

# Resonance scattering formalism for the hydrogen lines in the presence of magnetic and electric fields

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We derive a formalism for the computation of resonance-scattering polarization of hydrogen lines in the presence of simultaneous magnetic and electric fields, within a framework of the quantum theory of polarized line formation in the limit of complete frequency redistribution and of collisionless regime. Quantum interferences between fine-structure levels are included in this formalism. In the presence of a magnetic field, these interferences affect, together with the magnetic Hanle effect, the polarization of the atomic levels. In the presence of an electric field, interferences between distinct orbital configurations are also induced, further affecting the polarization of the hydrogen levels. In turn, the electric field is expected to affect the polarization of the atomic levels (*electric Hanle effect*), in a way analogous to the magnetic Hanle effect. We find that the simultaneous action of electric and magnetic fields give rise to complicated patterns of polarization and depolarization regimes, for varying geometries and field strengths.

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## I. INTRODUCTION

In this paper, we adopt the quantum theory of polarized line formation in the limit of complete frequency redistribution (CRD) and of collisionless regime, developed by Landi Degl'Innocenti [1–4], in order to derive a general formalism for the description of resonance scattering polarization from a hydrogen atom subject to simultaneous magnetic and electric fields. This formalism allows for the possibility that quantum interferences between distinct  $J$  levels within an atomic term (generally specified by its  $L$  and  $S$  values, assuming that the approximation of  $L$ - $S$  coupling is valid, and by the principal quantum number  $n$ ) may be important. In the additional presence of an electric field, in the case of the hydrogen atom (and, more generally, of all one-electron atoms, which are sensitive to the linear Stark effect), quantum interferences between different  $L$  levels are also induced, due to the dipolar character of the electric Hamiltonian. A study of the effect of electric fields on the atomic polarization of hydrogen (*electric Hanle effect*) was presented in Ref. [5], for the special case of Lyman  $\alpha$ .

In order to meet the requirements of CRD, we must assume that the incident radiation field has no spectral structure over the frequency interval of the separation between interfering levels (*flat-spectrum approximation*). In the case of hydrogen, CRD can be considered a reasonably good approximation, e.g., in solar plasmas, for the lines of the Balmer and the Paschen series, because the line absorption spectrum coming from the solar atmosphere typically is Doppler broadened to widths much larger than the frequency structure of the line multiplets. This is true also in the presence of external fields, so far that  $|H_{E,B}| \lesssim |H_{FS}|$ , where  $H_E$  and  $H_B$  are the electric- and magnetic-field Hamiltonians, respectively, and  $H_{FS}$  is the fine-structure (FS) contribution

to the atomic Hamiltonian. (We neglect in this paper the effect of the hyperfine structure; see the conclusive section.) In the case of Lyman  $\alpha$ , instead, partial redistribution (PRD) is known to play a role in the formation of the line emission spectrum even in the solar atmosphere.<sup>1</sup> Another limitation of this work is that the formalism is developed for the collisionless regime, so the statistical equilibrium of the atomic system is completely determined by the incident radiation only.<sup>2</sup>

In the CRD approximation, the problem of computing the polarization of the scattered radiation by an atom subject to external fields can be separated in two stages. The first stage involves the solution of the statistical equilibrium of the atom in the external fields, illuminated by the incident radiation. Once this solution, which provides the excitation state of the atom, is known, we are able to compute the re-emitted (scattered) radiation and its polarization characteristics. We remark that the possibility of separating the scattering problem in such two-stage problem is a direct consequence of the hypothesis of CRD. In fact, since CRD is equivalent to non-coherent scattering of radiation, it becomes possible to describe scattering as the temporal succession of two first-order atom-photon processes, one of absorption of radiation (which determines the excitation state of the atomic gas), and one of re-emission (where the excitation state previously determined is used to calculate the scattered radiation). PRD requires instead the possibility of describing both coherent and noncoherent scattering, and therefore higher-order atom-photon processes must also be taken into account.

<sup>1</sup>Inclusion of PRD effects in a self-consistent theory of resonance scattering polarization for complex atoms is an ongoing effort. Until it is accomplished, CRD is a necessary limitation of any theoretical modeling.

<sup>2</sup>Within the limit of the impact approximation [6], the extension of the formalism to include collisions is straightforward, since the collisional rates can simply be added to the rates for the corresponding radiative processes.

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Following this scheme, in Sec. II, we derive the expressions of the radiative rates that enter the equations of the statistical equilibrium (SE) of a one-electron atom with FS. The solution of the SE problem is given in terms of the density-matrix elements of the atomic system, which describes both the population of the atomic levels, and their state of polarization and of mutual interference. Next, in Sec. III, we derive the expressions of the coefficients entering the transfer matrix in the radiative-transfer (RT) equation for polarized radiation. This equation governs the production and the transport of polarized radiation through a gas that is generally optically thick, and it allows the computation of the intensity and the polarization state of the scattered radiation, when the density matrix of the atomic system is known at every point in the gas, by having previously solved the SE problem. In both Secs. II and III, we adopt the formalism of the irreducible spherical tensors [7,8], because it provides a more direct interpretation of the various quantities in terms of physical symmetries of the interaction processes of the atomic system with the radiation field. Similarly, for the description of the polarization state of the incident and scattered radiation, we adopt the formalism of the Stokes parameters [9,10].

In Sec. IV, we present some numerical examples of the application of our formalism. We show how the simultaneous action of electric and magnetic fields (parallel to each other, in the special case of these examples) modifies the atomic orientation of the hydrogen levels, and how this translates into a characteristic polarization signature of the scattered radiation in the limit of an optically thin gas permeated by both fields. In particular, we show how the additional presence of even small electric fields can act as a “catalyst” of the atomic orientation induced by a magnetic field. This atomic orientation must translate observationally in an amount of net circular polarization of the scattered radiation significantly larger than in the case where only a magnetic field is present, as it is clearly shown in Sec. IV. Finally, in Sec. V, we summarize the main results of this paper.

## II. EQUATIONS OF STATISTICAL EQUILIBRIUM

In this section we derive the statistical-equilibrium (SE) equations. These represent the quantum-mechanical evolution equation for the statistical operator of the atomic system  $\rho_A$ , projected on some basis of choice for the Hilbert space of the atom. If we select the basis of eigenstates  $|\nu\rangle$  of the atomic Hamiltonian  $H_A$ , where  $\nu$  specifies a complete set of commuting observables of the atomic system, the SE equations have the following form [1,4]:

$$\begin{aligned} \frac{d}{dt}\rho_{\nu\nu'} = & -i\omega_{\nu\nu'}\rho_{\nu\nu'} - \sum_{\nu''\nu'''} [R_A(\nu, \nu'; \nu'', \nu''') \\ & + R_S(\nu, \nu'; \nu'', \nu''') + R_E(\nu, \nu'; \nu'', \nu''')] \rho_{\nu''\nu'''} \\ & + \sum_{\nu_1\nu_1'} T_A(\nu, \nu'; \nu_1, \nu_1') \rho_{\nu_1\nu_1'} + \sum_{\nu_u\nu_u'} [T_S(\nu, \nu'; \nu_u, \nu_u') \\ & + T_E(\nu, \nu'; \nu_u, \nu_u')] \rho_{\nu_u\nu_u'} \end{aligned} \quad (1)$$

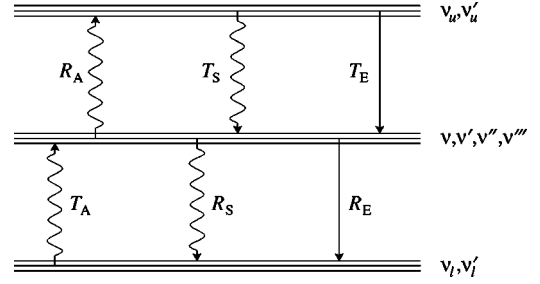


FIG. 1. Grotrian diagram illustrating the different radiative processes in the multiterm atom. Straight arrows indicate spontaneous emission processes, whereas wiggly lines indicate radiation-induced processes (absorption and stimulated emission).

In Eq. (1), only radiative processes are considered. The meaning of the different *transfer rates*,  $T_{A,E,S}$ , and *relaxation rates*,  $R_{A,E,S}$ , associated with these processes is illustrated in Fig. 1, in terms of absorption and emission (spontaneous and stimulated) processes.

We will develop our formalism for the specific case of one-electron atoms (typically, the hydrogen atom), because of the particular importance that electric fields have for such systems, because of their sensitivity to the linear Stark effect. For one-electron atoms, the general ket of the standard basis of eigenstates of  $H_A$ , in the absence of external fields, can be chosen in the form

$$|\nu\rangle = |nLSJM\rangle, \quad (2)$$

having assumed that the conditions for  $L$ - $S$  coupling are valid and that the effects of hyperfine structure are negligible (see also the conclusive section). Obviously  $S=1/2$  in the case of one-electron atoms.

When both magnetic and electric fields are present, only the Bohr configuration number  $n$  remains a good quantum number (if we limit our investigation to field strengths such that configuration mixing can be neglected; see, e.g., Ref. [11]), besides the fixed spin quantum number  $S$ . In other words,  $H_B + H_E$  “mixes” states of the form (2) having different  $L$ ’s,  $J$ ’s, and  $M$ ’s. Because of this, the eigenstates of  $H_A$  in the presence of arbitrarily oriented  $\mathbf{B}$  and  $\mathbf{E}$  can be written as

$$|nS, \mu\rangle = \sum_{LJM} C_{\mu}^{LJM}(nS) |nLSJM\rangle, \quad (3)$$

with eigenvalues  $\lambda_{\mu}(nS)$ , where  $\mu$  is a discrete index spanning the dimension of the Hilbert subspace of the level  $n$ . If we choose the frame of reference for the description of the atomic states such that the  $z$  axis (the quantization axis) is directed like  $\mathbf{B}$ , and the  $x$  axis lies in the plane determined by the two vectors  $\mathbf{B}$  and  $\mathbf{E}$  (see Fig. 2 later on), then we can prove that  $H_A$  is real symmetric [11], so the coefficients  $C_{\mu}^{LJM}(nS)$  can also be chosen to be real quantities, satisfying the following orthogonality properties (valid within the Hilbert subspace of each Bohr level  $n$ ):

$$\sum_{\mu} C_{\mu}^{LJM}(nS) C_{\mu}^{L'J'M'}(nS) = \delta_{LL'} \delta_{JJ'} \delta_{MM'}, \quad (4a)$$

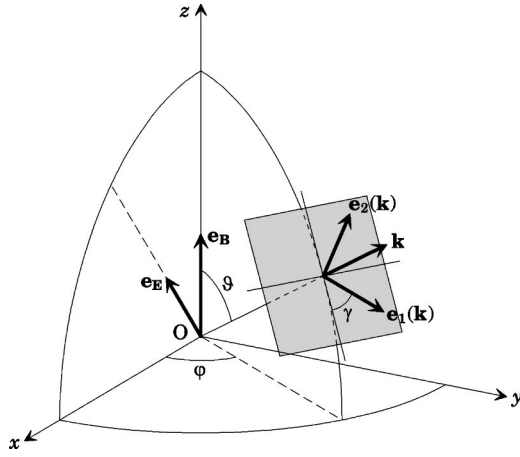


FIG. 2. Geometry defining the propagation vector  $\mathbf{k}$  and the reference direction of linear polarization  $\mathbf{e}_1(\mathbf{k})$  in the frame of reference adopted for the diagonalization of the atomic Hamiltonian. The plane of polarization, generated by the basis vectors for linear polarization,  $\mathbf{e}_1(\mathbf{k})$  and  $\mathbf{e}_2(\mathbf{k})$ , is normal to the propagation vector  $\mathbf{k}$ . The position angle  $\gamma$  corresponds to the position of the acceptance axis of the linear polarizer through which  $\frac{1}{2}(I+Q)$  is measured [see after Eq. (25) for the definition of the Stokes parameters  $I$  and  $Q$ ].

$$\sum_{LJM} C_{\mu}^{LJM}(nS) C_{\mu'}^{LJM}(nS) = \delta_{\mu\mu'}. \quad (4b)$$

