# Entropy exchange and entanglement in the Jaynes-Cummings model

E. Boukobza and D. J. Tannor
*Department of Chemical Physics, Weizmann Institute of Science, Rehovot 76100, Israel*
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The Jaynes-Cummings model (JCM) is the simplest fully quantum model that describes the interaction between light and matter. We extend a previous analysis by Phoenix and Knight [Ann. Phys. **186**, 381 (1988)] of the JCM by considering mixed states of both the light and matter. We present examples of qualitatively different entropic correlations. In particular, we explore the regime of entropy exchange between light and matter, i.e., where the rate of change of the two are anticorrelated. This behavior contrasts with the case of pure light-matter states in which the rate of change of the two entropies are positively correlated and in fact identical. We give an analytical derivation of the anticorrelation phenomenon and discuss the regime of its validity. Finally, we show a strong correlation between the region of the Bloch sphere characterized by entropy exchange and that characterized by minimal entanglement as measured by the negative eigenvalues of the partially transposed density matrix.

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## I. INTRODUCTION

Quantum entropy was first formulated by von Neumann [1] as an extension of the Gibbs entropy in classical statistical mechanics. The foundations of modern information theory were established by Shannon [2] using a definition of entropy which is essentially identical to the Gibbs entropy. Similarly, quantum entropy plays a central role in the theory of quantum information [3–5]. Modern definitions of quantum entropy include those of Rényi [6] and Tsallis [7].

In recent years, entropy of quantum systems has been discussed in the context of entanglement. The Horodecki's [8] derived Rényi entropy inequalities and discussed their implications for nonlocality. They showed that the entropic inequalities are violated by any pure entangled state. They also argued that by considering Rényi entropy inequalities one can limit teleportation [9] of several states. Cerf *et al.* [10] gave an interesting interpretation to independent, correlated, and entangled two qubit systems by considering conditional and mutual entropies. For example, a maximally entangled (nonseparable) two qubit system possesses negative conditional entropies and excessive mutual entropy (which is related to unaccessible information), while a maximally correlated (separable) two qubit system possesses zero conditional entropies and maximal classical mutual entropy. Phoenix and Knight [11] explored the dynamics of a single cavity mode in resonance with a single atom by comparing von Neumann entropy, Shannon entropy, and photon number variances.

In this paper we focus not on the entropy of the individual subsystems, but on the entropy *correlations* between subsystems in a bipartite system. This question has been almost completely overlooked in the literature, and for a good reason: for a bipartite system that starts in a pure state, the partial entropies of the two subsystems are equal at all times. This result is a consequence of the equality between the eigenvalues of the partial density matrices in a pure bipartite system, which in turn is a consequence of the Schmidt decomposition. In this case the correlations are trivial. However, there are two qualifications in obtaining these trivial correlations. First, that the system is indeed bipartite. Second, that the joint state of the two subsystems is pure. If either of these conditions is not met the correlation between the entropies of the subsystems is nontrivial. In particular, in this paper we focus on the question of whether and under what conditions it is possible to get anticorrelated behavior of the entropy of the subsystems. This is interesting because it implies that there is an entropy transfer process going on, consistent with classical thermodynamic concepts but opposed to the default result obtained for entangled bipartite systems. In this paper, for definiteness, we focus on entropy correlations between an atom and a single quantized cavity mode in the framework of the Jaynes-Cummings model (JCM). We present qualitatively different entropy correlations between the atom and the field, and we demonstrate a regime of entropy exchange between them, both numerically and analytically.

This paper is arranged in the following manner. Section I is a brief introduction. It is devoted to a description of the density matrix of bipartite systems, the definition of quantum entropy, and the issue of separability and entanglement of bipartite systems. In this section we also review the JCM Hamiltonian in the context of quantum entropy. Section II is devoted to calculating entropy correlations for mixed states of light and matter with qualitatively different behaviors. We give examples of entropy exchange between light and matter, and explore the parameter range of this regime. In Sec. III we give analytical proof of the entropy exchange effect and an analysis of the regime of its validity. In Sec. IV we apply two different entanglement tests to determine whether the atomic-field system is entangled for all types of entropy correlations. Section V concludes.

A bipartite system is described by a density matrix of a $C^m \otimes C^n$ Hilbert space. The partial density matrix of one part is obtained by tracing over the other:

$$\rho_{A(B)} = \mathrm{Tr}_{B(A)}(\rho_{AB}). \qquad (1)$$

The entropy of a quantum system is given by the von Neumann entropy [1]

 

$$S = - k_B \mathrm{Tr}(\rho \ln \rho). \quad (2)$$

The purity of a quantum system is given by $\mathrm{Tr}(\rho^2)$, and it is bounded: $0 < \mathrm{Tr}(\rho^2) \leqslant 1$. Purity is related to the $q=2$ Tsallis entropy [7]

$$S_2 = 1 - \mathrm{Tr}(\rho^2). \quad (3)$$

Qualitatively, purity oppositely tracks the von Neumann entropy: an increase in the von Neumann entropy is parallel to a decrease in purity (or an increase in the Tsallis entropy).

