=1/kT$. Again, independently of the precise meaning of each term (which can be found in Ref. [27]), it is obvious that the $\rho(\mathbf{p})$ factor has a pole in the ω_p complex plane and therefore the initial condition is not pole free. Then, defining as before

$$\nu = \omega_p - \omega_{p'}, \quad 2\lambda = \omega_p + \omega_{p'}, \quad \omega_p = \lambda + \frac{\nu}{2}, \quad \text{etc.}, \tag{40}$$

the terms of Eq. (37) have the form

$$\int \cdots \frac{1}{e^{\beta[\lambda+(\nu/2)]}-1} e^{i(\nu t/\hbar)} d\lambda\, d\nu, \tag{41}$$

where the dots symbolize other factors coming from the interaction (which may have poles not considered here). When we introduce the complex variable $z=\omega+i\gamma$, the poles of the factor $\rho(\mathbf{p})$ turn out to be located where the following equation is satisfied:

$$e^{\beta[\lambda+(z/2)]} = e^{\beta[\lambda+(\omega/2)]}\left(\cos\frac{\gamma}{2kt} + i\sin\frac{\gamma}{2kt}\right) = 1. \tag{42}$$

Then, the poles are located in the coordinates

$$\omega = -2\lambda, \quad \gamma = 4\pi nkT, \tag{43}$$

where n is an integer number. For $n=1$ we obtain the decoherence time

$$t_D = \frac{\hbar}{4\pi kT} \tag{44}$$

which, for room temperature, $T\sim 10^2$ K, gives $t_{D_1}=10^{-13}$ s for a single-particle system, and $t_D=t_{D_1}/N=10^{-37}$ s for a mol-particle system.

Moreover, if $S_1=\hbar$ is the characteristic action for a particle system,[11] and $S=ML^2/\Upsilon$ is the action of a macroscopic system (where M, L, and Υ are the characteristic mass, length, and time), the particle number can be estimated as

$$N = \frac{S}{S_1} = \frac{ML^2}{\hbar\Upsilon} \tag{45}$$

and the decoherence time reads

$$t_D \sim \Upsilon\frac{\hbar^2}{ML^2kT}. \tag{46}$$

In the particular case that $\Upsilon=\gamma_0^{-1}$, $L=L_0$ and $M=M$ are the characteristic time, length, and mass of the model of Sec. 4.1 of Ref. [11], when we introduce the de Broglie length $\lambda_{DB}=\hbar/\sqrt{3MkT}\sim\hbar/\sqrt{MkT}$, we obtain

$$t_D \sim \gamma_0^{-1}\left(\frac{\lambda_{DB}}{L_0}\right)^2, \tag{47}$$

namely, Eq. (4.10) of Ref. [11] which represents the decoherence time for the model of Sec. 4.1. This shows, again in the spirit of Comment 2 of the Introduction, the agreement between the orders of magnitude of the decoherence times obtained by the SID and the EID approaches.

VIII. CONCLUSION

In a series of previous papers we have developed an approach to decoherence that avoids the drawbacks of the einselection approach. In the present paper we have shown that our formalism supplies a precise method for computing the decoherence time, and that the results obtained with such a method are physically meaningful.

However, we are aware of the fact that a great number of results have been obtained in the context of the einselection program when compared with the cases treated by means of the self-induced approach. Therefore our future work has to be directed to enlarge the set of applications of our theory. We consider that this task will be worth the effort to the extent that decoherence is a key element in the explanation of the emergence of classicality from the quantum world.

APPENDIX A: HARTOG THEOREM

A necessary mathematical remark is in order: from the Hartog theorem, we know that the realm of analytical functions of more than one complex variable is much more involved than that of just one variable. In Refs. [26,44,45], we have considered the analytical continuation of two variables, z and z', in products like $f_1(z)\,f_2(z')$; this suggests that we should have worked with the theory of analytical functions of two complex variables. However, this is not the case because the only relevant variable for the problems considered in the quoted papers is the difference $Z=z-z'$ that appears in the evolution factor $e^{i(z-z')t}$; then, we can always introduce a change of variables,

$$z+z' = 2\lambda, \quad z-z' = Z,$$

[11] E.g., (i) In the harmonic oscillator, the unidimensional coordinates are $q=(M\omega/\hbar)^{1/2}Q$ and $p=(1/M\omega\hbar)^{1/2}P$; then, by making $q=p=1$ we obtain $S_1=QP=\hbar$. (ii) In a model with spherical symmetry we have $L_zY(\theta,\varphi)=m\hbar Y(\theta,\varphi)$; then, by making $m=1$ we obtain $S_1=\varphi L_z=\hbar$, and so forth.