The atomic Hamiltonian can then be expressed through its spectral representation,

$$\begin{aligned} H_A &= \sum_n \sum_{\mu} \lambda_{\mu}(nS) |nS, \mu\rangle \langle nS, \mu| \\ &= \sum_n \sum_{\mu} \sum_{LJM} \sum_{L'J'M'} \lambda_{\mu}(nS) C_{\mu}^{LJM}(nS) C_{\mu'}^{L'J'M'}(nS) \\ &\quad \times |nLSJM\rangle \langle nL'SJ'M'|, \end{aligned} \quad (5)$$

from which we derive the important sum rule

$$\sum_{\mu} \lambda_{\mu}(nS) C_{\mu}^{LJM}(nS) C_{\mu'}^{L'J'M'}(nS) = \langle nLSJM | H_A | nL'SJ'M' \rangle. \quad (6)$$

In order to derive explicit expressions for the SE equations, we introduce the following correspondences:

$$\nu \equiv nS, \mu, \quad \nu' \equiv nS, \mu', \quad \nu'' \equiv nS, \mu'', \quad \nu''' \equiv nS, \mu''',$$

$$\nu_l \equiv n_l S, \mu_l, \quad \nu'_l \equiv n_l S, \mu'_l,$$

$$\nu_u \equiv n_u S, \mu_u, \quad \nu'_u \equiv n_u S, \mu'_u$$

by which the atomic density matrix becomes, with evident notation,

$$\begin{aligned} \rho_{\nu\nu'} &\equiv {}^{nS}\rho(\mu, \mu') \equiv \langle nS, \mu | \rho | nS, \mu' \rangle \\ &= \sum_{LJM} \sum_{L'J'M'} C_{\mu}^{LJM}(nS) C_{\mu'}^{L'J'M'}(nS) {}^{nS}\rho(LJM, L'J'M'), \end{aligned} \quad (7a)$$

and conversely, using Eq. (4a),

$${}^{nS}\rho(LJM, L'J'M') = \sum_{\mu\mu'} C_{\mu}^{LJM}(nS) C_{\mu'}^{L'J'M'}(nS) {}^{nS}\rho(\mu, \mu'). \quad (7b)$$

One task is to express the SE equations for the density matrix in the standard representation,  ${}^{nS}\rho(LJM, L'J'M')$ . In order to do so, we must apply the double summation of Eq. (7b),

$$\sum_{\mu\mu'} C_{\mu}^{LJM}(nS) C_{\mu'}^{L'J'M'}(nS) \times, \quad (8)$$

to each side of Eq. (1). Ultimately we want to work with the irreducible spherical-tensor representation of the statistical operator. For this, we will also need the transformation formula,

$$\begin{aligned} {}^{nS}\rho(LJM, L'J'M') &= \sum_{KQ} (-1)^{J-M} \Pi_K \begin{pmatrix} J & J' & K \\ M & -M' & -Q \end{pmatrix} \\ &\quad \times {}^{nS}\rho_Q^K(LJ, L'J'), \end{aligned} \quad (9a)$$

and its inverse

$$\begin{aligned} {}^{nS}\rho_Q^K(LJ, L'J') &= \sum_{MM'} (-1)^{J-M} \Pi_K \begin{pmatrix} J & J' & K \\ M & -M' & -Q \end{pmatrix} \\ &\quad \times {}^{nS}\rho(LJM, L'J'M'), \end{aligned} \quad (9b)$$

having adopted the shorthand notation [12]

$$\Pi_{ab \cdots n} = \sqrt{(2a+1)(2b+1) \cdots (2n+1)}. \quad (10)$$

Both operations (8) and

$$\sum_{MM'} (-1)^{J-M} \Pi_K \begin{pmatrix} J & J' & K \\ M & -M' & -Q \end{pmatrix} \quad (11)$$

must then be applied to each side of Eq. (1).

#### A. Evaluation of the depolarization kernel

We first concentrate on the imaginary term in Eq. (1),

$$\omega_{\nu\nu'} \rho_{\nu\nu'} \equiv \hbar^{-1} [\lambda_{\mu}(nS) - \lambda_{\mu'}(nS)] {}^{nS}\rho(\mu, \mu').$$

To this expression, we consecutively apply the substitutions (7a) and (9a), along with operations (8) and (11). For this transformation, the identities (4a) and (6) are also needed. We obtain in the end

$$\begin{aligned} \omega_{\nu\nu'} \rho_{\nu\nu'} &\rightarrow \sum_{L''J''} \sum_{L'''J'''} \sum_{K'Q'} N(nS; L''J''L'J'KQ; L''J''L'''J'''K'Q') \\ &\quad \times {}^{nS}\rho_{Q'}^{K'}(L''J'', L'''J'''), \end{aligned} \quad (12)$$

where

$$\begin{aligned}
N(nS;LJL'J'KQ;L''J''L'''J'''K'Q') &= \hbar^{-1} \sum_{MM'} \Pi_{KK'} \begin{pmatrix} J & J' & K \\ M & -M' & -Q \end{pmatrix} \\
&\times \left[ \delta_{L'L''} \delta_{J'J''} \sum_{M''} (-1)^{J-M+J''-M''} \begin{pmatrix} J'' & J' & K' \\ M'' & -M' & -Q' \end{pmatrix} \langle nLSJM|H_A|nL''SJ''M'' \rangle \right. \\
&\quad \left. - \delta_{LL''} \delta_{JJ''} \sum_{M''} \begin{pmatrix} J & J'' & K' \\ M & -M'' & -Q' \end{pmatrix} \langle nL'''SJ'''M'''|H_A|nL'SJ'M' \rangle \right], \quad (13)
\end{aligned}$$

is the *depolarization kernel* (see, e.g., Ref. [4], Chap. 7). In order to explicitly evaluate it, we need the expression for the matrix elements of the atomic Hamiltonian in the presence of the external fields,  $\mathbf{B}$  and  $\mathbf{E}$ , for the particular reference frame described in the previous section. From the Wigner-Eckart theorem and its corollaries [13], this is given by

$$\begin{aligned}
\langle nLSJM|H_A|nL'SJ'M' \rangle &= \delta_{LL'} \delta_{JJ'} \delta_{MM'} \hbar \omega_{LJ}^{nS} \\
&+ \delta_{LL'} \delta_{MM'} \mu_0 B \left[ \delta_{JJ'} M + (-1)^{L+S+J+J'+M} \Pi_{JJ'S} \sqrt{S(S+1)} \begin{pmatrix} J & J' & 1 \\ -M & M & 0 \end{pmatrix} \begin{Bmatrix} J & J' & 1 \\ S & S & L \end{Bmatrix} \right] \\
&+ a_0 e_0 E (-1)^{S-M'} \Lambda(nL, nL') \Pi_{JJ'} \begin{Bmatrix} L & L' & 1 \\ J' & J & S \end{Bmatrix} \sum_Q \begin{pmatrix} J & J' & 1 \\ -M & M' & -Q \end{pmatrix} (\mathbf{e}_E)_Q, \quad (14)
\end{aligned}$$

where  $(\mathbf{e}_E)_Q$ , for  $Q=0, \pm 1$ , are the spherical components of the unit vector of the electric field, and where we also defined

$$\Lambda(nL, n'L') = \Pi_{LL'} \langle nL|r|n'L' \rangle \begin{pmatrix} L & L' & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

having expressed the dipole operator  $\mathbf{r}$  in units of the Bohr radius  $a_0$ .

We notice that the function  $\Lambda(nL, n'L')$  is purely real, and also symmetric, the dipole matrix element in it being calculated through the well-known Gordon formula [14], valid for  $|L-L'|=1$ . In the case of the electric contribution to Eq. (13), this function is evaluated only for  $n=n'$ , in which case the dipole matrix element in Eq. (15) has the much simpler expression [14]

$$\langle nL|r|nL-1 \rangle = \frac{3}{2} n \sqrt{n^2 - L^2} = \langle nL-1|r|nL \rangle. \quad (16)$$

The FS term in Eq. (14) determines a diagonal contribution to the depolarization kernel, which is simply evaluated using the orthogonality properties of the  $3j$  symbols, in order to perform the summation over the magnetic quantum numbers. We find

$$\begin{aligned}
N(nS;LJL'J'KQ;L''J''L'''J'''K'Q')_{\text{FS}} \\
= \delta_{LL''} \delta_{JJ''} \delta_{L'L''} \delta_{J'J''} \delta_{KK'} \delta_{QQ'} \omega_{LJ}^{nS} \omega_{L'L''}^{nS}. \quad (17a)
\end{aligned}$$

The nondiagonal contribution to the depolarization kernel, determined by the external fields, is instead evaluated with the help of Eq. (6) on p. 454 of Ref. [12]. The magnetic and electric contributions are, respectively,

$$\begin{aligned}
N(nS;LJL'J'KQ;L''J''L'''J'''K'Q')_B &= \delta_{LL''} \delta_{L'L''} \delta_{QQ'} \omega_B (-1)^{J'+J''-Q} \Pi_{KK'} \begin{pmatrix} K & K' & 1 \\ -Q & Q & 0 \end{pmatrix} \left[ \delta_{J'J''} \Gamma_{LS}(J, J'') \begin{Bmatrix} K & K' & 1 \\ J'' & J & J' \end{Bmatrix} \right. \\
&\quad \left. + \delta_{JJ''} (-1)^{K+K'} \Gamma_{L'S}(J''', J') \begin{Bmatrix} K & K' & 1 \\ J''' & J' & J \end{Bmatrix} \right], \quad (17b)
\end{aligned}$$

and

$$\begin{aligned}
N(nS;LJL'J'KQ;L''J''L'''J'''K'Q')_E &= \omega_E (-1)^{S-J-Q'+1} \Pi_{KK'} \sum_{Q''} \begin{pmatrix} K & K' & 1 \\ -Q & Q' & -Q'' \end{pmatrix} (\mathbf{e}_E)_{Q''} \\
&\times \left[ \delta_{L'L''} \delta_{J'J''} (-1)^{J'-J''} \Lambda(nL, nL'') \Pi_{JJ''} \begin{Bmatrix} J & J'' & 1 \\ L'' & L & S \end{Bmatrix} \begin{Bmatrix} K & K' & 1 \\ J'' & J & J' \end{Bmatrix} \right. \\
&\quad \left. + \delta_{LL''} \delta_{JJ''} (-1)^{K+K'} \Lambda(nL', nL''') \Pi_{J'J''} \begin{Bmatrix} J' & J''' & 1 \\ L''' & L' & S \end{Bmatrix} \begin{Bmatrix} K & K' & 1 \\ J''' & J' & J \end{Bmatrix} \right], \quad (17c)
\end{aligned}$$

where we introduced the frequencies  $\omega_B = \mu_0 B / \hbar$  and  $\omega_E = a_0 e_0 E / \hbar$ , and the function

$$\Gamma_{LS}(J, J') = \delta_{JJ'} \Pi_J \sqrt{J(J+1)} - (-1)^{L+S+J} \Pi_{JJ'S} \sqrt{(S+1)S} \begin{Bmatrix} J & J' & 1 \\ S & S & L \end{Bmatrix}. \quad (18)$$

### B. Evaluation of the radiative rates

If we make the assumption that the radiation illuminating the atom is spectrally flat over the frequency range where

quantum interferences within a term exist (so to comply with the limit of complete frequency redistribution, in which the present theory is known to be valid; see Sec. I), then the radiative rates are independent of the strength of the external fields, and we can evaluate their expressions directly in the standard representation, without passing preliminarily through the energy-eigenstate representation. Below, we give the formal proof of this property in the case of the absorption transition rate  $T_A$ , but the same proof holds also for the other transition rates, and for the relaxation rates as well, with only a small modification.

