

Multipartite entanglement in a one-dimensional time-dependent Ising modelArul Lakshminarayan^{1,*} and V. Subrahmanyam^{2,†}¹*Department of Physics, Indian Institute of Technology Madras, Chennai, 600036, India*²*Department of Physics, Indian Institute of Technology, Kanpur, 208016, India*

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We study multipartite entanglement measures for a one-dimensional Ising chain that is capable of showing both integrable and nonintegrable behavior. This model includes the kicked transverse Ising model, which we solve exactly using the Jordan-Wigner transform, as well as nonintegrable and mixing regimes. The cluster states arise as a special case and we show that while one measure of entanglement is large, another measure can be exponentially small, while symmetrizing these states with respect to up and down spins produces those with large entanglement content uniformly. We also calculate exactly some entanglement measures for the nontrivial but integrable case of the kicked transverse Ising model. In the nonintegrable case we begin on extensive numerical studies that show that large multipartite entanglement is accompanied by diminishing two-body correlations, and that time averaged multipartite entanglement measures can be enhanced in nonintegrable systems.

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I. INTRODUCTION

The strictly quantum-mechanical property of entanglement has attracted much attention recently, mainly due to its role in quantum protocols such as teleportation, dense coding, and other processes that involve transfer of quantum information. Entanglement has thus been thought of as a resource for quantum information processing, and perhaps quantum computing. While there is an understanding of what entanglement is, measures of the same are not so obvious, or well established.

Entanglement as quantum correlation has also been recently studied with the help of a slew of well-known models from condensed-matter physics, such as the Ising and the Heisenberg models [1–7]. Mainly, two-body correlations characterized by the concurrence [8] have been studied in these systems. Also these were concerned mostly with stationary state properties, especially ground states. The entanglement content of a spin chain, consisting of many spins, could be potentially much more than those that are present in two-body correlations, and nonstationary states are of potential interest in small chains, such as those that may be realized in ion trap experiments. The difficulty is in defining proper measures of global entanglement content in such chains. Also it is important to note that much of the work has centered around those models that are completely integrable, mostly solvable by the Bethe ansatz or by the Jordan-Wigner transform [9].

Two-body or bipartite entanglement in pure states and its relation to chaos has been investigated more thoroughly mainly due to the von Neumann entropy of the reduced density matrices being an unambiguous measure of entanglement. One of the first works to find that chaos leads to larger

entanglement production in this case linked the classical Lyapunov exponent with the rate of entropy production [10]. In this case it has been generally found that chaos encourages entanglement [11–16], and that complete chaos leads to a universal distribution of the eigenvalues of the reduced density matrices giving rise to a universal entanglement that depends only on the Hilbert space dimensions [14,15]. An early study that addressed the role of nonintegrability in many-body entanglement used the Harper model [17], while later works used the quantum baker map [18], the Frenkel-Kontorova model [19] and disordered spin chains [20].

While the relation between entanglement and chaos or nonintegrability is subtle even in bipartite systems, it gets even more so in the case of many-body systems. It has been claimed that opposite effects have been observed in this case, namely a decrease of entanglement with chaos [20]. However, in the case of one-particle states it has been observed that the average of all the two-body correlations present in the system does increase with chaos [17,19], while near-neighbor correlations decrease with chaos, where the nearness of the neighbor depends on a kind of quantum correlation length [17]. Thus it would seem that chaos in these cases can encourage distant entanglement, even of a two-body type. However, most studies have addressed two-body entanglements, and not global or multipartite entanglements. The exceptions are recent works of Scott and Caves [18] that make use of a measure due to Meyer and Wallach [21], called here the Q measure, and indeed show, using the examples of a quantum kicked rotor and the quantum baker map, that an increase in chaos entails larger global entanglement.