A bipartite quantum system is considered separable if it can be written as [12]

$$\rho_{AB} = \sum P^i \rho_A^i \otimes \rho_B^i, \quad (4)$$

where $P^i \geqslant 0$, $\Sigma P^i = 1$, and $\rho^i_{A(B)}$ are individual partial density matrices. A system that cannot be factored into the form above is said to be entangled. In practice, determining whether a general $C^m \otimes C^n$ bipartite system can be factored into the form above is very hard. Thus, several tests have been introduced in order to determine whether a system is entangled. One test originates from quantum information theory, and it relies on calculating conditional entropies. Conditional entropy indicates the entropy of one subsystem after measuring the other, and it is given by

$$S(A|B) = S_{AB} - S_B. \quad (5)$$

A bipartite system with at least one degree of freedom having negative conditional entropy is entangled. Therefore a necessary condition for separability is that the conditional entropies are positive. Alternatively, one can calculate the mutual entropy. Mutual entropy indicates the entropy shared between the two subsystems, and it is given by

$$S(A{:}B) = S_A + S_B - S_{AB}. \quad (6)$$

Mutual entropy is bounded:

$$0 \leqslant S(A{:}B) \leqslant 2 \min[S_A, S_B]. \quad (7)$$

A bipartite system whose mutual entropy is excessive: $\min[S_A, S_B] < S(A{:}B) \leqslant 2 \min[S_A, S_B]$ is entangled. Therefore a necessary condition for separability is $0 < S(A{:}B) \leqslant \min[S_A, S_B]$.

A second powerful test introduced, originally by Peres [13], relies on partial transposition. Partial transposition is a blockwise transposition of a matrix and it is given by

$$\rho^{T_2}_{i\alpha, j\beta} \equiv \rho_{i\beta, j\alpha}. \quad (8)$$

A system whose partially transposed density matrix is negative is entangled. Therefore a necessary condition for separability is the positivity of the partially transposed density matrix (the Horodecki's [14] showed that it is also a sufficient condition for separability of $C^m \otimes C^n$ systems with $mn \leqslant 6$). Kraus and co-workers [15] analyzed systems supported on $C^2 \otimes C^N$. They gave necessary and sufficient conditions for separability of systems with positive partially transposed (PPT) density matrices. They showed that if the rank of density matrix is equal to $N$, the density matrix is separable. However, other criteria they devised are very complicated to implement for a generic density matrix.

The JCM is the simplest fully quantum model that describes the interaction between light and matter. The model consists of a single quantized two level atom interacting with a single quantized electromagnetic cavity mode under the rotating wave approximation (RWA) and the dipole approximation. The resonant JCM Hamiltonian [16] in the interaction representation is given by

$$H = \hbar \lambda (\sigma_+ a + \sigma_- a^\dagger). \quad (9)$$

For a system that starts in a pure state partial entropies of the field and the atom are equal at all times for a system that starts in a pure state [11]. As already indicated above, this result is a consequence of the equality between the eigenvalues of the partial density matrices in a bipartite system, which in turn is a consequence of the Schmidt decomposition. Since the partial entropies fluctuate together in time, their sum does not conserve the total entropy

$$S_a + S_f \geqslant S_{af}. \quad (10)$$

Thus, in this case partial entropies are not an additive (extensive) quantity. Nevertheless, knowledge of partial entropies gives information about the dynamics of the total system. The question that we address here is whether we can find states of light and matter for which the partial entropies are quasiadditive. Mixed states of light and matter can be natural candidates in this respect.

## II. RESULTS

We consider various combinations of initial pure and mixed states of light and matter. The mixed state for the electromagnetic field is chosen to be a Planck distribution for a single cavity mode $\rho_f(0) = \Sigma_{n=0}^{\infty} P_n |n\rangle\langle n|$, $P_n = [\bar{n}^n / (\bar{n}+1)^{n+1}]$, where $\bar{n}$ is the average number of photons in the cavity. The mixed state for the atom has the following general form $\rho_a(0) = P_e |e\rangle\langle e| + (1-P_e)|g\rangle\langle g|$, $0 < P_e < 1$. When the cavity mode is in exact resonance with the atomic transition, one can derive an analytic expression for the propagator and obtain analytic expressions for the full and partial density matrices at any time. The general solution for the full density matrix has the following form:

$$\rho_{af}(t) = e^{-iHt} \rho_{af}(0) e^{iHt}, \quad (11)$$

where $H$ is the interaction Hamiltonian mentioned above. By expanding $U(t) = e^{-iHt}$ in a Taylor series one obtains the analytic form of the propagator

$$U(t) = \begin{pmatrix} \cos(\lambda t \sqrt{aa^\dagger}) & -i \dfrac{\sin(\lambda t \sqrt{aa^\dagger})}{\sqrt{aa^\dagger}} a \\ -i \dfrac{\sin(\lambda t \sqrt{a^\dagger a})}{\sqrt{a^\dagger a}} a^\dagger & \cos(\lambda t \sqrt{a^\dagger a}) \end{pmatrix}.$$