Under the flat-spectrum approximation,  $T_A$  satisfies the following transformation property:

$$T_A(nS, \mu\mu'; n_l S, \mu_l \mu'_l) = \sum_{LJM} \sum_{L'J'M'} \sum_{L''J''M''} \sum_{L'''J'''M'''} C_{\mu}^{LJM}(nS) C_{\mu'}^{L'J'M'}(nS) C_{\mu_l}^{L''J''M''}(n_l S) C_{\mu'_l}^{L'''J'''M'''}(n_l S) \times T_A(nS, LJM, L'J'M'; n_l S, L''J''M'', L'''J'''M''').$$

The above expression must be multiplied by  $\rho_{\nu\nu'} \equiv \rho(\mu_l, \mu'_l)$ , expressed through Eq. (7a), and summed over the states  $\nu_l$  and  $\nu'_l$ . Use of the orthogonality property (4a) gives

$$\sum_{\nu_l \nu'_l} T_A(\nu\nu'; \nu_l \nu'_l) \rho_{\nu\nu'} \equiv \sum_{\bar{L}\bar{J}\bar{M}} \sum_{\bar{L}'\bar{J}'\bar{M}'} C_{\mu}^{\bar{L}\bar{J}\bar{M}}(nS) C_{\mu'}^{\bar{L}'\bar{J}'\bar{M}'}(nS) \times \sum_{n_l} \sum_{L_l J_l M_l} \sum_{L'_l J'_l M'_l} T_A(nS, \bar{L}\bar{J}\bar{M}, \bar{L}'\bar{J}'\bar{M}'; n_l S, L_l J_l M_l, L'_l J'_l M'_l) \rho(L_l J_l M_l, L'_l J'_l M'_l),$$

where for convenience we renamed  $(LJM) \rightarrow (\bar{L}\bar{J}\bar{M})$  and  $(L'J'M') \rightarrow (\bar{L}'\bar{J}'\bar{M}')$ . We then apply Eq. (8) to both sides of Eq. (1) to pass to the standard representation, and use again the orthogonality property (4a), which brings a product of Kronecker  $\delta$ 's between barred and unbarred quantum numbers. Therefore we have proved our initial statement that, after operation (8),

$$\sum_{\nu_l \nu'_l} T_A(\nu\nu'; \nu_l \nu'_l) \rho_{\nu\nu'} \rightarrow \sum_{n_l} \sum_{L_l J_l M_l} \sum_{L'_l J'_l M'_l} T_A(nS, LJM, L'J'M'; n_l S, L_l J_l M_l, L'_l J'_l M'_l) \rho(L_l J_l M_l, L'_l J'_l M'_l). \quad (19)$$

For notational convenience, we now introduce the following correspondences:

$$\tilde{n} \equiv nLSJM, \quad \tilde{n}' \equiv nL'SJ'M', \quad \tilde{n}'' \equiv nL''SJ''M'', \\ \tilde{n}''' \equiv nL'''SJ'''M''',$$

$$\tilde{n}_l \equiv n_l L_l S J_l M_l, \quad \tilde{n}'_l \equiv n_l L'_l S J'_l M'_l,$$

$$\tilde{n}_u \equiv n_u L_u S J_u M_u, \quad \tilde{n}'_u \equiv n_u L'_u S J'_u M'_u.$$

Limiting the present derivation to electric-dipole transitions, and expressing again the dipole operator in units of  $a_0$ , we have

$$T_A(\tilde{n}, \tilde{n}'; \tilde{n}_l, \tilde{n}'_l) = 16\pi^3 \frac{e_0^2 a_0^2}{\hbar^2 c} \sum_{qq'} \langle nLSJM | r_q | n_l L_l S J_l M_l \rangle \langle nL'SJ'M' | r_{q'} | n_l L'_l S J'_l M'_l \rangle^* (-1)^{q+q'} J_{-q-q'}(\omega_{nn_l}), \quad (20a)$$

$$T_S(\tilde{n}, \tilde{n}'; \tilde{n}_u, \tilde{n}'_u) = 16\pi^3 \frac{e_0^2 a_0^2}{\hbar^2 c} \sum_{qq'} \langle n_u L_u S J_u M_u | r_q | nL'SJ'M' \rangle \langle n_u L_u S J_u M_u | r_{q'} | nLSJM \rangle^* (-1)^{q+q'} J_{-q-q'}(\omega_{n_u n}), \quad (20b)$$

$$T_E(\tilde{n}, \tilde{n}'; \tilde{n}_u, \tilde{n}'_u) = \frac{4}{3} \frac{e_0^2 a_0^2}{\hbar c^3} \omega_{n_u n}^3 \sum_q \langle n_u L_u' S J_u' M_u' | r_q | n L' S J' M' \rangle \langle n_u L_u S J_u M_u | r_q | n L S J M \rangle^*, \quad (20c)$$

$$R_A(\tilde{n}, \tilde{n}'; \tilde{n}'', \tilde{n}''') = 8 \pi^3 \frac{e_0^2 a_0^2}{\hbar^2 c} \sum_{(u)} \sum_{qq'} (-1)^{q+q'} J_{-q-q'}(\omega_{n_u n}) [\delta_{LL''} \delta_{J J''} \delta_{M M''} \langle n_u L_u S J_u M_u | r_q | n L' S J' M' \rangle \langle n_u L_u S J_u M_u | r_{q'} | n L''' S J''' M''' \rangle^* \\ + \delta_{L' L''} \delta_{J' J''} \delta_{M' M''} \langle n_u L_u S J_u M_u | r_q | n L'' S J'' M'' \rangle \langle n_u L_u S J_u M_u | r_{q'} | n L S J M \rangle^*], \quad (20d)$$

$$R_S(\tilde{n}, \tilde{n}'; \tilde{n}'', \tilde{n}''') = 8 \pi^3 \frac{e_0^2 a_0^2}{\hbar^2 c} \sum_{(l)} \sum_{qq'} (-1)^{q+q'} J_{-q-q'}(\omega_{n l}) [\delta_{LL''} \delta_{J J''} \delta_{M M''} \langle n L''' S J''' M''' | r_q | n_l L_l S J_l M_l \rangle \langle n L' S J' M' | r_{q'} | n_l L_l S J_l M_l \rangle^* \\ + \delta_{L' L''} \delta_{J' J''} \delta_{M' M''} \langle n L S J M | r_q | n_l L_l S J_l M_l \rangle \langle n L'' S J'' M'' | r_{q'} | n_l L_l S J_l M_l \rangle^*], \quad (20e)$$

$$R_E(\tilde{n}, \tilde{n}'; \tilde{n}'', \tilde{n}''') = \frac{2}{3} \frac{e_0^2 a_0^2}{\hbar c^3} \sum_{(l)} \omega_{n l}^3 \sum_q [\delta_{LL''} \delta_{J J''} \delta_{M M''} \langle n L''' S J''' M''' | r_q | n_l L_l S J_l M_l \rangle \langle n L' S J' M' | r_q | n_l L_l S J_l M_l \rangle^* \\ + \delta_{L' L''} \delta_{J' J''} \delta_{M' M''} \langle n L S J M | r_q | n_l L_l S J_l M_l \rangle \langle n L'' S J'' M'' | r_q | n_l L_l S J_l M_l \rangle^*]. \quad (20f)$$

The general matrix element of the dipole operator can be evaluated through the Wigner-Eckart theorem and its corollaries [13],

$$\langle n L S J M | r_q | n' L' S J' M' \rangle = (-1)^{L+S-M} \Pi_{JJ'L} \begin{pmatrix} J & J' & 1 \\ -M & M' & q \end{pmatrix} \begin{Bmatrix} L & L' & 1 \\ J' & J & S \end{Bmatrix} \langle n L \| \mathbf{r} \| n' L' \rangle = (-1)^{S-M} \Lambda(n L, n' L') \Pi_{JJ'} \begin{pmatrix} J & J' & 1 \\ -M & M' & q \end{pmatrix} \\ \times \begin{Bmatrix} J & J' & 1 \\ L' & L & S \end{Bmatrix}. \quad (21)$$

Equation (21) in particular shows that the electric-dipole matrix elements in Eqs. (20a)–(20f) are purely real quantities (see comment at the end of Sec. II A). We see that this time we need to evaluate the quantity  $\Lambda(n L, n' L')$  for  $n' \neq n$ , and therefore we must adopt the Gordon formula to compute the dipole matrix element. This quantity is proportional to the square root of the Einstein coefficients for the transition  $(n L) - (n' L')$ ,

$$A(n L, n' L') = \frac{4}{3} \frac{e_0^2 a_0^2}{\hbar c^3} \omega_{n L, n' L'}^3 |\langle n L \| \mathbf{r} \| n' L' \rangle|^2 \\ = \frac{\hbar}{4 \pi^3 c^2} \omega_{n L, n' L'}^3 B(n L, n' L'), \quad (22)$$

when we consider that, because of Eq. (21),

$$\Lambda(n L, n' L') = (-1)^L \Pi_L \langle n L \| \mathbf{r} \| n' L' \rangle. \quad (23)$$

The polarization tensors of the incident radiation field,  $J_{qq'}(\omega_{ul})$ , must be expressed in terms of their irreducible spherical components,

$$J_{qq'}(\omega_{ul}) = \sum_{KQ} (-1)^{q+1} \frac{\Pi_{Kq}}{\sqrt{3}} \begin{pmatrix} 1 & 1 & K \\ q & -q' & -Q \end{pmatrix} J_{QK}^K(\omega_{ul}). \quad (24)$$

The definition of the irreducible spherical tensors  $J_{QK}^K(\omega)$  is the following:

$$J_{QK}^K(\omega) = \sum_{i=0}^3 \oint \frac{d\hat{\mathbf{k}}}{4\pi} T_Q^K(i, \hat{\mathbf{k}}) S_i(\omega, \hat{\mathbf{k}}), \quad (25)$$

where  $T_Q^K(i, \hat{\mathbf{k}})$  are irreducible spherical tensors defining the propagation direction  $\hat{\mathbf{k}}$  and the reference direction of linear polarization in the particular reference frame of the external fields previously introduced, in which the solution density matrix is calculated (see Fig. 2). These tensors are given in Table I. Finally,  $(S_0, S_1, S_2, S_3) \equiv (I, Q, U, V)$  is the Stokes vector of the radiation incident on the atom, where  $I$  is the radiation intensity,  $Q$  and  $U$  are the two parameters specifying the linear polarization of the radiation, and  $V$  is the Stokes parameter of circular polarization [9,10].

We illustrate in some detail the steps to calculate the expressions of the radiative rates in the irreducible spherical-tensor representation. We explicitly refer to the case of  $T_A$ , however, an analogous procedure applies also to the other transition rates, and also to the relaxation rates with only minor changes. We first apply Eq. (11) to the right-hand side of Eq. (19), with  ${}^{n_i S} \rho(L_l J_l M_l, L_l' J_l' M_l')$  expressed through Eq. (9a), and  $T_A$  given by Eq. (20a). We then use Eq. (21) to express the product of dipole matrix elements, and Eq. (24) for the radiation-field tensors. We thus obtain a product of five  $3j$  symbols summed upon the sextuplet  $(M, M', M_l, M_l', q, q')$ , which is evaluated through Eq. (14) on p. 456 of Ref. [12]. The procedure for  $T_S$  is identical,

TABLE I. Components of the polarization tensors  $T_Q^K(i, \hat{\mathbf{k}})$ , for  $i=0, 1, 2, 3$ , and  $Q \geq 0$  [4]. The components with negative  $Q$  are obtained from the conjugation property,  $T_{-Q}^K(i, \hat{\mathbf{k}}) = (-1)^Q T_Q^K(i, \hat{\mathbf{k}})^*$ . The angular parameters  $\vartheta$ ,  $\varphi$ , and  $\gamma$ , for a given propagation direction  $\hat{\mathbf{k}}$  and reference direction of linear polarization are defined in Fig. 2.

