

# Coherent-state superpositions in cavity quantum electrodynamics with trapped ions

F. L. Semião and A. Vidiella-Barranco

*Instituto de Física “Gleb Wataghin,” Universidade Estadual de Campinas, 13083-970 Campinas, São Paulo, Brazil*

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We investigate how superpositions of motional coherent states naturally arise in the dynamics of a two-level trapped ion coupled to the quantized field inside a cavity. We extend our considerations, including a more realistic setup where the cavity is not ideal and photons may leak through its mirrors. We found that a detection of a photon outside the cavity would leave the ion in a pure state. The statistics of the ionic state still keeps some interference effects that might be observed in the weak-coupling regime.

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There has been a great deal of interest in the coherent manipulation of simple quantum systems [1–3], mainly in the high degree of control necessary for the implementation of quantum-information-processing tasks [4–6]. In particular, the study of trapped ions interacting with laser beams has attracted much attention due to the significant experimental advances in the generation of quantum states in such a system [7,8]. The interaction of trapped ions with laser beams is well understood in terms of a semiclassical model with the electromagnetic field being treated as a  $c$  number, but additional features might be revealed due to the field quantization. The entanglement between photons and ions is a remarkable consequence of that quantization and its potential applications have been motivating the experimental work in cavity quantum electrodynamics with trapped particles [9]. For instance, there have been reported schemes for the generation of specific entangled states such as Greenberger-Horne-Zeilinger (GHZ) states [10] as well as Bell states [11].

One of the reasons for interest in studying and experimentally coupling photons with material particles comes from the fact that in order for the quantum-information processing to be used in its full extent, one should be able to interconvert stationary and flying qubits and also faithfully transmit the flying qubits between given positions. Those two statements are part of what are known as DiVincenzo’s requirements for the physical implementation of quantum computation and information [12]. The entanglement present in the system consisting of cavities and trapped ions may be useful in the propagation of information carried by photons between two distant locations [13].

It is not just entangled states involving either two-level systems or Fock states of the electromagnetic field that find applications in quantum information. Another interesting class of nonclassical states with high potential applications is the one formed by linear superpositions of coherent states. This class of states has been considered for quantum teleportation [14,15], logic gate implementation [16,17], and tests of local realism [18], for instance. In this paper, we show that superpositions of motional coherent states may be generated in the framework of cavity electrodynamics with trapped ions by letting the system evolve in the resonant carrier dynamics and by performing a measurement of the internal state of the ion. We apply the formalism of quantum jumps to study the nonideal case including damping in the cavity and show that the detection of photons outside the cavity could be used to generate nonclassical states of the motion of the

trapped ion. More precisely, we show that the statistics of the generated state keeps track of the coherence displayed in the oscillatory behavior of the phonon number distribution and the variation of its width from Poissonian to sub- or super-Poissonian [19]. Although that is not a deterministic protocol (it depends on the random event of the leaking of a photon from the cavity), it might be of interest because it could be implemented in current experimental systems. Experiments involving trapped ions and optical cavity fields have been performed only in the weak-coupling regime in which the cavity damping is stronger than the ion-cavity coupling [9].

In this work we consider a single two-level ion trapped in a Paul trap and placed inside an optical cavity. The cavity mode couples to the ionic internal degrees of freedom  $\{|e\rangle, |g\rangle\}$  and the system Hamiltonian is given by [20]

$$\hat{H} = \hbar \nu \hat{a}^\dagger \hat{a} + \hbar \omega \hat{b}^\dagger \hat{b} + \hbar (\omega_0/2) \hat{\sigma}_z + \hbar g (\hat{\sigma}_+ + \hat{\sigma}_-) (\hat{b}^\dagger + \hat{b}) \cos \eta (\hat{a}^\dagger + \hat{a}), \quad (1)$$

where  $\hat{a}^\dagger(\hat{a})$  denotes the creation (annihilation) operator of the center-of-mass vibrational motion of the ion (frequency  $\nu$ ),  $\hat{b}^\dagger(\hat{b})$  is the creation (annihilation) operator of photons in the field mode (frequency  $\omega$ ), the  $\hat{\sigma}$  operators are the usual Pauli matrices for the two internal levels of the ion,  $\omega_0$  is the atomic frequency transition,  $g$  is the ion-field coupling constant, and  $\eta = 2\pi a_0/\lambda$  is the Lamb-Dicke parameter,  $a_0$  being the amplitude of the harmonic motion and  $\lambda$  the wavelength of the cavity field.