To simulate the evolution of the entropic quantities, we truncated the set of Fock states that compose the Planck distribution at some $n_f$, where $\Sigma_{n=0}^{n_f} P_n \cong 1$. The accuracy of the results we obtained was tested by adding more Fock states to the Planck distribution to see if the simulated values changed.

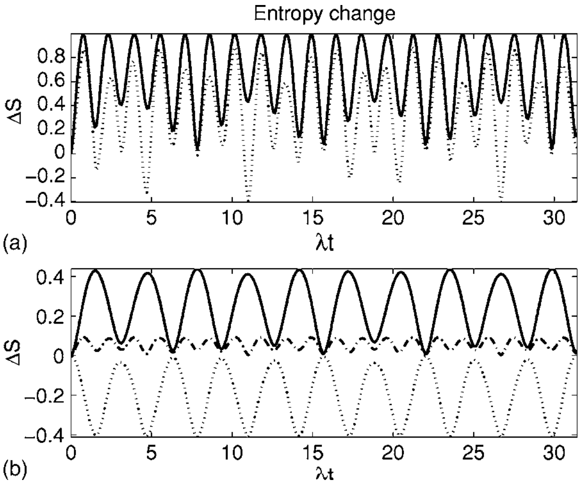


FIG. 1. Partial entropy change plots $\Delta S = S - S(0)$ (field dotted line, atomic solid line). (a) Atom initially in the excited state and a weakly excited thermal field, $\rho(0) = (|e\rangle\langle e|)_a \otimes (\Sigma_{n=0}^{\infty} P_n |n\rangle\langle n|; \bar{n} = 0.1)_f$. (b) Atom initially in the ground state and a weakly excited thermal field $\rho(0) = (|g\rangle\langle g|)_a \otimes (\Sigma_{n=0}^{\infty} P_n |n\rangle\langle n|)_f; \bar{n} = 0.1)_f$. The dash-dot curve represents the sum of atomic and field partial entropy changes, and it is quasiconserved.

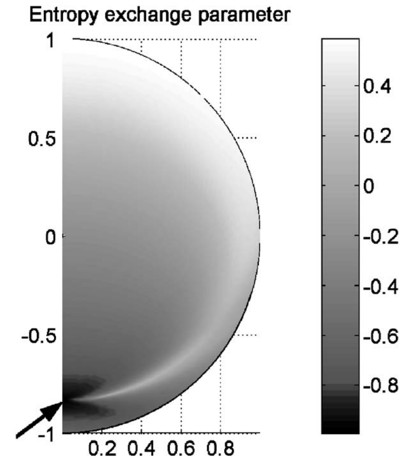


FIG. 2. Entropy exchange parameter for a weakly excited thermal field and various initial atomic density matrices, characterized by the longitudinal angle $\theta$ and the Bloch vector length $r$. The dark region in parameter space is characterized by significant entropy exchange ($P < -0.8$). Note that this region is centered around a "fixed" point where the partial density matrices are stationary (black arrow; see text).

In Fig. 1 the changes in atomic and field partial entropies $[S(t) - S(0)]$ vs the dimensionless quantity $\lambda t$ are plotted. In Fig. 1(a) the atom is initially excited and the field is in a weakly excited thermal state ($\bar{n} = 1$). This case is similar to a situation where both the field and atom are initially in a pure state, because both partial entropies rise and lower together (although they are not equal). In Fig. 1(b) the atom is initially in the ground state and the field is in a weakly excited thermal state. It is clear that there is entropy exchange between the atom and field (although the exchange is not complete). The sum of the field and atomic entropy changes (dash-dot curve) is quasiconserved as seen by its amplitude of fluctuations, which is significantly smaller than each of the partial entropy changes.

When the field is excited more, i.e., $\bar{n} = 1$, or $\bar{n} = 10$, no substantial entropy exchange between the atom and field occurs. In particular, when the field is highly excited ($\bar{n} \geq 10$) and the atom is close to a pure state, there is a sharp and rapid collapse of the atomic purity (sharp and rapid rise in the von Neumann entropy) with no substantial revival.