$T_0^0(0, \hat{\mathbf{k}}) = 1$	$T_0^0(1, \hat{\mathbf{k}}) = 0$
$T_0^1(0, \hat{\mathbf{k}}) = 0$	$T_0^1(1, \hat{\mathbf{k}}) = 0$
$T_1^1(0, \hat{\mathbf{k}}) = 0$	$T_1^1(1, \hat{\mathbf{k}}) = 0$
$T_0^2(0, \hat{\mathbf{k}}) = \frac{1}{2\sqrt{2}}(3 \cos^2 \vartheta - 1)$	$T_0^2(1, \hat{\mathbf{k}}) = -\frac{3}{2\sqrt{2}} \cos 2\gamma \sin^2 \vartheta$
$T_1^2(0, \hat{\mathbf{k}}) = -\frac{\sqrt{3}}{2} \sin \vartheta \cos \vartheta e^{i\varphi}$	$T_1^2(1, \hat{\mathbf{k}}) = -\frac{\sqrt{3}}{2} (\cos 2\gamma \cos \vartheta + i \sin 2\gamma) \sin \vartheta e^{i\varphi}$
$T_2^2(0, \hat{\mathbf{k}}) = \frac{\sqrt{3}}{4} \sin^2 \vartheta e^{i2\varphi}$	$T_2^2(1, \hat{\mathbf{k}}) = -\frac{\sqrt{3}}{4} [\cos 2\gamma (1 + \cos^2 \vartheta) + i2 \sin 2\gamma \cos \vartheta] e^{i2\varphi}$
$T_0^0(2, \hat{\mathbf{k}}) = 0$	$T_0^0(3, \hat{\mathbf{k}}) = 0$
$T_0^1(2, \hat{\mathbf{k}}) = 0$	$T_0^1(3, \hat{\mathbf{k}}) = \sqrt{\frac{3}{2}} \cos \vartheta$
$T_1^1(2, \hat{\mathbf{k}}) = 0$	$T_1^1(3, \hat{\mathbf{k}}) = -\frac{\sqrt{3}}{2} \sin \vartheta e^{i\varphi}$
$T_0^2(2, \hat{\mathbf{k}}) = \frac{3}{2\sqrt{2}} \sin 2\gamma \sin^2 \vartheta$	$T_0^2(3, \hat{\mathbf{k}}) = 0$
$T_1^2(2, \hat{\mathbf{k}}) = \frac{\sqrt{3}}{2} (\sin 2\gamma \cos \vartheta - i \cos 2\gamma) \sin \vartheta e^{i\varphi}$	$T_1^2(3, \hat{\mathbf{k}}) = 0$
$T_2^2(2, \hat{\mathbf{k}}) = \frac{\sqrt{3}}{4} [\sin 2\gamma (1 + \cos^2 \vartheta) - i2 \cos 2\gamma \cos \vartheta] e^{i2\varphi}$	$T_2^2(3, \hat{\mathbf{k}}) = 0$

whereas for  $T_E$  we obtain a contraction of only four  $3j$  symbols, which is evaluated through Eq. (8) on p. 454 of Ref. [12]. In the evaluation of  $R_A$  and  $R_S$ , we also obtain contractions of five  $3j$  symbols, but these are instead evaluated through Eq. (13) on p. 456 of Ref. [12]. Finally, in the case of  $R_E$ , the contraction of four  $3j$  symbols is directly evaluated through the orthogonality relations. In the case of the relaxation rates, it is also possible to sum over the  $J$  quantum

number associated with the summations over terms that appear in Eqs. (20d) and (20e). This is accomplished via Eq. (18) on p. 466 of Ref. [12], in the case of  $R_A$  and  $R_S$ , and through the orthogonality properties of  $6j$  symbols in the case of  $R_E$ .

Following this calculation procedure, the SE equations in the formalism of the irreducible spherical tensors can finally be written in the following form:

$$\begin{aligned}
 \frac{d}{dt} n^S \rho_Q^K(LJ, L'J') = & - \sum_{L''J''} \sum_{L'''J'''} \sum_{K'Q'} [iN(nS; LJJ'L'KQ; L''J''L'''J'''K'Q') + R_A(nS; LJJ'L'KQ; L''J''L'''J'''K'Q') \\
 & + R_S(nS; LJJ'L'KQ; L''J''L'''J'''K'Q') + R_E(nS; LJJ'L'KQ; L''J''L'''J'''K'Q')] n^S \rho_Q^{K'}(L''J'', L'''J''') \\
 & + \sum_{n_l} \sum_{L_l J_l} \sum_{L_l' J_l'} \sum_{K_l Q_l} T_A(nS, LJJ'L'KQ; n_l S, L_l J_l L_l' J_l' K_l Q_l) n_l^S \rho_{Q_l}^{K_l}(L_l J_l, L_l' J_l') \\
 & + \sum_{n_u} \sum_{L_u J_u} \sum_{L_u' J_u'} \sum_{K_u Q_u} [T_S(nS, LJJ'L'KQ; n_u S, L_u J_u L_u' J_u' K_u Q_u) \\
 & + T_E(nS, LJJ'L'KQ; n_u S, L_u J_u L_u' J_u' K_u Q_u)] n_u^S \rho_{Q_u}^{K_u}(L_u J_u, L_u' J_u'). \tag{26}
 \end{aligned}$$

In conclusion, we find the following expressions for the various radiative rates:

$$\begin{aligned}
 T_A(nS, LJJ'L'KQ; n_l S, L_l J_l L_l' J_l' K_l Q_l) = & \frac{16\pi^3 e_0^2 a_0^2}{3 \hbar^2 c} \Lambda(n_l L_l, nL) \Lambda(n_l L_l', nL') (-1)^{J_l' - J_l + K_l - Q_l} \Pi_{JJ'l'J_l'} \left\{ \begin{matrix} J & J_l & 1 \\ L_l & L & S \end{matrix} \right\} \left\{ \begin{matrix} J' & J_l' & 1 \\ L_l' & L' & S \end{matrix} \right\} \\
 & \times \sum_{K_r Q_r} \sqrt{3} \Pi_{KK_l K_r} \left\{ \begin{matrix} K & K_l & K_r \\ J & J_l & 1 \\ J' & J_l' & 1 \end{matrix} \right\} \left( \begin{matrix} K & K_l & K_r \\ -Q & Q_l & -Q_r \end{matrix} \right) J_{Q_r}^{K_r}(\omega_{nn_l}), \tag{27a}
 \end{aligned}$$

$$\begin{aligned}
T_S(nS;LJL'J'KQ;n_uS,L_uJ_uL'_uJ'_uK_uQ_u) &= \frac{16\pi^3}{3} \frac{e_0^2 a_0^2}{\hbar^2 c} \Lambda(n_u L_u, nL) \Lambda(n_u L'_u, nL') (-1)^{J'_u - J_u + K_u - Q_u} \Pi_{JJ_u J'_u} \begin{Bmatrix} J & J_u & 1 \\ L_u & L & S \end{Bmatrix} \begin{Bmatrix} J' & J'_u & 1 \\ L'_u & L' & S \end{Bmatrix} \\
&\times \sum_{K_r Q_r} (-1)^{K_r} \sqrt{3} \Pi_{KK_u K_r} \begin{Bmatrix} K & K_u & K_r \\ J & J_u & 1 \\ J' & J'_u & 1 \end{Bmatrix} \begin{pmatrix} K & K_u & K_r \\ -Q & Q_u & -Q_r \end{pmatrix} J_{Q_r}^{K_r}(\omega_{n_u n}), \quad (27b)
\end{aligned}$$

$$\begin{aligned}
T_E(nS;LJL'J'KQ;n_uS,L_uJ_uL'_uJ'_uK_uQ_u) &= \frac{4}{3} \frac{e_0^2 a_0^2}{\hbar c^3} \omega_{n_u n}^3 \Lambda(n_u L_u, nL) \Lambda(n_u L'_u, nL') \delta_{KK_u} \delta_{QQ_u} (-1)^{J'+J_u+K+1} \Pi_{JJ_u J'_u} \begin{Bmatrix} J & J_u & 1 \\ L_u & L & S \end{Bmatrix} \\
&\times \begin{Bmatrix} J' & J'_u & 1 \\ L'_u & L' & S \end{Bmatrix} \begin{Bmatrix} J & J' & K \\ J'_u & J_u & 1 \end{Bmatrix}, \quad (27c)
\end{aligned}$$

$$\begin{aligned}
R_A(nS;LJL'J'KQ;L''J''L'''J'''K'Q') &= \frac{16\pi^3}{3} \frac{e_0^2 a_0^2}{\hbar^2 c} \sum_{n_u L_u} \sum_{K_r Q_r} (-1)^{L_u - S + J - Q' + 1} \sqrt{3} \Pi_{KK'K_r} \begin{pmatrix} K & K' & K_r \\ Q & -Q' & Q_r \end{pmatrix} J_{Q_r}^{K_r}(\omega_{n_u n}) \\
&\times \frac{1}{2} \left[ \delta_{LL''} \delta_{J''} (-1)^{L'+L''} \Pi_{J'J''} \Lambda(nL', n_u L_u) \Lambda(nL''', n_u L_u) \right. \\
&\times \begin{Bmatrix} L' & L''' & K_r \\ J''' & J' & S \end{Bmatrix} \begin{Bmatrix} L' & L''' & K_r \\ 1 & 1 & L_u \end{Bmatrix} \begin{Bmatrix} K & K' & K_r \\ J''' & J' & J \end{Bmatrix} \\
&+ \delta_{L'L''} \delta_{J'J''} (-1)^{L+L''+J'-J''+K+K'+K_r} \Pi_{JJ''} \Lambda(nL, n_u L_u) \Lambda(nL'', n_u L_u) \\
&\times \left. \begin{Bmatrix} L & L'' & K_r \\ J'' & J & S \end{Bmatrix} \begin{Bmatrix} L & L'' & K_r \\ 1 & 1 & L_u \end{Bmatrix} \begin{Bmatrix} K & K' & K_r \\ J'' & J & J' \end{Bmatrix} \right], \quad (27d)
\end{aligned}$$

$$\begin{aligned}
R_S(nS;LJL'J'KQ;L''J''L'''J'''K'Q') &= \frac{16\pi^3}{3} \frac{e_0^2 a_0^2}{\hbar^2 c} \sum_{n_l L_l} \sum_{K_r Q_r} (-1)^{L_l - S + J + K_r - Q' + 1} \sqrt{3} \Pi_{KK'K_r} \begin{pmatrix} K & K' & K_r \\ Q & -Q' & Q_r \end{pmatrix} J_{Q_r}^{K_r}(\omega_{n n_l}) \\
&\times \frac{1}{2} \left[ \delta_{LL''} \delta_{J''} (-1)^{L'+L''} \Pi_{J'J''} \Lambda(nL', n_l L_l) \Lambda(nL''', n_l L_l) \right. \\
&\times \begin{Bmatrix} L' & L''' & K_r \\ J''' & J' & S \end{Bmatrix} \begin{Bmatrix} L' & L''' & K_r \\ 1 & 1 & L_l \end{Bmatrix} \begin{Bmatrix} K & K' & K_r \\ J''' & J' & J \end{Bmatrix} \\
&+ \delta_{L'L''} \delta_{J'J''} (-1)^{L+L''+J'-J''+K+K'+K_r} \Pi_{JJ''} \Lambda(nL, n_l L_l) \Lambda(nL'', n_l L_l) \\
&\times \left. \begin{Bmatrix} L & L'' & K_r \\ J'' & J & S \end{Bmatrix} \begin{Bmatrix} L & L'' & K_r \\ 1 & 1 & L_l \end{Bmatrix} \begin{Bmatrix} K & K' & K_r \\ J'' & J & J' \end{Bmatrix} \right], \quad (27e)
\end{aligned}$$

$$R_E(nS;LJL'J'KQ;L''J''L'''J'''K'Q') = \frac{4}{3} \frac{e_0^2 a_0^2}{\hbar c^3} \delta_{LL''} \delta_{L'L''} \delta_{J'J''} \delta_{J'J''} \delta_{KK'} \delta_{QQ'} \sum_{n_l L_l} \omega_{n n_l}^3 \frac{1}{2} [\Pi_L^{-2} \Lambda^2(nL, n_l L_l) + \Pi_{L'}^{-2} \Lambda^2(nL', n_l L_l)]. \quad (27f)$$

We notice that the last equation can also be expressed in terms of the Einstein  $A$  coefficients,  $A(nL, n_l L_l)$  and  $A(nL', n_l L_l)$ , if we recall Eqs. (22) and (23), and also observe that  $\omega_{nL, n_l L_l} \approx \omega_{nL', n_l L_l} \approx \omega_{n n_l}$ . In fact, if only a magnetic field is present, we can assume the diagonality over  $L$  of the above

equations.<sup>3</sup> This implies  $L=L'$  and  $L_{u,l}=L'_{u,l}$  in the expressions of the transition rates, and  $L=L'=L''=L'''$  in the expres-

<sup>3</sup>This is only approximately true. In fact, even in the absence of external fields, it is possible in principle to excite atomic coherences between two distinct upper levels  $L-1$  and  $L+1$ , through absorption of radiation from a lower level  $L$ . However, in the absence of external electric fields, these coherences are normally negligible.



sions of the relaxation rates. In such case, all products of two  $\Lambda$ 's become perfect squares that can be replaced by the corresponding Einstein  $A$  and  $B$  coefficients [cf. Eqs. (22) and (23)]. It is then immediate to verify that Eqs. (27a)–(27f) reduce to the SE equations for the multiterm atom in the  $L$ - $S$  coupling scheme [4].