For our purposes here we may work in the Lamb-Dicke regime ( $\eta \ll 1$ ), i.e., the situation in which the spatial extent of the motion of the trapped ion is much smaller than the wavelength of the cavity field. In this regime, we may perform an approximation that simplifies the original Hamiltonian (1) as follows:

$$\cos \eta (\hat{a}^\dagger + \hat{a}) \approx 1 - \eta^2 (1 + 2\hat{a}^\dagger \hat{a})/2 - \eta^2 (\hat{a}^{\dagger 2} + \hat{a}^2)/2. \quad (2)$$

If we tune the light field so that it exactly matches the atomic transition, i.e.,  $\omega_0 - \omega = 0$  (carrier transition), we obtain the interaction Hamiltonian in the Lamb-Dicke regime, which, after discarding rapidly oscillating terms, reads

$$\hat{H}_I = \hbar g [1 - \eta^2 (1 + 2\hat{a}^\dagger \hat{a})/2] (\hat{\sigma}_- \hat{b}^\dagger + \hat{\sigma}_+ \hat{b}). \quad (3)$$

The resulting Hamiltonian in Eq. (3) is similar to the Jaynes-Cummings Hamiltonian but having an effective coupling

constant which in our case depends on the excitation number of the ionic oscillator,  $\hat{m} = \hat{a}^\dagger \hat{a}$ . Such a dependence on the intensity has already been demonstrated [21] to be related to the occurrence of super-revivals (revivals taking place at long times) of the atomic inversion.

We now consider that the system is initially prepared in such a way that the ion is in its excited state  $|e\rangle$  (internal level), the cavity in the vacuum state  $|0\rangle_c$ , and the vibrational motion in the coherent state  $|\alpha\rangle_v$ , i.e.,  $|\psi(0)\rangle = |\alpha\rangle_v |0\rangle_c |e\rangle$ . Under the Hamiltonian (3), the state  $|\psi(0)\rangle$  evolves to

$$|\psi(t)\rangle = \cos\{gt[1 - \eta^2(1 + 2\hat{a}^\dagger \hat{a})/2]\} |\alpha\rangle_v |0\rangle_c |e\rangle - i \sin\{gt[1 - \eta^2(1 + 2\hat{a}^\dagger \hat{a})/2]\} |\alpha\rangle_v |1\rangle_c |g\rangle. \quad (4)$$

We still have to apply the functions of the operator  $\hat{a}^\dagger \hat{a}$  in the coherent state  $|\alpha\rangle_v$ . This may be easily done by moving to the Fock basis, and the result is given by

$$|\psi(t)\rangle = [\cos(\omega_\eta t) |\Phi_+\rangle_v - i \sin(\omega_\eta t) |\Phi_-\rangle_v] |0\rangle_c |e\rangle + [\cos(\omega_\eta t) |\Phi_-\rangle_v - i \sin(\omega_\eta t) |\Phi_+\rangle_v] |1\rangle_c |g\rangle, \quad (5)$$

where  $\omega_\eta \equiv g(1 - \eta^2/2)$  and  $|\Phi_\pm\rangle_v$  are general superpositions of coherent states given by

$$|\Phi_\pm\rangle_v \equiv (|\alpha e^{i\phi}\rangle_v \pm |\alpha e^{-i\phi}\rangle_v)/2, \quad (6)$$

where we defined the time-dependent real phase  $\phi = \eta^2 gt$ . The state (5) is an entangled state involving superpositions of motional coherent states of the trapped ion, its internal electronic states, and Fock states of the cavity field. It is noteworthy that for interaction times given by  $t_k = k\pi$ , with  $k$  being an integer number, the state of the system reduces to

$$|\psi(t)\rangle = |\Phi_+\rangle_v |0\rangle_c |e\rangle + |\Phi_-\rangle_v |1\rangle_c |g\rangle. \quad (7)$$