In order to determine which initial atomic states can exchange entropy with a weakly excited thermal field we considered different initial atomic states by discretizing the Bloch sphere with a longitudinal angle ($-\pi/2 \leq \theta \leq \pi/2$, for positive $\theta$ the atom is more excited) and a Bloch vector length ($0 < r \leq 1$), disregarding the azimuthal angle ($\phi$). We introduce a time averaged entropy exchange parameter defined by

$$P = \overline{\left( \frac{\Delta S_{a(f)}}{\Delta S_{f(a)}} \right)}, \tag{12}$$

where $\Delta S_{a(f)} = S_{a(f)}(t_j) - S_{a(f)}(t_{j-1})$, and in the numerator the smaller of the two instantaneous partial entropy changes was substituted. The time averaged entropy exchange parameter is suitable for quantifying the extent of entropy flow between a weakly excited thermal field and an atom due to the oscillatory nature of the partial entropies. Furthermore, the entropy exchange parameter is bounded: $-1 \leq P \leq 1$; when it tends to $-1$ there is high degree of entropy exchange, and when it tends to $+1$ the two partial entropies rise and lower together.

In Fig. 2 the entropy exchange parameter is plotted for a weakly excited thermal field and various initial atomic states. The dark region in parameter space is characterized by significant entropy exchange ($P < -0.8$); and in this region the atom is mostly in the ground state (ground state probability is more than 80%). Thus entropy exchange occurs most effectively when the atom is initially close to the ground state. An example for an almost complete entropy exchange is given in Fig. 3, where the changes in atomic and field partial entropies are shown for an atom initially in the state $\rho_a\left(r = \sqrt{\frac{7}{10}}, \theta = -\pi/2\right)$ ($P \cong 0.99$). In this case the sum of the two partial entropies is almost completely conserved as seen by the amplitude of fluctuation of the sum of the partial entropy changes, which is two orders of magnitude smaller than each of the partial entropy changes.

## III. ANALYTICAL DERIVATION OF THE ENTROPY EXCHANGE EFFECT

We now provide an analytical derivation for the entropy exchange effect. We first note that contours of significant entropy exchange ($P < -0.8$, Fig. 2) center around a point on the central vertical axis of the Bloch sphere. This is a "fixed" point where the partial density matrices are stationary, and it corresponds to a situation where the field temperature is identical to the atomic Boltzman temperature as we shall now show. Consider a situation where

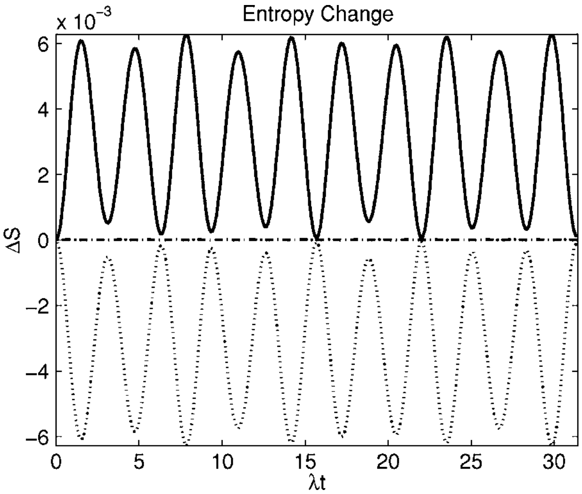


FIG. 3. Partial entropy change (field dotted line, atomic solid line, atomic+field dash dot curve) for a weakly excited thermal field ($\bar{n}=0.1$) and atom initially in the state $\rho_a(r=\sqrt{7/10};\theta=-\pi/2)$. Entropy exchange between the atom and field is almost complete ($P \approxeq 0.99$).

$$\frac{P_e}{P_g}=\frac{P_{n+1}}{P_n}=\frac{\bar{n}}{\bar{n}+1}=e^{-\hbar\omega/k_BT}, \tag{13}$$

where $\omega$ is the transition frequency. The full density matrix where both the atom and field are in mixed states (with no initial coherence) is given by

$$\rho_{af}(t)=U\begin{pmatrix} P_e\otimes\mathbf{P_n} & 0 \\ 0 & P_g\otimes\mathbf{P_n}\end{pmatrix}U^\dagger,$$

where $(\mathbf{P_n})_{mn}=\delta_{mn}P_n$. The partial density matrices are in diagonal form:

$$\rho_a(t)=\begin{pmatrix} A(t) & 0 \\ 0 & B(t)\end{pmatrix},$$

$$\rho_f^{nn}(t)=C(t)+D(t), \tag{14}$$

where

$$A(t)=P_e\sum P_n\cos^2(\alpha_n t)+P_g\sum P_{n+1}\sin^2(\beta_{n+1}t),$$

$$B(t)=P_e\sum P_{n-1}\sin^2(\alpha_{n-1}t)+P_g\sum P_n\cos^2(\beta_n t),$$

$$C(t)=P_e(P_n\cos^2(\alpha_n t)+P_{n-1}\sin^2(\alpha_{n-1}t)),$$

$$D(t)=P_g[P_{n+1}\sin^2(\beta_{n+1}t)+P_n\cos^2(\beta_n t)],$$

$$\alpha_{n-1}=\beta_n=\lambda\sqrt{n}. \tag{15}$$

By substituting Eq. (13) into Eq. (14) one obtains

$$\rho_a(t)=\rho_a(0)=\begin{pmatrix} P_e & 0 \\ 0 & P_g\end{pmatrix},$$

$$\rho_f^{nn}(t)=\rho_f^{nn}(0)=P_n. \tag{16}$$

For example, for a weakly excited thermal field with $\langle\bar{n}\rangle=0.1$, this point is located at $r=\frac{5}{6};\theta=-\frac{\pi}{2}$ (as indicated by the black arrow in Fig. 2).

We now proceed to analyze the vicinity around the fixed point, where substantial entropy exchange occurs. For a weakly excited thermal field the atomic fixed point is close to the ground state (see Fig. 2). Therefore, we consider an initial state where an atom in the ground state interacts with a thermal cavity mode: $\rho_{af}(0)=(|g\rangle\langle g|)_a\otimes(\sum_{n=0}^{\infty}P_n|n\rangle\langle n|)_f$. In this case the full density matrix is given by

$$\rho_{af}(t)=\begin{pmatrix} S_1\rho_f(0)S_1^\dagger & -iS_1\rho_f(0)C_1^\dagger \\ iC_1\rho_f S_1^\dagger & C_1\rho_f(0)C_1^\dagger\end{pmatrix},$$

where $C_1=\cos(\lambda t\sqrt{a^\dagger a})$, and $S_1=[\sin(\lambda t\sqrt{aa^\dagger})/\sqrt{aa^\dagger}]a$. The partial density matrices are in diagonal form

$$\rho_a(t)=\begin{pmatrix} \sum P_{n+1}\sin^2(\beta_{n+1}t) & 0 \\ 0 & \sum P_n\cos^2(\beta_n t)\end{pmatrix},$$

$$\rho_f^{nn}(t)=P_n\cos^2(\beta_n t)+P_{n+1}\sin^2(\beta_{n+1}t). \tag{17}$$

Since the purity is qualitatively similar to the von Neumann entropy we can analyze the time dependence of the partial purities. The time derivatives of the partial purities are given by:

$$\frac{d(\mathrm{Tr}(\rho_a^2))}{dt}=2\sum P_{n+1}\sin^2(\beta_{n+1}t)\sum P_{n+1}\beta_{n+1}\sin(2\beta_{n+1}t) - 2\sum P_n\cos^2(\beta_n t)\sum P_n\beta_n\sin(2\beta_n t)$$

$$\frac{d(\mathrm{Tr}(\rho_f^2))}{dt}=2\sum (P_{n+1}\beta_{n+1}\sin(2\beta_{n+1}t)-P_n\beta_n\sin(2\beta_n t)) \times[P_n\cos^2(\beta_n t)+P_{n+1}\sin^2(\beta_{n+1}t)]. \tag{18}$$

One can rearrange the time derivatives in Eq. (18) in terms of $P_iP_j$ multiplying sums of oscillatory functions. For a weakly excited thermal cavity ($\bar{n}=0.1$) the $P_iP_j$ decrease by approximately an order of magnitude for every increment in either $i$ or $j$. Approximating the derivatives of the partial purities by the first nonvanishing term ($P_0P_1=0.0751$) one obtains

$$\frac{d(\mathrm{Tr}(\rho_a^2))}{dt}\approx -2P_0P_1\beta_1\sin(2\beta_1 t),$$

$$\frac{d(\mathrm{Tr}(\rho_f^2))}{dt}\approx 2P_0P_1\beta_1\sin(2\beta_1 t). \tag{19}$$

The above approximation shows explicitly that there is entropy exchange between a weakly excited thermal cavity mode and an atom initially in the ground state. The leading terms in the time derivatives of the partial purities have the same functional dependence on time but with opposite signs.