### III. RADIATIVE TRANSFER EQUATION

In the previous section, we presented the equations that govern the statistical equilibrium (SE) of a one-electron atom subject to simultaneous electric and magnetic field, and illuminated by a radiation field that can be both anisotropic and polarized. The solution of this problem, which is achieved by solving Eq. (26), is represented by the irreducible spherical components of the density matrix operator for the atomic system,  ${}^{nS}\rho_Q^K(LJ, L'J')$ , which specifies the populations of the atomic levels, but also the polarization of those levels, as well as their mutual interference due to the effect of the atomic fine structure and the applied fields.

When the atomic density matrix is known, it is possible to compute the radiation that is emitted by an ensemble of atoms through both spontaneous and stimulated processes. In a gas that is optically thick—that is, such that radiation is absorbed and re-emitted more than once before finally escaping the gas towards the observer—the production and transport of polarized radiation of angular frequency  $\omega$  in the direction  $\hat{\mathbf{k}}$  are governed by the following radiative transfer (RT) vector equation:

$$\frac{d}{ds}\mathbf{I}(\omega, \hat{\mathbf{k}}) = -[\mathbf{K}^{(a)}(\omega, \hat{\mathbf{k}}) - \mathbf{K}^{(s)}(\omega, \hat{\mathbf{k}})]\mathbf{I}(\omega, \hat{\mathbf{k}}) + \mathbf{J}(\omega, \hat{\mathbf{k}}), \quad (28)$$

where  $\mathbf{I} \equiv (I, Q, U, V)$  is the Stokes vector of the polarized radiation [9,10],  $\mathbf{K}^{(a)}$  and  $\mathbf{K}^{(s)}$  are the absorption and stimulated-emission matrices,

$$\mathbf{K}^{(a,s)} \equiv \begin{pmatrix} \eta_I^{(a,s)} & \eta_Q^{(a,s)} & \eta_U^{(a,s)} & \eta_V^{(a,s)} \\ \eta_Q^{(a,s)} & \eta_I^{(a,s)} & \rho_V^{(a,s)} & -\rho_U^{(a,s)} \\ \eta_U^{(a,s)} & -\rho_V^{(a,s)} & \eta_I^{(a,s)} & \rho_Q^{(a,s)} \\ \eta_V^{(a,s)} & \rho_U^{(a,s)} & -\rho_Q^{(a,s)} & \eta_I^{(a,s)} \end{pmatrix}, \quad (29)$$

and finally  $\mathbf{J} \equiv (\varepsilon_I, \varepsilon_Q, \varepsilon_U, \varepsilon_V)$  is the emissivity vector. The length parameter  $s$  measures the distance along the line of sight, in the direction of the propagation unit vector  $\hat{\mathbf{k}}$ .

If the gas is optically thin, one can rely on the so-called *single-scattering approximation*, meaning that the radiation incident on the gas of atoms is absorbed only once before escaping the gas towards the observer. In such case, it is easy to show that the Stokes vector of the radiation received by the observer is simply proportional to the emissivity vector  $\mathbf{J}$ . In the illustrative examples presented in the next section,

we will only consider optically thin gases, and therefore we will only use the expressions of the emission coefficients,  $\varepsilon_i$  ( $i=0, 1, 2, 3$ , for the four Stokes parameters  $I, Q, U$ , and  $V$ ). However, because in this paper we are presenting the general formalism, we derive the explicit expressions of all the quantities involved in Eq. (28).

The expressions for the various radiative coefficients,  $\eta_i, \rho_i$ , and  $\varepsilon_i$  ( $i=0, 1, 2, 3$ ), for a transition between a lower level  $n_l$  and an upper level  $n_u$ , and for an atomic gas density  $N$ , are the following [1,4]:

$$\begin{aligned} \eta_i^{(a)}(\omega, \hat{\mathbf{k}}) &= 4\pi^2 \frac{e_0^2 a_0^2}{\hbar c} N \omega \sum_{\mu_u} \sum_{\mu_l} \sum_{qq'} (-1)^{q+q'} \\ &\times \text{Re}\{T_{-q-q'}(i, \hat{\mathbf{k}})^{n_l S} \rho(\mu_l, \mu_l') \Phi(\omega_{n_u \mu_u, n_l \mu_l} - \omega) \\ &\times \langle n_u S, \mu_u | r_q | n_l S, \mu_l \rangle \langle n_u S, \mu_u | r_{q'} | n_l S, \mu_l' \rangle^*\}, \end{aligned} \quad (30a)$$

$$\begin{aligned} \eta_i^{(s)}(\omega, \hat{\mathbf{k}}) &= 4\pi^2 \frac{e_0^2 a_0^2}{\hbar c} N \omega \sum_{\mu_l} \sum_{\mu_u} \sum_{qq'} (-1)^{q+q'} \\ &\times \text{Re}\{T_{-q-q'}(i, \hat{\mathbf{k}})^{n_u S} \rho(\mu_u', \mu_u) \Phi(\omega_{n_u \mu_u, n_l \mu_l} - \omega) \\ &\times \langle n_u S, \mu_u | r_q | n_l S, \mu_l \rangle \langle n_u S, \mu_u' | r_{q'} | n_l S, \mu_l' \rangle^*\}, \end{aligned} \quad (30b)$$

and

$$\rho_i^{(a,s)}(\omega, \hat{\mathbf{k}}) = \eta_i^{(a,s)}(\omega, \hat{\mathbf{k}}) \{\text{Re} \rightarrow \text{Im}\} \quad (i = 1, 2, 3), \quad (31a)$$

$$\varepsilon_i(\omega, \hat{\mathbf{k}}) = \frac{\hbar}{4\pi^3 c^2} \omega_{n_u n_l}^3 \eta_i^{(s)}(\omega, \hat{\mathbf{k}}) \equiv \frac{A_{n_u n_l}}{B_{n_u n_l}} \eta_i^{(s)}(\omega, \hat{\mathbf{k}}). \quad (31b)$$

The geometric tensors  $T_{qq'}(i, \hat{\mathbf{k}})$  are expressed in terms of their irreducible spherical components  $T_Q^K(i, \hat{\mathbf{k}})$  (see Table I) through a transformation formula identical to Eq. (24).

The line profile  $\Phi(\omega)$  is determined by the thermodynamic properties of the emitting gas, and is generally given by the normalized Voigt profile characteristic of the temperature and density of the gas. If pressure broadening is negligible, the Lorentzian contribution to the Voigt profile must be the one corresponding to the natural width of the spectral line, which is determined by the sum of the Einstein  $A$  coefficients of the atomic levels  $n_l$  and  $n_u$  (see, e.g., Sec. 63 of Ref. [15]).

Using the same substitutions as in the derivation of the radiative rates of the SE equations, we get, after some tedious but straightforward algebra,

$$\begin{aligned}
\eta_i^{(a)}(\omega, \hat{\mathbf{k}}) &= \frac{4\pi^2 e_0^2 a_0^2}{3 \hbar c} N \omega \sum_{L_l J_l M_l} \sum_{L'_l J'_l M'_l} \sum_{L''_l J''_l M''_l} \sum_{L_u J_u M_u} \sum_{L'_u J'_u M'_u} \sum_{q q'} (-1)^{J''_l - M''_l + M_u - M'_u + q' + 1} \\
&\times \sum_{\mu_l \mu_u} C_{\mu_l}^{L_l J_l M_l}(n_l S) C_{\mu_l}^{L'_l J'_l M'_l}(n_l S) C_{\mu_u}^{L_u J_u M_u}(n_u S) C_{\mu_u}^{L'_u J'_u M'_u}(n_u S) \sum_{KQ} \sum_{K_l Q_l} \sqrt{3} \Pi_{KK_l} \begin{pmatrix} J_u & J_l & 1 \\ -M_u & M_l & q \end{pmatrix} \begin{pmatrix} J'_u & J'_l & 1 \\ -M'_u & M'_l & q' \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 1 & K \\ -q & q' & -Q \end{pmatrix} \begin{pmatrix} J''_l & J'_l & K_l \\ M''_l & -M'_l & -Q_l \end{pmatrix} \begin{Bmatrix} L_u & L_l & 1 \\ J_l & J_u & S \end{Bmatrix} \begin{Bmatrix} L'_u & L'_l & 1 \\ J'_l & J'_u & S \end{Bmatrix} \Pi_{J_u J'_u J_l J'_l} \Lambda(n_u L_u, n_l L_l) \Lambda(n_u L'_u, n_l L'_l) \\
&\times \text{Re}\{T_Q^K(i, \hat{\mathbf{k}}) {}^{n_l S} \rho_{Q_l}^{K_l}(L'_l J'_l, L'_l J'_l) \Phi(\omega_{n_u \mu_u, n_l \mu_l} - \omega)\}, \tag{32a}
\end{aligned}$$

$$\begin{aligned}
\eta_i^{(s)}(\omega, \hat{\mathbf{k}}) &= \frac{4\pi^2 e_0^2 a_0^2}{3 \hbar c} N \omega \sum_{L_u J_u M_u} \sum_{L'_u J'_u M'_u} \sum_{L''_u J''_u M''_u} \sum_{L_l J_l M_l} \sum_{L'_l J'_l M'_l} \sum_{q q'} (-1)^{J'_u - M'_u + q' + 1} \\
&\times \sum_{\mu_u \mu_l} C_{\mu_u}^{L_u J_u M_u}(n_u S) C_{\mu_u}^{L'_u J'_u M'_u}(n_u S) C_{\mu_l}^{L_l J_l M_l}(n_l S) C_{\mu_l}^{L'_l J'_l M'_l}(n_l S) \sum_{KQ} \sum_{K_u Q_u} \sqrt{3} \Pi_{KK_u} \begin{pmatrix} J_u & J_l & 1 \\ -M_u & M_l & q \end{pmatrix} \begin{pmatrix} J'_u & J'_l & 1 \\ -M'_u & M'_l & q' \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 1 & K \\ -q & q' & -Q \end{pmatrix} \begin{pmatrix} J'_u & J''_u & K_u \\ M'_u & -M''_u & -Q_u \end{pmatrix} \begin{Bmatrix} L_u & L_l & 1 \\ J_l & J_u & S \end{Bmatrix} \begin{Bmatrix} L'_u & L'_l & 1 \\ J'_l & J'_u & S \end{Bmatrix} \Pi_{J_u J'_u J_l J'_l} \Lambda(n_u L_u, n_l L_l) \Lambda(n_u L'_u, n_l L'_l) \\
&\times \text{Re}\{T_Q^K(i, \hat{\mathbf{k}}) {}^{n_u S} \rho_{Q_u}^{K_u}(L'_u J'_u, L''_u J''_u) \Phi(\omega_{n_u \mu_u, n_l \mu_l} - \omega)\}. \tag{32b}
\end{aligned}$$