$$z = \lambda + \frac{Z}{2}, \quad z' = \lambda - \frac{Z}{2}, \tag{A1}$$

and functions $f(z,z')$, like the product $f_1(z)\,f_2(z')$, have to be considered as functions $f(z,z')=f[\lambda+(Z/2),\lambda-(Z/2)]$ where $Z \in \mathbb{C}$; but since in all the cases we have taken $\lambda \in \mathbb{R}$, $f(z,z')$ is really a function of only one complex variable Z.

For instance, in Eq. (71) of Ref. [45], the symbol $\mathrm{cont}_{\omega\to\bar{z}_0}\mathrm{cont}_{\omega'\to z_0}$ should be understood as

$$\mathrm{cont}_{\lambda+(\nu/2)\to\tilde{\omega}+i(\gamma/2)}\mathrm{cont}_{\lambda-(\nu/2)\to\tilde{\omega}-i(\gamma/2)} = \mathrm{cont}_{\lambda\to\tilde{\omega}}\mathrm{cont}_{\nu\to i\gamma}, \tag{A2}$$

where λ is a real number. In this equation, only the second continuation of the rhs is an analytical continuation in the complex plane, being the first one a simple change of a real variable, from λ to $\tilde{\omega}$.

These considerations show that Refs. [26,44,45] and, in general, the method presented in this paper, are free from Hartog's objection.

APPENDIX B: TWO-TIMES EVOLUTION

Let us now generalize Eq. (21) by adding two interactions,

$$H = H_0 + V = \int_0^\infty d\omega\omega|\omega\rangle\langle\omega| + \int_0^\infty\int_0^\infty [V^{(1)}(\omega,\omega') + V^{(2)} \times(\omega,\omega')]|\omega\rangle\langle\omega'|d\omega d\omega', \tag{B1}$$

where $V^{(1)}$ represents a macroscopic interaction and $V^{(2)}$ represents a microscopic interaction: $V^{(1)}(\omega,\omega') \gg V^{(2)}(\omega,\omega')$. As a consequence, in a first step we can neglect $V^{(2)}(\omega,\omega')$ and repeat what was said in Sec. IV. Then, we can begin with considering a Hamiltonian

$$H^{(1)} = H_0 + V_1 = \int_0^\infty \omega|\omega\rangle\langle\omega|d\omega + \int_0^\infty\int_0^\infty V^{(1)}(\omega,\omega')|\omega\rangle \times\langle\omega'|d\omega d\omega'. \tag{B2}$$

If we change the basis to $\{|\omega\rangle^+_{(1)}\}$, we obtain

$$|\omega\rangle^+_{(1)} = |\omega\rangle + \frac{1}{\omega + i0 - H}V^{(1)}|\omega\rangle. \tag{B3}$$

Since the interaction $V^{(1)}$ is macroscopic, the dumping of the off-diagonal terms has a characteristic time of $10^{-37}-10^{-39}$ s. Therefore, after an initial period much larger than this magnitude, the state can be considered nearly diagonal for all practical purposes.

However, the state is not yet in complete equilibrium, because the interaction $V^{(2)}$ is always present: now it becomes relevant. Then, after the initial period we can consider the total Hamiltonian as

$$H = H^{(1)} + V_2 = \int_0^\infty \omega|\omega\rangle^+_{(1)}\langle\omega|^+_{(1)}d\omega + \int_0^\infty\int_0^\infty V^{(2)\prime}(\omega,\omega') \times|\omega\rangle^+_{(1)}\langle\omega'|^+_{(1)}d\omega d\omega',$$

where $V^{(2)\prime}(\omega,\omega')$ is $V^{(2)}(\omega,\omega')$ in the new basis $\{|\omega\rangle^+_{(1)}\}$. When we make a final change of basis

$$|\omega\rangle^+ = |\omega\rangle^+_{(1)} + \frac{1}{\omega + i0 - H}V^{(2)}|\omega\rangle^+_{(1)}$$

we can compute the characteristic time in this case. But now this time may be of the order of 1 s, that is, the relaxation time of a macroscopic body. In this way we can describe a two-times process, with a extremely short decoherence time and a long relaxation time.

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