Consider now an initial state of an excited atom interacting with a thermal cavity mode $\rho_{af}(0)=(|e\rangle\langle e|)_a \otimes(\Sigma_{n=0}^{\infty}P_n|n\rangle\langle n|)_f$. In this case the full density matrix is given by

$$\rho_{af}(t)=\begin{pmatrix} C_2\rho_f(0)C_2^\dagger & iC_2\rho_f(0)S_2^\dagger \\ -iS_2\rho_f C_2^\dagger & S_2\rho_f(0)S_2^\dagger \end{pmatrix},$$

where $C_2=\cos(\lambda t\sqrt{aa^\dagger})$, and $S_2=[\sin(\lambda t\sqrt{a^\dagger a})/\sqrt{a^\dagger a}]a^\dagger$. Again the partial density matrices are in diagonal form

$$\rho_a(t)=\begin{pmatrix} \sum P_n\cos^2(\alpha_n t) & 0 \\ 0 & \sum P_{n-1}\sin^2(\alpha_{n-1}t) \end{pmatrix},$$

$$\rho_f^{nn}(t)=P_n\cos^2(\alpha_n t)+P_{n-1}\sin^2(\alpha_{n-1}t). \tag{20}$$

The time derivatives of the partial purities are given by

$$\frac{d(\mathrm{Tr}(\rho_a^2))}{dt}=-2\sum P_n\alpha_n\sin(2\alpha_n t)\sum P_n\cos^2(\alpha_n t)$$
$$+2\sum P_{n-1}\sin^2(\alpha_{n-1}t)\sum P_{n-1}\alpha_{n-1}\sin(2\alpha_{n-1}t),$$

$$\frac{d(\mathrm{Tr}(\rho_f^2))}{dt}=2\sum[P_{n-1}\alpha_{n-1}\sin(2\alpha_{n-1}t)-P_n\alpha_n\sin(2\alpha_n t)]$$
$$\times[P_n\cos^2(\alpha_n t)+P_{n-1}\sin^2(\alpha_{n-1}t)]. \tag{21}$$

Rearranging the time derivatives in Eq. (21) in terms of $P_iP_j$, as in the case when the atom was initially in the ground state, and approximating them by the first nonvanishing term ($P_0^2=0.8264$) one obtains

$$\frac{d(\mathrm{Tr}(\rho_a^2))}{dt}\approx-P_0^2\alpha_0\sin(4\alpha_0 t),$$

$$\frac{d(\mathrm{Tr}(\rho_f^2))}{dt}\approx-P_0^2\alpha_0\sin(4\alpha_0 t). \tag{22}$$

The above approximation shows that in this case the partial purities oscillate together in time. The leading terms in the time derivatives of the partial purities are identical.

When the cavity mode is excited thermally with more photons, i.e., $\bar{n}\geqslant 1$, the terms in the time derivatives of the partial purities rearranged according to $P_iP_j$ no longer decrease by an order of magnitude as $i,j$ increase. Now many terms contribute substantially to the partial entropy behavior; since these terms are not identical in both degrees of freedom, observing entropy exchange becomes increasingly less probable as $\bar{n}$ increases.

## IV. ENTROPY CORRELATIONS AND ENTANGLEMENT

We consider now the relationship between entropy correlations developed between the atom and field discussed in previous sections, and entanglement of the various atomic-field states. One would expect that as $S_A+S_B\to S_{AB}$ the system becomes separable. This is motivated by the following formula from statistical mechanics [17]:

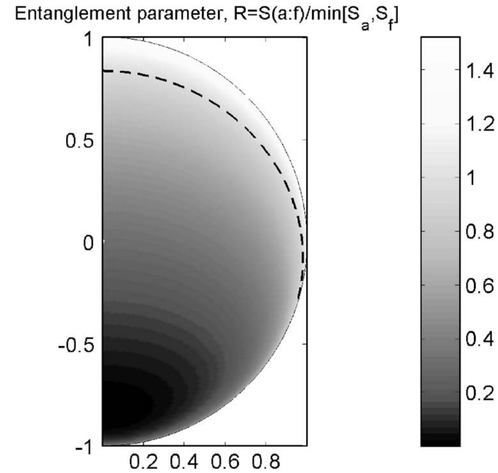


FIG. 4. Time averaged entanglement parameter based on the ratio between mutual entropy and partial entropies for different initial atomic states coupled to a weakly excited thermal field ($\bar{n}=0.1$). The region in parameter space where $\bar{R}\leqslant 1$ (dotted line) encompasses the entire region of entropy exchange ($P<-0.8$), but in addition includes significant area where there is no entropy exchange (even regions with $P>0$; compare with Fig. 2).

$$S_{AB}=S_A+S_B+k_B\sum_{i,j}\ln\frac{P_A^iP_B^j}{P_{AB}^{ij}}. \tag{23}$$

If $S_{AB}=S_A+S_B$, then $P_{AB}^{ij}=P_A^iP_B^j$ (since the sum over the natural logarithm terms has to be zero). This means that the two subsystems are independent (if $S_A+S_B\cong S_{AB}$ then the two subsystems are weakly classically correlated). In regions of the Bloch sphere where the entropy exchange is almost complete ($\Delta S_a=\simeq-\Delta S_f$):

$$S_a+S_f\simeq S_a(0)+S_f(0)=S_{af}(0)=S_{af}. \tag{24}$$

The last two equalities hold since the combined atomic-field system begins to evolve from a separable state, and since the evolution is unitary, respectively. Therefore, we would expect that in regions of the Bloch sphere where the entropy exchange is almost complete, there should be minimal entanglement. We now proceed with comparing our measure for entropy exchange with various entanglement measures.