The above equations simplify considerably if one is only interested in the polarization signal integrated across the line spectral range. In that case, if we approximate  $\omega \approx \omega_{n_u n_l}$  in the overall factor in front of Eqs. (32a) and (32b), the frequency integral of the line profile is equal to 1, so all dependences on the energy-eigenvector expansion coefficients disappear thanks to the orthogonality relation (4a). It then becomes possible to contract the product of four  $3j$  symbols (using Eq. (8) on p. 454 of Ref. [12]), and also to perform the summation over the  $J$  numbers of the final level of the transition (using Eq. (18) on p. 466 of Ref. [12]). We finally obtain (here and in the following, we place a bar over frequency-integrated quantities)

$$\begin{aligned}
\bar{\eta}_i^{(a)}(\hat{\mathbf{k}}) &= \frac{4\pi^2 e_0^2 a_0^2}{3 \hbar c} N \omega_{n_u n_l} \sum_{L_l L'_l L_u} \sum_{J_l J'_l} \sum_{KQ} (-1)^{L_l + L'_l + L_u + S + J'_l + 1} \Lambda(n_u L_u, n_l L_l) \Lambda(n_u L_u, n_l L'_l) \\
&\times \sqrt{3} \Pi_{J_l J'_l} \begin{Bmatrix} L_l & L'_l & K \\ 1 & 1 & L_u \end{Bmatrix} \begin{Bmatrix} L_l & L'_l & K \\ J'_l & J_l & S \end{Bmatrix} T_Q^K(i, \hat{\mathbf{k}}) {}^{n_l S} \rho_Q^K(L_l J_l, L'_l J'_l), \tag{33a}
\end{aligned}$$

$$\begin{aligned}
\bar{\eta}_i^{(s)}(\hat{\mathbf{k}}) &= \frac{4\pi^2 e_0^2 a_0^2}{3 \hbar c} N \omega_{n_u n_l} \sum_{L_u L'_u L_l} \sum_{J_u J'_u} \sum_{KQ} (-1)^{L_u + L'_u + L_l + S + J_u + K + 1} \Lambda(n_u L_u, n_l L_l) \Lambda(n_u L'_u, n_l L_l) \\
&\times \sqrt{3} \Pi_{J_u J'_u} \begin{Bmatrix} L_u & L'_u & K \\ 1 & 1 & L_l \end{Bmatrix} \begin{Bmatrix} L_u & L'_u & K \\ J'_u & J_u & S \end{Bmatrix} T_Q^K(i, \hat{\mathbf{k}}) {}^{n_u S} \rho_Q^K(L'_u J'_u, L_u J_u). \tag{33b}
\end{aligned}$$

Using the conjugations properties,

$$T_Q^K(i, \hat{\mathbf{k}})^* = (-1)^Q T_{-Q}^K(i, \hat{\mathbf{k}}),$$

$$\rho_Q^K(LJ, L'J')^* = (-1)^{J-J'+Q} \rho_{-Q}^K(L'J', LJ),$$

we can easily demonstrate that the integrated quantities of Eqs. (33a) and (33b) are purely real. In particular, this implies that the frequency-integrated coefficients  $\bar{\rho}_{Q,U,V}$ , which are responsible for magneto-optical effects in the line profile, vanish identically [see Eq. (31a)].

Equation (33b), together with Eq. (31b), will be used in the following section to compute the integrated linear and circular polarization of the radiation scattered by an ensemble of hydrogen atoms subject to simultaneous electric and magnetic fields.

Finally, we want to comment on the fact that the calculation of the radiative coefficients entering the RT equation for polarized radiation requires the preliminary solution of the SE equations, in order to determine the density matrix of the atomic system. Because the SE of the atom is generally dependent on the local radiation field, the self-consistent solution of the scattering problem typically entails an iteration

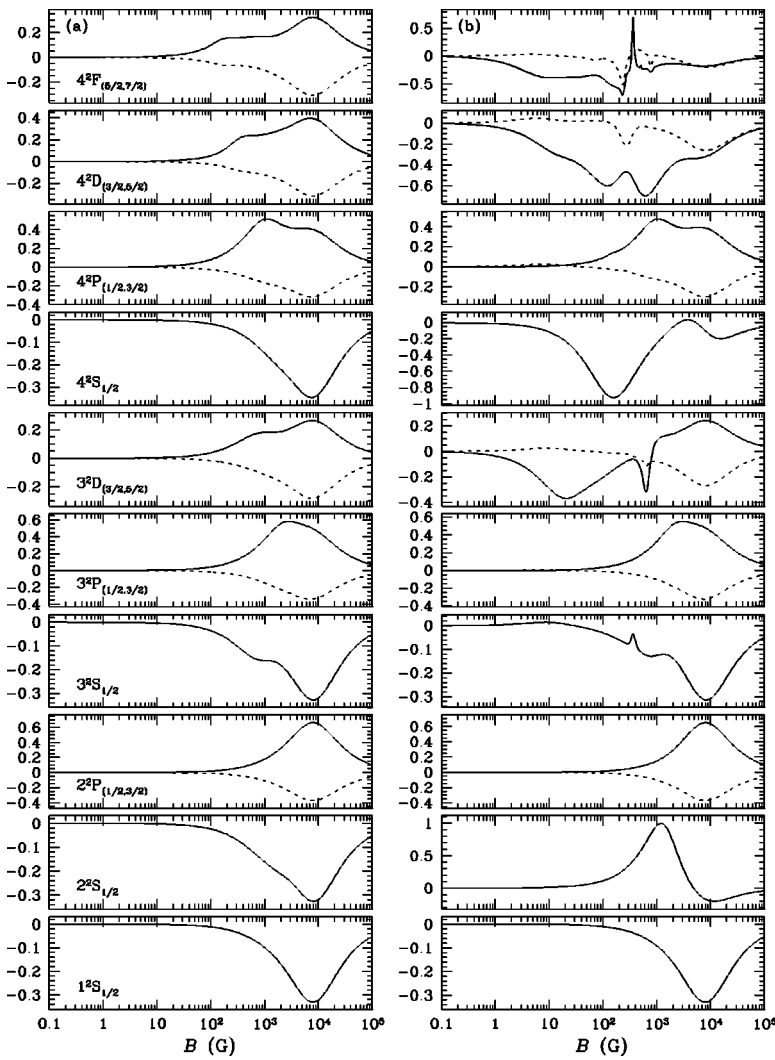


FIG. 3. Atomic orientation,  $\rho_0^1(LJ)/\rho_0^0(LJ)$ , of the levels  $n=1, 2, 3, 4$  of hydrogen, as a function of the strength  $B$ , of a vertical magnetic field (i.e., parallel to the direction of the incident radiation). (a) without electric fields; (b) with  $E=1 \text{ V cm}^{-1}$  parallel to  $\mathbf{B}$ . The dotted lines indicate the higher values of  $J$  in the corresponding panels.

scheme where the SE equations are solved at step  $i$  after the radiation tensors  $J_Q^K$  have been recalculated from the solution of the RT equation at step  $i-1$ . If the gas is only slightly polarizing, a typical approach to this problem is to solve the scalar transfer problem (i.e., for the radiation intensity only) through ordinary iteration schemes, and then use the spatial distribution of the radiation intensity in the gas to solve the SE equations only once. The solution density matrix so calculated is finally entered in the expressions of the radiative coefficients to compute the scattered radiation at each point in the gas.

#### IV. ILLUSTRATIVE RESULTS

In this section, we present some examples of the effect of simultaneous magnetic and electric fields on the atomic polarization of hydrogen. In particular, we show how the additional presence of an electric field significantly modifies the amplitude and the magnetic regime at which atomic orientation is produced. For the calculations in this section, we adopted a model atom of hydrogen inclusive of all Bohr levels up to  $n=4$ , and an incident radiation field corresponding to a Planckian distribution at  $T=20\,000 \text{ K}$ . We also as-

sumed that the illumination is provided by a single pencil of radiation incident on the atom, so that the radiation anisotropy,  $J_0^2/J_0^0$ , attain its maximum value of  $1/\sqrt{2}$ .

We start by commenting that our formalism recovers all the results of Ref. [5] for the scattering polarization of Lyman  $\alpha$  in the presence of an electric field, once we restrict the model atom to the Bohr levels  $n=1, 2$ , and we impose that the ground level be naturally populated, so to match the approximations of that paper. The results that we present in this section concern mostly the  $H\alpha$  line instead, in the presence of both magnetic and electric fields, and also allowing for atomic polarization in the ground level. However, we limited our sample calculations to the case of parallel fields only, in order to reduce the number of free parameters.

In Fig. 3 we plot the atomic orientation  $\rho_0^1(LJ)/\rho_0^0(LJ)$ , for all Bohr levels in our hydrogen model, as a function of the magnetic-field strength. For this figure we assumed a magnetic field aligned with the direction of the incident radiation. In Fig. 3(a), we show the atomic orientation in the presence of only a magnetic field. In Fig. 3(b), we show how the orientation is modified by the additional presence of an electric field, parallel to  $\mathbf{B}$ , with strength  $E=1 \text{ V cm}^{-1}$ . We notice that the presence of an electric field modifies the magnetic regime at which a local maximum (in absolute value) of the

atomic orientation is attained. In the absence of electric fields, magnetic fields of the order of  $10^3$  G or larger are necessary to achieve a maximum of atomic orientation. However, at those magnetic strengths, the resonance polarization of hydrogen is typically dominated by the Zeeman effect, especially for the circular polarization. The rearrangement of the magnetic sublevels determined by the presence of an electric field introduces instead a different set of level crossings and anticrossings, and corresponding quantum interferences, so that a significant peak of the atomic orientation is already attained for  $B \approx 20$  G, in the case of the level  $3^2D_{3/2}$ . The reason for this can be seen in Fig. 4, where we show the energy of the  $3^2P_{3/2}$  and  $3^2D_{3/2}$  levels as a function of  $B$ , (a) in the absence of electric fields, and (b) in the presence of  $E=1$  V cm $^{-1}$  parallel to  $\mathbf{B}$ . In the absence of electric fields, the two levels are completely independent, because of the diagonality of the magnetic Hamiltonian with respect to  $L$ , so the level crossings that are visible in Fig. 4(a) do not contribute to the creation of atomic orientation in those levels. On the contrary, when an electric field is present, the two levels are coupled because of the dipolar form of the electric Hamiltonian, and a different set of level crossings is produced. In particular, we notice in Fig. 4(b) the anticrossings at  $B \approx 6$  G and  $B \approx 17$  G. The latter is responsible for the appearance of the significant peak of atomic orientation in the  $3^2D_{3/2}$  level for  $B \approx 20$  G. Because of the smaller magnetic strengths at which a maximum of atomic orientation is attained, the circular polarization emission this time is dominated by the atomic polarization rather than the Zeeman effect. Therefore one should expect strong deviations of the circular polarization line profile from the antisymmetric shape characteristic of the longitudinal Zeeman effect.

From Fig. 3, we also realize that the modifications of the atomic orientation, determined by the presence of an electric field, become more complex for larger principal quantum numbers. This is easily understood, when we consider that the complexity of the level structure increase with  $n$ , and also that the fine structure of the level becomes less important, so that more complicated patterns of level crossing and anti-crossing must be expected for a given electric strength. As an example, the secondary peak of the atomic orientation of the level  $3^2D_{3/2}$  is determined by a new anticrossing between the same sublevels responsible for the primary peak at  $B \approx 20$  G. This second anticrossing, at  $B \approx 620$  G, is shown in the right panel of Fig. 5 (marked as number 1). Another anticrossing visible in that same panel (at  $B \approx 360$  G; marked as number 2) is responsible for the sharp secondary peak of the atomic orientation of the level  $3^2S_{1/2}$ . In the left panel of Fig. 5, we plot the energy diagram of the Bohr level  $n=2$ . There we marked the anticrossing resonance that is responsible for the broad peak of the atomic orientation of the level  $2^2S_{1/2}$  centered at  $B \approx 1200$  G. (We notice that the  $x$  axis in the plots of Fig. 5 is drawn in logarithmic scale.)