Admittedly, global measures of entanglement are only now beginning to be explored and it is likely that the various proposed measures quantify different aspects of entanglement in multipartite states, aspects that need further elucidation. We now briefly recapitulate the definitions of the different entanglement measures used in this paper. We emphasize that throughout this paper we will deal exclusively with *pure* states.

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A. Concurrence

The concurrence in two qubits i and j that are in the joint state ρ_{ij} is given by the following procedure [8]: calculate the eigenvalues of the matrix $\rho_{ij}\tilde{\rho}_{ij}$, where $\tilde{\rho}_{ij}=\sigma^y \otimes \sigma^y \rho_{ij}^* \otimes \sigma^y$, and the complex conjugation is done in the standard computational basis. The eigenvalues are positive and when arranged in decreasing order if they are $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, the concurrence is $C_{i,j}=\max(\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}, 0)$. This is such that $0 \leq C_{i,j} \leq 1$, with the concurrence vanishing for unentangled states and reaches unity for maximally entangled ones. The entanglement of formation of the two qubits is known to be a monotonic function of $C_{i,j}$ and hence concurrence is itself a good measure of entanglement. In the case of many-qubit pure states, we study concurrence between any two qubits by tracing out the others qubits, and studying the resultant density matrix. Thus this is a ‘‘two-body’’ correlation. It is known that in typical states multipartite entanglement is shared among many qubits rather than in a pairwise manner [18]. We use the following two measures to study multipartite entanglement.

B. Residual tangle and N tangle

If two qubits i and j are in a pure state $|\psi\rangle$, the concurrence reduces to $|\langle \psi | \sigma^y \otimes \sigma^y | \psi^* \rangle|$. The state $|\psi^*\rangle$ is such that its components in the computational basis are the complex conjugates of those of $|\psi\rangle$. It was found that the square of the concurrence $\tau_{i,j}=C_{i,j}^2$ is a more natural measure and is now called the tangle [22]. We can also define the tangle between one spin (say the k th) and the rest of the spins if the overall state is pure. This is because in this case, the Schmidt decomposition gives two unique eigendirections to the rest of the spins corresponding to those eigenvalues of the reduced density matrix that are nonzero. There will be at most only two such values as the nonzero eigenvalues of the two parts are identical. Thus the rest of the qubits can also be effectively thought of as a two-state system. The tangle between spin k and the rest, the one tangle, is $\tau_{k,(rest)}=4 \det(\rho_k)$, where ρ_k is the reduced density matrix of the k th qubit (we will also call this simply τ_k , not to be confused with the n tangle introduced below). This was used to define a purely three-way entanglement measure in a pure state of three qubits as

$$\tau_{1,(23)} - \tau_{1,2} - \tau_{1,3}. \tag{1}$$

This quantity, called the residual tangle, is independent of the focus qubit, in the above this being the qubit numbered 1 [22], and hence stakes its claim as a pure three-way entanglement measure. The construction used here to define the residual tangle was generalized to N qubits in Ref. [23], effectively defining a measure of multipartite entanglement, the n tangle as $\tau_N=|\langle \psi | \sigma^y \otimes \dots \otimes \sigma^y | \psi^* \rangle|^2$. This is evidently the tangle for $N=2$, for $N=3$ this is the residual tangle, while for $N>3$ and odd this vanishes. Thus this measure is used only for N even, with the exception of $N=3$. It has been shown to be an entanglement monotone [23] and hence is a candidate for measuring multipartite entanglement. It is maximal (unity) for GHZ-type states, but can also be maximum for states such as the product state of two groups of four spins in the 4-GHZ

states. It is of course zero for completely unentangled states. It must be noted that the direct generalization of the residual tangle in Eq. (1) is conjectured to be positive [22], and is not the same as the n tangle for $n>3$.