As discussed in the Introduction, bipartite systems with excessive mutual entropy are entangled. We introduce an entanglement parameter $R$ based on the ratio between mutual entropy and partial entropies

$$R=\frac{S(a{:}f)}{\min[S_a,S_f]}. \tag{25}$$

It follows from Eq. (7) that $0\leqslant R\leqslant 2$, with $1<R\leqslant 2$ being a sufficient condition for entanglement and conversely $0\leqslant R\leqslant 1$ being a necessary condition for separability. In Fig. 4 the time averaged entanglement parameter, $\bar{R}$, is plotted for different initial atomic states coupled to a weakly excited thermal field. The region in parameter space with $\bar{R}\leqslant 1$ (bounded by a dashed line) indicates that the *necessary* condition for separability is fulfilled for most of the evolution

time. This region is large, occupying about 75% of the interior of the Bloch sphere. It encompasses the entire region of entropy exchange ($P < -0.8$), but in addition includes significant area where there is no entropy exchange (even regions with $P > 0$). This is consistent with the observation that $0 < \bar{R} \leq 1$ is a *necessary* but not a *sufficient* condition for separability. Although the region defined by this necessary condition is too broad to be informative, note that in cases where there is substantial entropy exchange as seen in Fig. 2, $\bar{R}$ approaches zero in Fig. 4.

We consider now the PPT test, where the existence of even a single negative eigenvalue of the partially transposed density matrix is a *sufficient* criterion for entanglement. Since the PPT test can only be applied for finite dimensional density matrices, in order to apply it to the Jaynes-Cummings model we need to truncate the infinite set of Fock states. We do this according to the following procedure:

$$\rho_f(0) = \sum_{n=0}^{\infty} P_n |n\rangle\langle n| \tag{26}$$

$$\rightarrow \sum_{n=0}^{n_f} P_n |n\rangle\langle n| + \left(1 - \sum_{n=0}^{n_f} P_n\right) |n_{f+1}\rangle\langle n_{f+1}|; \tag{27}$$

in words, all residual probability from the truncation is placed in the $(n_f+1)$st state. The criterion for truncating the infinite set of Fock states is that the partial entropies do not change by more than $10^{-14}$. We find that no matter what the initial state of the atom is, there is always at least one negative eigenvalue of the partially transposed density matrix. However, further investigation shows that this result is in essence due to the truncation of the infinite Fock basis to a finite size. Specifically, we find that in all regions of the Bloch sphere there is always one negative eigenvalue of extremely small magnitude whose size is on the order of the probabilities tail of the infinite Fock basis that gets truncated. Since our primary interest here is in the original and not in the truncated Jaynes-Cummings model, we are inclined to ignore this negative eigenvalue of extremely small magnitude and to interpret the test of partial transposition in terms of the remaining negative eigenvalues. The remaining negative eigenvalues are all of sizable magnitude and robust with respect to the size of the Fock basis.

With that introduction, we define $\lambda_m^{T_2}$ as the largest (in absolute value) negative eigenvalue of the partially transposed density matrix, $\rho^{T_2}$. We then introduce as an entanglement measure

$$E = \ln(|\overline{\lambda_m^{T_2}}|), \tag{28}$$

where the overbar signifies the time average. In Fig. 5 we plot the entanglement parameter $E$ for different initial states for $n_f = 13$. Clearly, there are two well-defined regions in atomic parameter space. The dark region corresponds to a single negative eigenvalue of $\rho^{T_2}$ on the order of $10^{-16} - 10^{-18}$. This region is characterized by marginal entanglement, corresponding in all likelihood to completely separable evolution in the untruncated Jaynes-Cummings model. Notice that the region of atomic parameter space with substantial entropy exchange ($P < -0.8$ in Fig. 2, shown as a solid white line in Fig. 5) falls precisely within the region with minimal entanglement parameter $E$. The lighter shade region in Fig. 5 is characterized by two or more negative eigenvalues of $\rho^{T_2}$. We believe that this region corresponds to significant entanglement in the original, untruncated Jaynes-Cummings model. This region is clearly orthogonal to that of the region of entropy exchange.