Finally, we notice that the atomic orientation of the ground level and of the  $n^2P_J$  levels is mostly unaffected by the additional presence of a small electric field. This is because the resonant Lyman transitions strongly couple the population and atomic polarization of those levels with the intensity and polarization of the incident radiation. Thus the

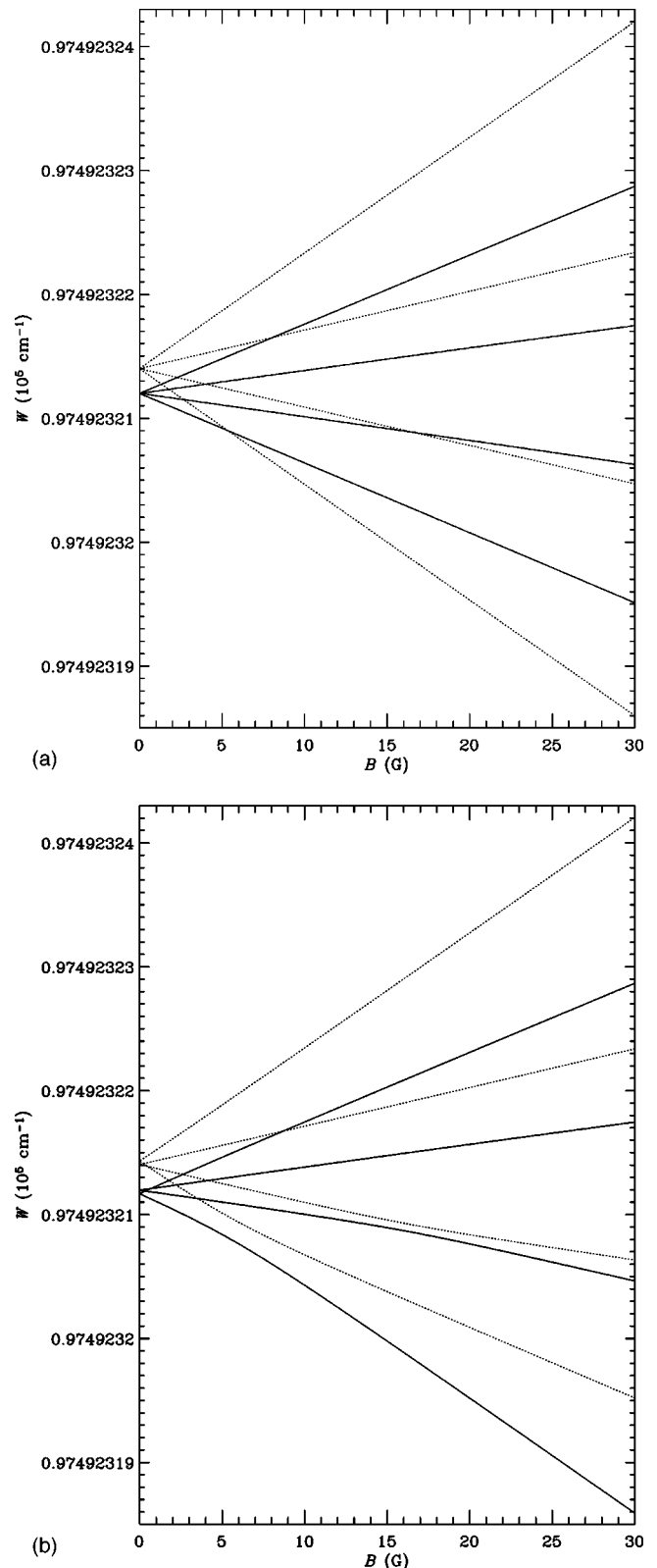


FIG. 4. Energy diagrams of the levels  $3^2P_{3/2}$  (continuous line) and  $3^2D_{3/2}$  (dotted line) as function of the magnetic strength. (a) without electric fields; (b) with  $E=1$  V cm $^{-1}$ . The additional presence of an electric field determines a new set of level crossings and anticrossings for the two multiplets, ultimately affecting the dependence of atomic polarization on the field strengths.

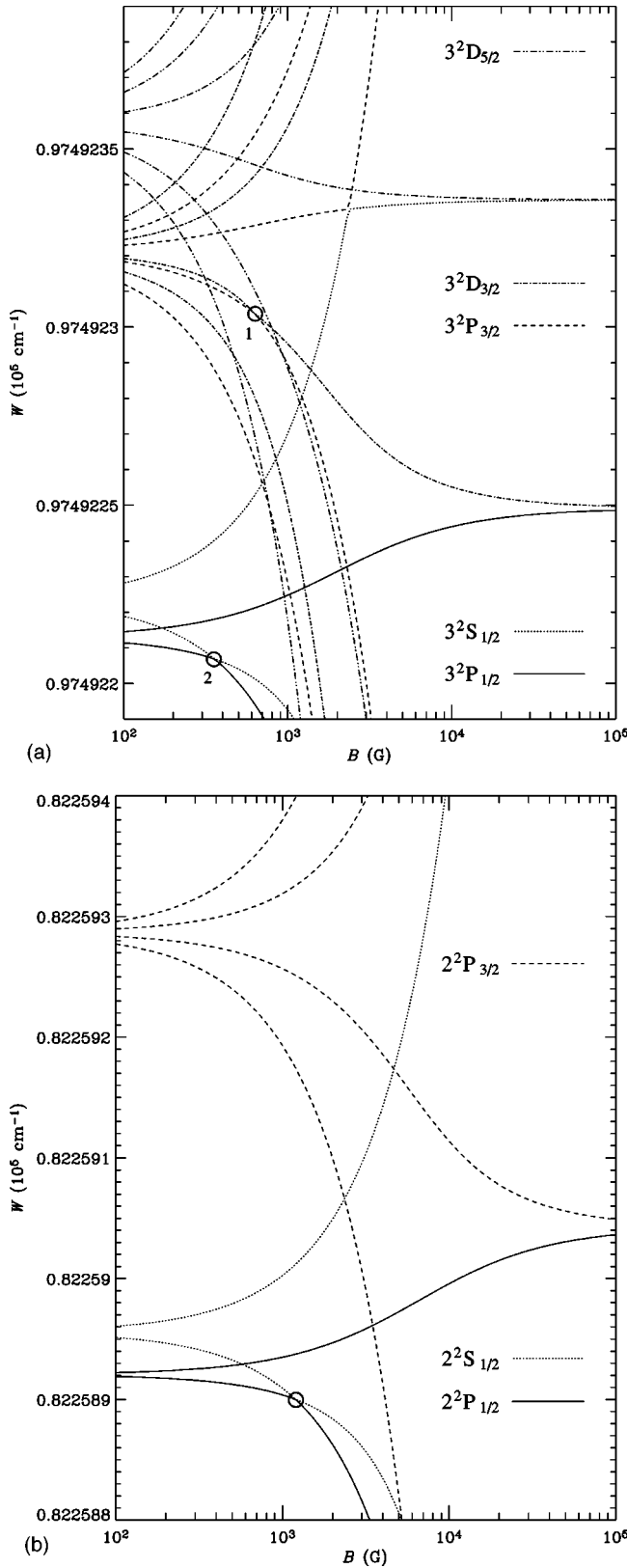


FIG. 5. Energy diagrams of  $n=2$  (top panel) and of  $n=3$  (bottom panel), for the same conditions of Fig. 4(b). These plots show some anticrossing resonances that are responsible for significant peaks of the atomic orientation of the hydrogen levels involved (see text for more details).

modifications to the atomic orientation determined by the additional presence of a small electric field turn out to be significant only for those levels that are connected to the  $n^2P_J$  levels via secondary (i.e., nonresonant) transitions.

Figures 6–8 show surface plots for the integrated linear polarization,  $P_L = \sqrt{\bar{Q}^2 + \bar{U}^2}/\bar{I}$  (left panels), and for the net circular polarization,  $P_V = \bar{V}/\bar{I}$  (right panels), as functions of both magnetic and electric fields. For the calculation of these plots, we only considered the expression of the emission vector  $\mathbf{J}$  [see Eqs. (31b) and (33b)]. This is equivalent to assuming that the emitting gas is optically thin, and that its thermodynamic properties, and the electric and magnetic fields in it, are constant along the line of sight. As we anticipated in Sec. III, under these conditions, we do not need to solve Eq. (28), since the emerging radiation is simply proportional to  $\mathbf{J}$ .

In Figs. 6–8, the case in which only a magnetic field is present is obviously represented by the intersection of the surfaces with the plane  $E=0$ . In particular, in such case we notice the monotonic negative trend of the linear polarization degree, and the very small net circular polarization that can be expected in the presence of magnetic fields up to 100 G. The modifications introduced by the additional presence of an electric field are striking. First of all, for a given geometry of the fields, one finds certain ranges of the electric field where the linear polarization degree increases again with the magnetic field after the initial descent from the field-free value. Second, and more importantly, from the plots of the net circular polarization we see that the presence of even a very small electric field (nonhorizontal) greatly enhances the possibility of creating significant asymmetries in the Stokes- $V$  profile of  $H\alpha$ , without requiring very strong magnetic fields. This is in agreement with the argument presented in the discussion of Fig. 3. However, the surface plots of the net circular polarization provide a more complete picture of this phenomenon, as they also take into account mixed-level interferences of the form  $\rho_Q^1(LJ, LJ')$ .

We want to remark that the creation of a significant level of atomic orientation at small magnetic strengths, induced by the presence of an electric field, and the ensuing asymmetries in the circular-polarization profile, truly are an effect of the simultaneous action of the applied electric and magnetic fields. We could say that the electric field acts here as a “catalyst” for the creation of significant levels of atomic orientation for relatively small magnetic strengths. This is particularly evident in the case of Fig. 3, since the electric field by itself, no matter how strong, is unable to create any amount of atomic orientation for that particular field geometry. Even for field directions such that the electric field can directly induce atomic orientation in the system, the strength that are necessary to attain a significant level of orientation in the absence of magnetic fields are of the order of  $50 \text{ V cm}^{-1}$  (see Figs. 7 and 8).

The fact that a small electric field can alter so radically the amount of atomic orientation that can be created by anisotropic irradiation, for a given strength of a pre-existent magnetic field, is of great diagnostic interest for the inference of magnetic fields in laboratory and astrophysical plasmas using polarimetric analysis of the scattered radiation in hy-

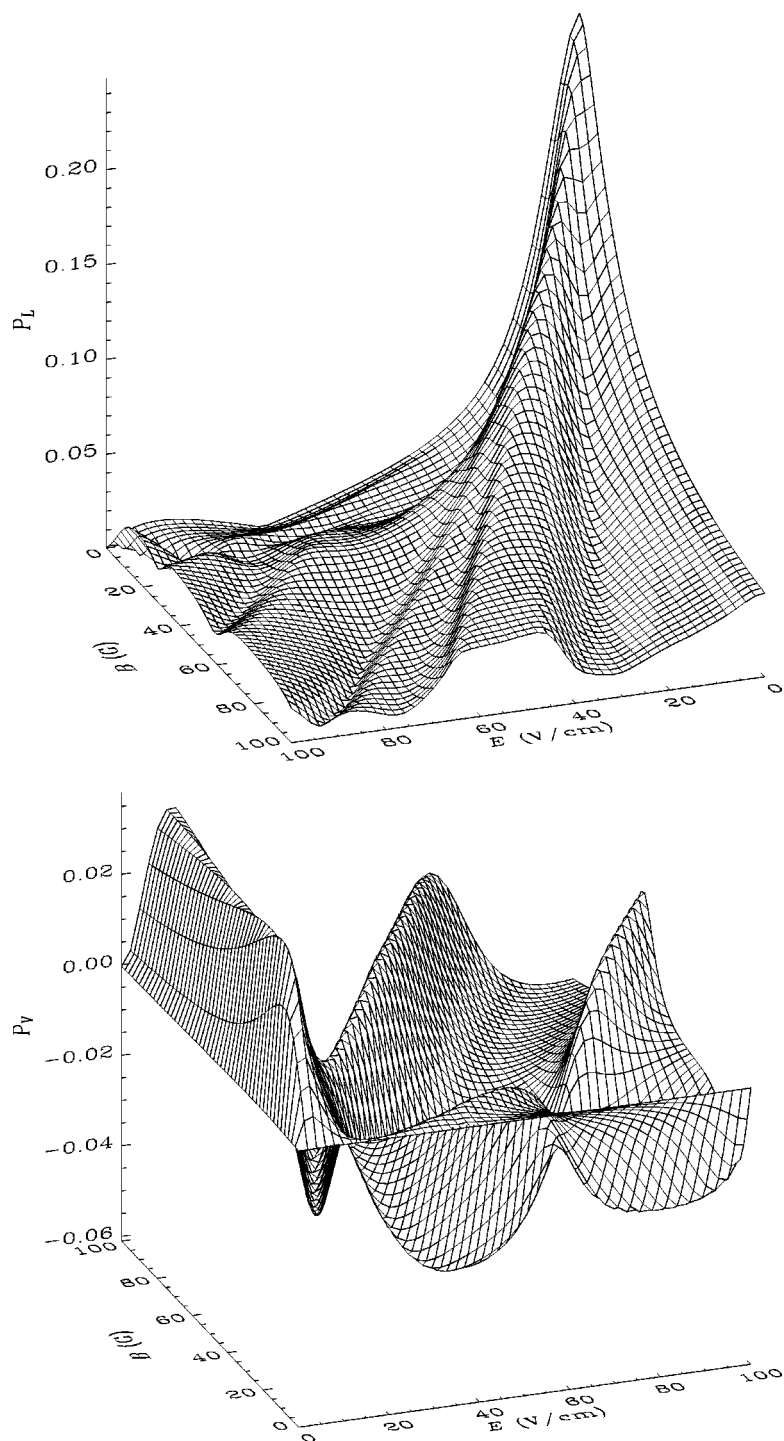


FIG. 6. Surface plots of the linear polarization degree,  $P_L = \sqrt{\overline{Q}^2 + \overline{U}^2} / \overline{I}$  (top), and of the net circular polarization,  $P_V = \overline{V} / \overline{I}$  (bottom), for varying magnetic and electric fields, and  $90^\circ$  scattering. For this figure the fields are inclined of  $45^\circ$  from the direction of the incident radiation, and are lying on the scattering plane formed by this direction and the line of sight ( $\varphi_{E,B} = 0^\circ$ , where  $\varphi_{E,B}$  is the azimuth angle of the fields around the incident direction).

hydrogen lines. A characteristic example is represented by recent observations of an anomalously large degree of net circular polarization in  $H\alpha$  radiation emitted by solar prominences [16], which cannot be explained in terms of atomic orientation induced by the presence of magnetic fields of strengths typical of these solar structures ( $B \lesssim 100$  G). The additional presence of small electric fields in the partly ionized hydrogen gas, of which these structures are mostly composed, could provide a possible explanation for the anomalous circular polarization signal observed.