C. Meyer and Wallach Q measure

The geometric multipartite entanglement measure Q [21] has been shown to be simply related to one-qubit purities [24], which makes their calculation and interpretation straightforward. It also seems to have the potential for being experimentally measurable. This is defined as

$$Q(\psi) = 2 \left(1 - \frac{1}{L} \sum_{k=1}^L \text{Tr}(\rho_k^2) \right). \tag{2}$$

From the unit trace of density matrices, it follows immediately that for qubits $1 - \text{Tr}(\rho_k^2) = 2 \det(\rho_k)$. Thus we get that

$$Q(\psi) = \frac{1}{L} \sum_{k=1}^L \tau_k. \tag{3}$$

This measure is therefore simply the average of the tangle between a given qubit and the rest, averaged over this ‘‘focus’’ qubit. The relationship between Q and single spin reduced density matrix purities has led to a generalization of this measure to higher dimensional systems and taking various other bipartite splits of the chain [25]. In some ways, for many states, the n tangle and the Q measure seem to be measuring quite ‘‘orthogonal’’ aspects of entanglement, as we see below, even though we can and do construct GHZ-type states that maximize both these measures.

It needs to be stressed that it is not true that nonintegrability in general produces more entanglement for arbitrary states. There are very simple operators, trivially integrable ones, that can create maximally entangled states out of particular unentangled initial states. However, apart from being true only for particular initial states, the entanglement will oscillate in time and can be completely destroyed once again. An example is provided by the Hamiltonian $H=S_A^x \otimes S_B^x$ of two spin-half particles. With $\hbar=1$ we get

$$\exp(-iJS_A^x \otimes S_B^x t)|11\rangle = \cos(Jt/4)|11\rangle - i \sin(Jt/4)|00\rangle, \tag{4}$$

where the states are in the standard S^z diagonal basis and $|1\rangle$ is the state with eigenvalue $1/2$. The two tangle is simple to calculate and is

$$\tau = |\langle \psi | \sigma_A^y \otimes \sigma_B^y | \psi^* \rangle|^2 = \sin^2(Jt/2), \tag{5}$$

which follows on substituting the above state. The quantity τ is unity for maximally entangled states and is zero for unentangled states. Thus after a time $t=\pi/J$, the spins will be maximally entangled, whereas after twice that time they would be totally unentangled once more. Generalizations of such Hamiltonians, and states, to larger spins and to larger numbers of qubits also yield similar results, and is elaborated upon later below.

In this paper we study issues related to entanglement sharing in spin chains that can range from the integrable to the

nonintegrable, but which nevertheless involve only nearest-neighbor interactions and are translationally invariant. This is in contrast to models which have been studied so far, which are essentially single body dynamics, such as the Harper or the quantum baker map, that have been mapped onto many qubit systems by means of an isomorphism of the Hilbert space. This implies that the interactions need not be nearest neighbor and can in fact involve all to all interactions. In particular the model we study is a kicked Ising model of which the kicked transverse Ising model is a special case. The kicked transverse Ising model is integrable and we solve it using the Jordan-Wigner transform, and thereby study the entanglement generated by this evolution. The zero-field version of this is trivially solvable and a class of states that follow in this case have been previously studied as the “cluster states” [26]. We show that while the cluster states have large entanglement as measured by one entanglement measure, Q measure, it has an exponentially small (in number of spins) n tangle. We also show how symmetrizing the states produces those that have large entanglement according to both these measures. We also emphasize that the kicking is unlikely to be a crucial aspect for the issues discussed here, and on the contrary is more suitable for implementations in say ion-trap experiments.

We numerically study the nonintegrable case and compare it with the integrable one. We find that while nonintegrability does discourage two-body entanglement, multipartite entanglement is increased on the average. In fact this entanglement comes at the cost of two-body correlations. Again, non-integrability is not required to produce maximally entangled states, but produces states that retain large entanglement without disentangling. Thus we study the time-averaged entanglement measures starting with the vacuum (all spin down) state and study it as a function of the strength and tilt of the external field. It is seen that the parameter space corresponding to nonintegrable chains is capable of having substantial entanglement.