One could then obtain a disentangled motional state by performing a measurement on the internal state of the ion. The experimental discrimination between the two electronic levels may be done using the very efficient electron shelving method [22]. Depending on the measurement outcome, the collapsed motional state may be either  $|\Phi_+\rangle_v$  or  $|\Phi_-\rangle_v$ . One of the main interesting characteristics of those superposition states is that their statistics are strongly sensitive to the value of the phase  $\phi$ . The trivial case takes place when  $\phi = 0$ , what leads the distribution  $P_m = |\langle m | \Phi_\pm \rangle_v|^2$  (phonon statistics) to be Poissonian. However, it is well known that there are domains in which it can be either sub- or super-Poissonian. As pointed out in [19], when the statistics is super-Poissonian, the distribution  $P_m$  displays an oscillatory behavior, this being a direct consequence of interference in phase space. Such a behavior is analogous to the oscillatory photon statistics of highly squeezed states [23]. Although similar superposition states may also be generated using classical fields [24], the possibility of entanglement with light is a unique feature related to the quantum nature of the electromagnetic field.

The scheme proposed here relies on a not very demanding initial preparation of the system. It requires the initial field to be in the vacuum state  $|0\rangle_c$ , i.e., there is no need to prepare or inject a coherent field state into the cavity. Additionally, the vibrational motion of the ion has to be prepared in a coherent

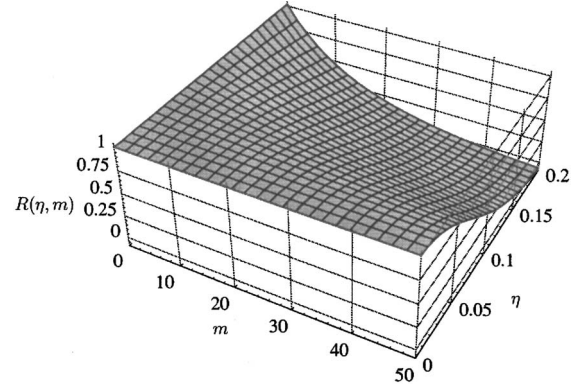


FIG. 1. Ratio between the exact coupling constant and the approximate one in the Lamb-Dicke regime. The approximation is valid in the region  $R(\eta, m) \approx 1$  where  $\eta$  and  $m$  are small enough.

state  $|\alpha\rangle_v$ , whose experimental realization for a  ${}^9\text{Be}^+$  ion trapped in a rf (Paul) trap has been already reported [7]. Regarding the internal ionic states, they need to be prepared in the excited state, which can be achieved by the application of laser pulses, for instance.

We would like to point out that the linear dependence of the ion-field coupling constant on the operator  $\hat{a}^\dagger \hat{a}$  is crucial for the generation of the states  $|\Phi_\pm\rangle_v$ . Therefore, it is very important to be aware of the limits where the parameter  $\eta$  and the initial magnitude  $\alpha$  may be varied and the approximation (2) is still valid. This limit is set by keeping the product  $\eta^2 \hat{a}^\dagger \hat{a}$  small enough, which allows us to neglect higher-order terms in the cosine expansion. If the Lamb-Dicke approximation was not performed, it would be necessary to work with the full nonlinear coupling constant  $\lambda \equiv \langle m | \cos \eta(\hat{a}^\dagger + \hat{a}) | m \rangle = e^{-\eta^2/2} L_m^0(\eta^2)$ . For convenient values of the product  $\eta^2 \hat{a}^\dagger \hat{a}$  this coupling constant reduces to  $\lambda_{LD} = 1 - \eta^2(1 + 2m)/2$ , which is the coupling constant we have used so far (Lamb-Dicke regime). In Fig. 1 we show the ratio between the exact and the approximate coupling constants,  $R(\eta, m) \equiv \lambda/\lambda_{LD}$ . We see that there are ranges of values of  $\eta$  and  $m$  for which  $R \approx 1$ . Under such circumstances, the Lamb-Dicke approximation is valid and the generation protocol proposed here is applicable.