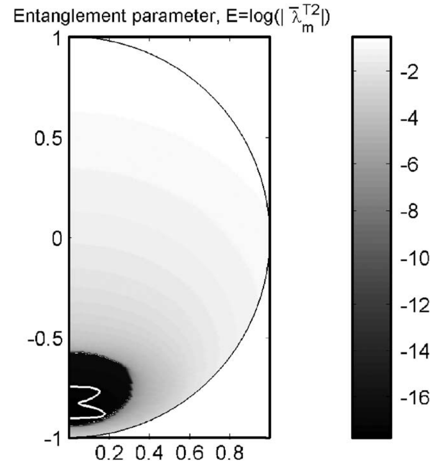


FIG. 5. Entanglement parameter based on the negativity of the eigenvalues of the partially transposed density matrix for a weakly excited thermal field ($\bar{n} \approx 0.1$) and various atomic states. The region of atomic parameter space with substantial entropy exchange ($P < -0.8$, bounded by a solid white line) falls within the region with minimal entanglement parameter (dark), which is dominated by anticorrelated partial entropies.

## V. CONCLUSION

We have explored entropy correlations between a quantized cavity mode and a single atom in the framework of the JCM by considering both pure and mixed atomic and field states. In particular, we explored the regime of entropy *exchange* between light and matter. We presented two qualitatively different entropy correlations. The first type of correlation is a case where both the atomic and field partial entropies fluctuate together in time. This is reminiscent of the case where both the atom and field start out in a pure state, and consequently their partial entropies are identical at all times. The second type of correlation is a case where the atomic and field partial entropies are anticorrelated. This implies that there is entropy exchange between the atom and the field. Since substantial entropy exchange occurs when the field is in a weakly excited thermal state ($\bar{n} = 0.1$) we introduced an entropy exchange parameter in order to determine which initial atomic states can efficiently exchange entropy with the field. Substantial entropy exchange occurs when the atom is initially close to the ground state. We showed that contours of substantial entropy exchange center around a stationary point in the Bloch sphere. This point corresponds to a situation where the initial field and atomic (Boltzman) temperature exactly match. By analyzing the partial purities we derived an analytic approximation for the entropy exchange phenomenon. The change in the atomic and field partial purities have the same functional dependence on time but with opposite signs.

It is natural to ask if there is any connection between the entropy exchange that we observe in this paper and a thermodynamic heat exchange process. Although we cannot rule out this possibility, we believe that the connection is unlikely. In classical thermodynamics, heat exchange is related to the transfer of energy that is in some sense degraded. In our case, because the entropy exchange is periodic and the system is small, there is no reason to believe *a priori* that the energy that is exchanged is unrecoverable for pure work. Nevertheless, if one examines the ratio $[(dE/dt)/(dS/dt)]$, where $dE/dt$ and $dS/dt$ are the instantaneous rates of energy and entropy change of each subsystem respectively, there is some overall correlation with the initial temperature $T$ of the field, in agreement with the classical result $dE/dS=T$.

A well-established measure of entanglement is the test of negative eigenvalues of the partial transpose of the density matrix (PPT test). Unfortunately, the PPT test cannot be applied directly to the Jaynes-Cummings model with its formally infinite Hilbert space for the field. We therefore applied the PPT test to a model in which the infinite Fock basis was truncated. We found that there is always at least one negative eigenvalue for the partially transposed density matrix. However, further investigation showed that this result is due to the truncation of the Fock basis. Since our primary interest here is in the original and not in the truncated Jaynes-Cummings model, we essentially discard this negative eigenvalue of extremely small magnitude and interpret the test of partial transposition in terms of only the remaining negative eigenvalues. We find that the region of the Bloch sphere in which there are no additional negative eigenvalues maps very closely onto the region where there is substantial entropy exchange between atom and field. This result is intuitively appealing, showing a strong negative correlation between entropy exchange and entanglement.

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[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).
[2] C. E. Shannon, Bell Syst. Tech. J. **27**, 379 (1948); **27**, 623 (1948).
[3] B. Schumacher, Phys. Rev. A **51**, 2738 (1995).
[4] B. Schumacher and M. D. Westmoreland, Phys. Rev. A **56**, 131 (1997).
[5] A. S. Holevo, IEEE Trans. Inf. Theory **44**, 269 (1998).
[6] A. Rényi, *Probability Theory* (North-Holland, Amsterdam, 1970).
[7] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[8] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996).
[9] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[10] N. J. Cerf and C. Adami, Phys. Rev. Lett. **79**, 5194 (1997).
[11] S. J. D. Phoenix and P. L. Knight, Ann. Phys. (N.Y.) **186**, 381 (1988).
[12] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[13] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[14] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[15] B. Kraus, J. I. Cirac, S. Karnas, and M. Lewenstein, Phys. Rev. A **61**, 062302 (2000).
[16] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
[17] F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965).