II. KICKED ISING MODEL

The model with which we principally study these issues in this paper is a variant [27] of the transverse Ising model, a variant that is at once both dynamically interesting and easier to implement with present day quantum technologies. The usual transverse Ising model has been studied in the context of both entanglement and state transport. It is an intriguing model that is integrable due to a mapping via the Jordan-Wigner transformation, from interacting spins to a collection of noninteracting spinless fermions. The relevance of this model to many physical systems has long been appreciated, and it is a well studied model, with a quantum phase transition separating ferromagnetic and paramagnetic phases at zero temperature as a parameter is varied. The Hamiltonian for L spin-1/2 particles is

$$H_I = J \sum_{n=1}^L S_n^x S_{n+1}^x + B \sum_{n=1}^L S_n^z, \quad (6)$$

where J is the local exchange coupling strength and B an external transversal field. For $J > 2B$, the system is in a fer-

romagnetic phase with nonzero expectation values of the S^x component of the spin, while for $J < 2B$ the system is paramagnetic with vanishing S^x spin expectation value, the point $J = 2B$ being a quantum critical point.

The variant mentioned above involves applying a *tilted* external field impulsively at regular intervals of time [27]. The operator that evolves states from one application of the field to the next is the quantum map or propagator whose spectral properties determine the time evolution. The Hamiltonian is

$$H = J \sum_{n=1}^L S_n^x S_{n+1}^x + B \sum_{k=-\infty}^{\infty} \delta\left(k - \frac{t}{T}\right) \sum_{n=1}^L [\sin(\theta) S_n^z + \cos(\theta) S_n^x], \quad (7)$$

while the unitary quantum map is (the time T between the kicks sets the time scale and is set to unity)

$$U = \exp\left(-iJ \sum_{n=1}^L S_n^x S_{n+1}^x\right) \exp\left(-iB \sum_{n=1}^L [\cos(\theta) S_n^x + \sin(\theta) S_n^z]\right). \quad (8)$$

When the field is transverse ($\theta = \pi/2$), due to the noncommutativity of the components of the spin operator, the above is not equal to $\exp(-iH_I)$, and gives rise to genuinely different dynamics. However, it has been shown that this “kicked” transverse Ising model is integrable [27] and there are suggestions to show that it also undergoes a quantum phase transition and belongs to the same universality class as the usual transverse Ising model [28]. In this integrable model too the key is the Jordan-Wigner transformation, and we solve the problem exactly as opposed to the assumption of the thermodynamic limit in Ref. [28]. For $0 < \theta < \pi/2$ it appears that the model is *nonintegrable* and capable of showing mixing behavior in the thermodynamic limit [27].

Define the following unitary operators:

$$U_{aa}(J_a) = \prod_{n=1}^{L_0} \exp(-iJ_a S_n^a S_{n+1}^a), \quad (9)$$

$$U_{x,z}(B, \theta) = \prod_{n=1}^L \exp\{-iB[\cos(\theta) S_n^x + \sin(\theta) S_n^z]\}. \quad (10)$$

Here $L_0 = L$ for periodic boundary conditions and is $L-1$ for open chains, and θ is an angle of tilt of the magnetic field in the x - z plane. The letter a can be x or z . For the most part we will consider the operator

$$U = U_{xx}(J_x) U_{x,z}(B, \theta). \quad (11)$$

This series of unitaries are quantum gates on nearest-neighbor pairs of qubits and on individual qubits. Ion-trap quantum computing provides one way of implementing the above. The two-qubit operator U_{xx} may be implemented as phase gates and the single one, which involves rotations, is implemented via a single Raman pulse. Thus these quantum maps may be experimentally implementable within these architectures in the immediate future. For further details and references we refer the reader to Ref. [28]. The tilted field

changes the character of the dynamics, the Jordan-Wigner transformation does not reduce the problem to one of noninteracting fermions, and there are features of quantum nonintegrability. This has been studied to some extent in the works of Prosen [27], where he has shown different parameter regimes where there is nonergodic to fully ergodic and mixing dynamics in the thermodynamic limit. This model with the tilted field is then one of substantial richness which deserves to be further studied in itself. We will use it as a simple and realizable model to study the entanglement issues that were discussed in the introduction. It is also worthwhile to mention that time evolution can be done with fast numerical algorithms, with a speed-up factor of the order of $2^L/L$ to evolve a state one time step, exactly as the fast Hadamard or the fast Fourier transform.