We are now interested in the more realistic setting where the cavity is not ideal and one could detect photons leaking through its mirrors. The setup we have in mind is depicted in Fig. 2. We still have a two-level trapped ion interacting resonantly with the cavity field but now we consider the cavity to

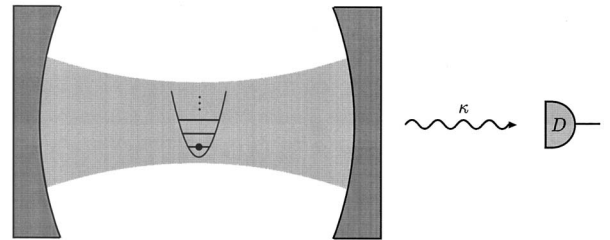


FIG. 2. Schematic experimental setup. The system consists of a single trapped ion placed in a lossy cavity having a decay rate  $\kappa$ . The detector  $D$  continuously monitors this decay channel.

be lossy, decaying at a rate  $\kappa$ . We assume that a detector  $D$  is placed outside the cavity in such a way that it may monitor the cavity decay.

The description of damping in quantum optical systems is usually made using master equations and its solution gives the time evolution of the system when the decay is not observed. However, the time evolution under continuous observation of photon counts may be adequately described by a pure state that evolves according to a non-Hermitian Hamiltonian. This approach is known as the quantum jumps or quantum trajectories [25] formalism. The idea of continuous observation of decaying channels in systems consisting of atoms or ions and cavities has proved itself useful to perform legitimate information-processing tasks such as teleportation [26] and generation of maximally entangled states [27,28] or quantum gates [29], for instance. We saw that the time evolution of the system under Hamiltonian (3) and the realization of a measurement on the electronic state may be used to generate the states (6). Now, instead of measuring the atomic state, we will show that a measurement of the photon outside the cavity collapses the state of the system into a state that keeps many of the characteristics of the state (6), namely, oscillatory behavior of the distribution  $P_m$ , as well as its narrowing and broadening [19]. For the sake of simplicity, we assume that the detector  $D$  is perfect. Otherwise we would just have to account for a finite probability that the detector fails in detecting an event of leaking of a photon, which would lead us to a description in terms of density matrices rather than state vectors. The time evolution of the system conditioned to no photon decay is given by

$$i\hbar d|\psi\rangle/dt = \hat{H}_{\text{eff}}|\psi\rangle, \quad (8)$$

where

$$\hat{H}_{\text{eff}} = -i\hbar \frac{\kappa \hat{b}^\dagger \hat{b}}{2} + \hbar g \left[ 1 - \frac{\eta^2 (1 + 2\hat{a}^\dagger \hat{a})}{2} \right] (\hat{\sigma}_- \hat{b}^\dagger + \hat{\sigma}_+ \hat{b}). \quad (9)$$

It is worth noticing that once the Hamiltonian (9) is not Hermitian, the norm of  $|\psi(t)\rangle$  is not constant in time. So it must be normalized in order to allow one to correctly evaluate any property of the system. It is clear that if the initial state of the system is the same as before, namely,  $|\psi(0)\rangle = |\alpha\rangle_v |0\rangle_c |e\rangle$ , the solution of Eq. (8) may be written as

$$|\psi(t)\rangle = \sum_m a_m(t) |m, 0, e\rangle + b_m(t) |m, 1, g\rangle. \quad (10)$$