## V. CONCLUSIONS

In this paper we derived a formalism for the description of resonance scattering in hydrogen lines, subject to the simultaneous presence of electric and magnetic fields. This work generalizes previous investigations on the polarized emission of hydrogen lines that were pursued in Ref. [11], for the case of naturally populated hydrogen atoms (i.e., without inclusion of atomic polarization) in the presence of both electric and magnetic fields of arbitrary geometry, and in Ref. [5], for the case of resonance scattering polarization of hydrogen Ly-

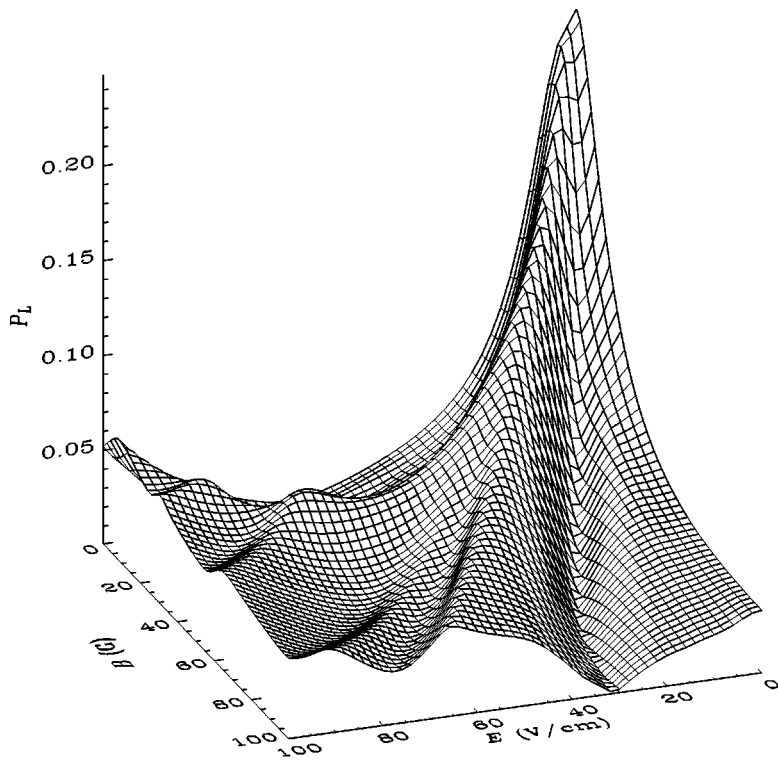
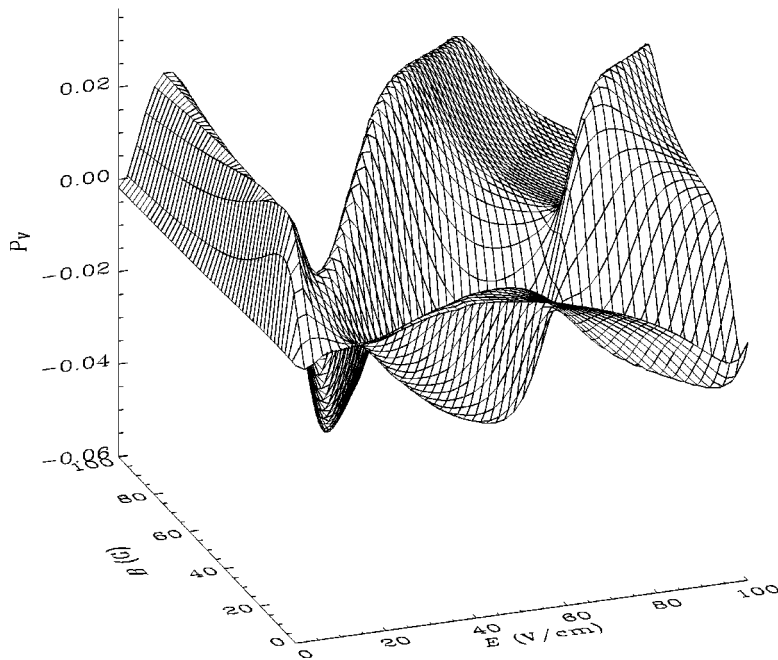


FIG. 7. Same as Fig. 6, but for  $\varphi_{E,B}=45^\circ$ .

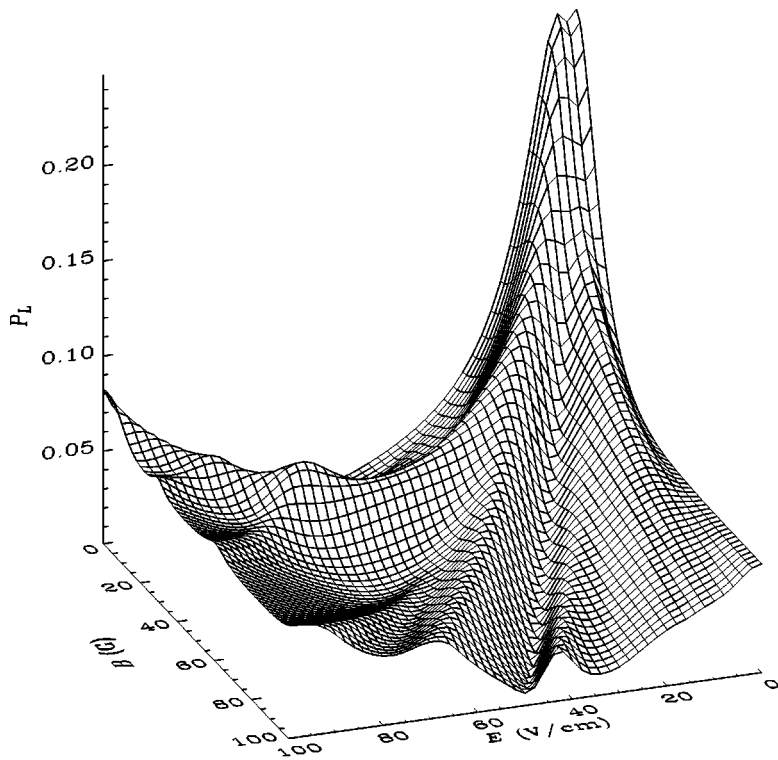
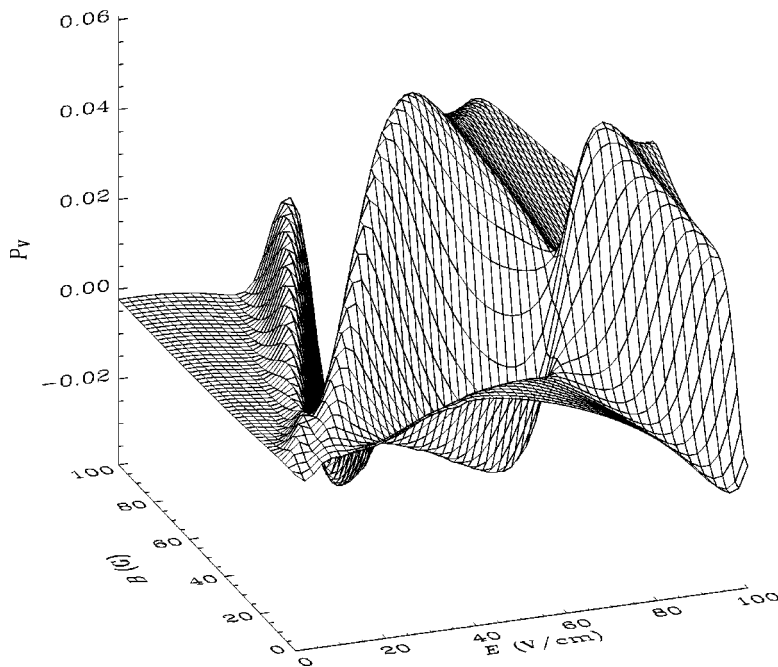


man  $\alpha$  in the presence of electric fields only (see also Ref. [17], which confirm the results on the linear polarization of Lyman  $\alpha$  resonance radiation derived by Ref. [5]).

For the derivation in this paper, we adopted the current theory of polarized line formation presented in Refs. [1–4], which is valid in the limit of complete frequency redistribution (noncoherent scattering). We only focused on the role of radiation in determining the statistical equilibrium of the hydrogen atom, so we did not derive a parallel formalism for the description of collisional polarization, although it would be possible in principle to do so in the limit of the impact approximation [6].

In the description of the hydrogen atom, we neglected the contribution of the hyperfine structure (HFS). This might be an important limitation for some applications, because of the known role of HFS depolarization in resonance scattering (e.g., Ref. [18]). There are no additional conceptual difficulties in rederiving the formalism presented in this paper also including the contribution of HFS, although the numerical problem becomes rapidly unmanageable for increasing values of  $n$ .

The main result of this work is the realization that the effect on resonance scattering polarization of simultaneous magnetic and electric fields cannot be reconciled with the

FIG. 8. Same as Fig. 6, but for  $\varphi_{E,B}=90^\circ$ .

simple superposition of the separate effects of magnetic and electric fields. The rearrangement of the atomic level structure in the presence of both fields, in fact, introduces a whole different set of level crossings and anticrossings, which ultimately determines completely different field-strength regimes of polarization and depolarization with respect to the magnetic-only and electric-only cases. A typical example is the creation of a significant amount of atomic orientation for relatively small fields, which would necessitate instead much larger magnetic or electric strengths if only one of the two fields were present (see Fig. 3, and Figs. 6–8). In particular,

we concluded that the electric field, even at small strengths, acts as a “catalyst” for the atomic orientation induced by the presence of a magnetic field (via the alignment-to-orientation conversion mechanism; see, e.g., Refs. [4,19]). Remarkably, this is true even for field directions such that the electric field by itself could not produce any atomic orientation (see Figs. 3 and 6).

This phenomenon has important observational effects, since atomic orientation translates into net circular polarization of the scattered radiation. As an illustrative example, we considered the case of hydrogen’s  $H\alpha$  line. When this line is



formed in the presence of only a magnetic field, a significant level of atomic orientation is only attained for magnetic strengths of the order of  $10^3$  G, at which the circular polarization signal is totally dominated by the antisymmetric signature of the Zeeman effect. If instead a small electric field (of the order of  $1 \text{ V cm}^{-1}$ ) is also present, then, the same amount of atomic orientation is attained for magnetic strengths of the order of 10 G. Since the amplitude of the Zeeman-effect signature in Stokes  $V$  is now two orders of magnitude smaller, the circular polarization this time is totally dominated by the symmetric signature of the atomic orientation.

We hope that this work will be useful to improve field diagnostics of both laboratory and astrophysical plasmas. In particular, it might provide an explanation of the anomalous (non-Zeeman-like) circular polarization signals of  $H\alpha$  recurrently observed in solar prominences [16].

## ACKNOWLEDGMENTS

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