III. ENTANGLEMENT IN THE INTEGRABLE CASES

A. Zero field

The simplest nontrivial special case of the models in this paper is an extension of what we discussed in the introduction to many qubits. Thus we first discuss the set of states

$$|\psi_L(t)\rangle = U_{xx}(J_x)^t |1\rangle^{\otimes L}. \quad (12)$$

Here for simplicity we have taken the state with all spins up rather than down. This set of states has been discussed earlier [26] and when $t/J_x = \pi, 3\pi, 5\pi, \dots$ the states are interesting examples of seemingly highly entangled states. We say ‘‘seemingly’’ as it is not clear that all measures of multipartite entanglement will be large for these states for instance we show below that the n tangle vanishes for these states; when the number of qubits is larger than 3. For the case of three spins, such states are local unitarily equivalent to the GHZ state. Expressing the initial state in the S_x basis we can easily time evolve, and converting back to the standard S_z basis we arrive at

$$|\psi_L(t)\rangle = \frac{1}{2^{L/2}} \sum_{a_k \in \{0,1\}} \exp\left(-\frac{iJt}{4} \sum_{k=1}^{L_0} (2a_k - 1)(2a_{k+1} - 1)\right) \otimes_{k=1}^L \left(\frac{|1\rangle + (-1)^{a_k} |0\rangle}{\sqrt{2}} \right). \quad (13)$$

The states $|\psi_L(\pi/J_x)\rangle$ are of special interest. For instance, for $L=2$ we have seen in the introduction that this is essentially one of the maximally entangled Bell states. Also up to an overall phase

$$|\psi_3(\pi/J_x)\rangle = \frac{1}{2}(|111\rangle - |100\rangle - |010\rangle - |001\rangle), \quad (14)$$

which after a local phase change $|0\rangle \rightarrow \sqrt{-1}|0\rangle, |1\rangle \rightarrow |1\rangle$, and a $\pi/4$ rotation (Hadamard transform) on each spin becomes the GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$. Similarly, up to overall phases we get

$$|\psi_4(\pi/J_x)\rangle = \frac{1}{2}(|0000\rangle - |1111\rangle - |1010\rangle - |0101\rangle), \quad (15)$$

$$|\psi_5(\pi/J_x)\rangle = \frac{1}{4}[|11111\rangle - \hat{\pi}(|11100\rangle + |10101\rangle + |10000\rangle)]. \quad (16)$$

The operation $\hat{\pi}$ on the states stand for all the five cyclic permutations of this one. For these states we have assumed periodic boundary conditions on the spins with $L_0=L$. We see, however, that open chains also give rise to similarly entangled states. It has been established earlier that these states with $L>3$ are *not* locally convertible to generalized GHZ states by means of local operations and classical communication. In some sense that has been termed persistence, these states possess higher entanglement content than these N -GHZ states or macroscopic superposition (cat) states which are $|0\rangle^{\otimes N} + |1\rangle^{\otimes N}$. Persistence is the minimum number of local measurements that render the state completely disentangled for all possible outcomes [26]. In terms of a multipartite generalization of the Schmidt numbers, these states seem to again have larger entanglement than the GHZ.