Substituting Eqs. (9) and (10) into Eq. (8) one obtains two coupled differential equations that may be easily solved, and the result is given by

$$a_m(\tau) = c_m(0) e^{-\Gamma\tau/4} (C(\tau) + (\Gamma/\sqrt{\Gamma^2 - 16\lambda_{LD}^2}) S(\tau)),$$

$$b_m(\tau) = -4 i c_m(0) e^{-\Gamma\tau/4} (\lambda_{LD}/\sqrt{\Gamma^2 - 16\lambda_{LD}^2}) S(\tau), \quad (11)$$

where  $c_m(0)$  are the coefficients of the expansion of the initial coherent state in the Fock basis,  $\Gamma = \kappa/g$ ,  $\tau = gt$ , and

$$C(\tau) = \cosh(\sqrt{\Gamma^2 - 16\lambda_{LD}^2} \tau/4), \quad (12)$$

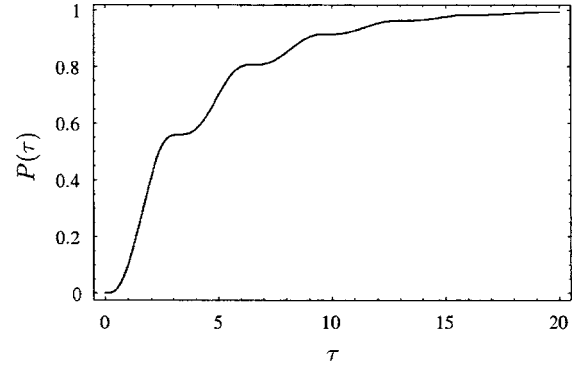


FIG. 3. Probability of detection of one photon outside the cavity. The system parameters are  $\Gamma=1$ ,  $\eta=0.05$ , and  $\alpha=2$ . This probability tends to 1 for higher values of  $\tau$ .

$$S(\tau) = \sinh(\sqrt{\Gamma^2 - 16\lambda_{LD}^2} \tau/4). \quad (13)$$

Now, we suppose that one photon is detected outside the cavity. This event would correspond to the destruction of one photon, leading the system to state  $\hat{b}|\psi(\tau)\rangle$ . Again, we remember that since the time evolution is not unitary the state must be normalized after this jump. In our case, the resulting state would be  $|\psi(\tau)\rangle_d = |\Phi(\tau)\rangle_v |0\rangle_c |e\rangle$ , i.e., a disentangled state having a normalized motional part given by

$$|\Phi(\tau)\rangle_v = \sum_{m=0}^{\infty} \left( b_m(\tau) / \sqrt{\sum_{p=0}^{\infty} |b_p(\tau)|^2} \right) |m\rangle_v. \quad (14)$$

Before investigating the statistical properties of that state, it is important to calculate the probability for photon emission because it is related to our probability of success in generating  $|\Phi(\tau)\rangle_v$ . The probability that at least one jump occurs between the initial instant 0 and the subsequent instant  $\tau$  is given by  $P(\tau) = 1 - \langle \psi(\tau) | \psi(\tau) \rangle$ , where  $|\psi(\tau)\rangle$  is the state in Eq. (10) with the coefficients (11). In Fig. 3 we have a plot showing the behavior of  $P(\tau)$  using parameters that are close to the ones in a current experimental situation, i.e., the weak-coupling regime.

Let us now start the analysis of the statistical properties of the vibrational state  $|\Phi(\tau)\rangle_v$ . The ion started in a coherent state which has a Poissonian distribution. We can describe its narrowing (or widening) via the normalized variance (also known as the *Fano factor*) defined as  $\sigma^2 = (\overline{m^2}/\overline{m}) - \overline{m}$  where  $\overline{m}$  and  $\overline{m^2}$  are the first and second moments of the distribution  $P_m = |\langle m | \Phi(\tau) \rangle|^2$ , respectively. Values of  $\sigma < 1$  indicate sub-Poissonian,  $\sigma > 1$  super-Poissonian, and  $\sigma = 1$  Poissonian statistics. The time evolution of  $\sigma(\tau)$  is shown in Fig. 4. The original Poissonian distribution naturally evolves to either sub- or super-Poissonian values. These changes in the width of the distribution could be observed even in a bad cavity that has a decay rate  $\kappa$  comparable to the coupling constant  $g$ , as we can see in Fig. 4. We would like also to show that strong signatures of nonclassical behavior, such as the oscillations in the phonon distribution  $P_m$  at times when the statistics is Poissonian, still persist in the weak-coupling regime. This may be seen in Fig. 5 where we show the

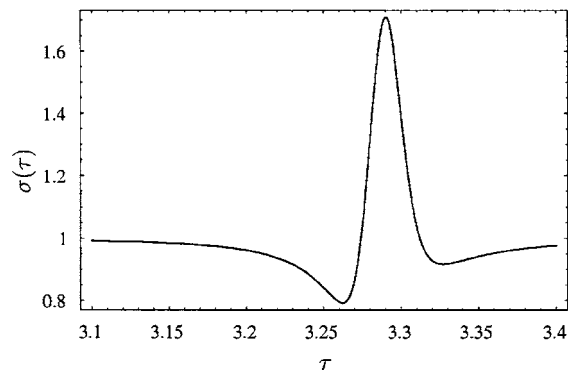


FIG. 4. Time evolution of the normalized variance. The system parameters are  $\Gamma=1$ ,  $\eta=0.05$ , and  $\alpha=2$ .

distribution at a time  $\tau=3.29$  and with  $\eta=0.05$ . Based on those considerations we conclude that general properties of coherent state superpositions, which arise in the lossless case, would still persist in our more realistic setup. This means that our proposal could be useful for the experimental investigation of certain nonclassical features.

We have investigated several aspects of the dynamics of a trapped ion inside a cavity. First we have considered a situation in which the unitary time evolution leads to a global entangled state involving superposed motional coherent states, Fock photon states, and the two internal electronic states. After the measurement of the internal state of the ion in a specific interaction time, the generation of quantum superposition of coherent states of motion of the ion is accomplished. Two different states may be generated (either  $|\Phi_+\rangle$  or  $|\Phi_-\rangle$ ) depending on the result of the measurement of the internal ionic state. The main requirement for such generation is the strong-coupling regime where the system may perform Rabi oscillations in the lifetime of the cavity photon. In the second part of our paper we consider the influence of cavity decay in the ionic dynamics. In fact, that itself represents a generation method, since a nonclassical state results

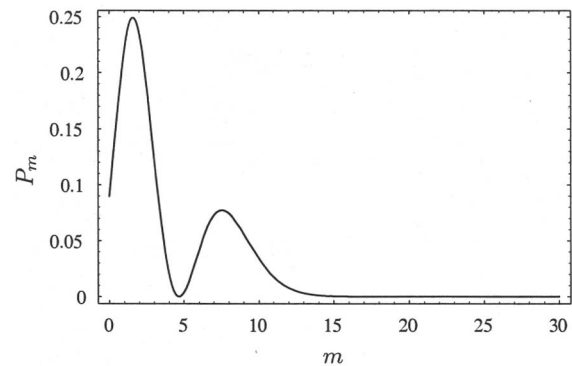


FIG. 5. Phonon distribution at  $\tau=3.29$  with system parameters  $\Gamma=1$ ,  $\eta=0.05$ , and  $\alpha=2$ .

from the dissipative evolution even with a photon decay rate of the same order as the ion-cavity coupling (weak-coupling regime). The cavity is continuously monitored by a detector, which causes the state of the system to be pure at any time. The measurement of the internal electronic state in the former suggestion is replaced now by the counting of a photon leaking out of the cavity. This collapses the entangled global state of the system onto a product state. Even though the cavity is not ideal, the ionic motional state still retains (after the photon decay) important nonclassical features that characterize quantum superposition of coherent states, such as, for instance, changes in the variance of the phonon distribution (sub- or super-Poissonian statistics) as well as its oscillatory behavior.

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