To calculate the Q measure we find the single qubit reduced density matrix ρ_k which is

$$\rho_k = \begin{pmatrix} \frac{1}{2} - \langle S_k^z \rangle & \langle S_k^+ \rangle \\ \langle S_k^- \rangle & \frac{1}{2} + \langle S_k^z \rangle \end{pmatrix}, \quad (17)$$

where the first element is $\langle 0|\rho_k|0\rangle$, etc., and the angular brackets are expectation values corresponding to the full pure state ψ we are interested in. The purity is easily expressed in terms of these expectation values from which we get the entanglement measure as

$$\begin{aligned} Q(\psi) &= 1 - \frac{4}{L} \sum_{k=1}^L (\langle S_k^z \rangle^2 + |\langle S_k^+ \rangle|^2) \\ &= 1 - \frac{4}{L} \sum_{k=1}^L (\langle S_k^x \rangle^2 + \langle S_k^y \rangle^2 + \langle S_k^z \rangle^2). \end{aligned} \quad (18)$$

For the states under consideration $|\psi_L(t)\rangle$ we may explicitly calculate these to get

$$\langle S_k^z \rangle = \frac{1}{2} \cos^2(J_x t/2), \langle S_k^+ \rangle = 0, \quad (19)$$

and hence

$$Q(\psi_L) = 1 - \cos^4(J_x t/2). \quad (20)$$

Thus this measure of entanglement for this class of states is independent of the length of the chain L , and periodically reaches a maximum at $t = \pi/J_x, 3\pi/J_x, \dots$, as indicated earlier, and this maximum is the highest possible. At $t = 0, 2\pi/J_x, \dots$ the state is completely unentangled and therefore in this simple time evolution we have large entangling and disentangling oscillations. The periodic boundary condition can be replaced by an open chain, in which case the entanglement content as measured by Q is

$$Q_{open} = 1 - \cos^4(J_x t/2) - \frac{1}{2L} \sin^2(J_x t), \quad (21)$$

implying again maximal entanglement at times that are odd multiples of π/J_x . Notice that for open chains there is marginal dependence of Q on the number of spins, and for $L=2$ this simplifies to $\sin^2(J_x t/2)$, which we have already derived as the two tangle for this state in the introduction.

If for these states there is high entanglement content as measured by Q , the two-spin correlations as measured by the concurrence are of interest. For $L=2$ the (square of the) concurrence coincides with Q , but for a higher number of spins, we find that while nearest-neighbor concurrences persist and oscillate in time, all other concurrences are perpetually and strictly zero. Also the times at which the nearest-neighbor concurrences vanish are periods when the multipartite entanglement content as measured by Q is maximized, indicating that two-body correlations are being distributed more globally. To calculate the concurrence between any two spins, say at positions i and j , of the chain, we need the two-spin reduced density matrix, which is

$$\begin{aligned} \langle ab|\rho_{ij}|cd\rangle &= \sum_{s_k \in \{0,1\}} \langle s_1 s_2 \cdots a \cdots b \cdots s_L | \psi \rangle \\ &\quad \times \langle \psi | s_1 s_2 \cdots c \cdots d \cdots s_L \rangle, \end{aligned} \quad (22)$$

where a and c are fixed states at position i (0 or 1) and similarly b and d are at position j . This matrix can also be written in terms of spin-expectation values as

$$\begin{aligned} \langle 00|\rho_{ij}|00\rangle &= \left\langle \left(\frac{1}{2} - S_i^z \right) \left(\frac{1}{2} - S_j^z \right) \right\rangle, \\ \langle 00|\rho_{ij}|01\rangle &= \left\langle \left(\frac{1}{2} - S_i^z \right) S_j^+ \right\rangle, \\ \langle 00|\rho_{ij}|10\rangle &= \left\langle S_i^+ \left(\frac{1}{2} - S_j^z \right) \right\rangle, \\ \langle 00|\rho_{ij}|11\rangle &= \langle S_i^+ S_j^+ \rangle, \\ \langle 01|\rho_{ij}|01\rangle &= \left\langle \left(\frac{1}{2} - S_i^z \right) \left(\frac{1}{2} + S_j^z \right) \right\rangle, \\ \langle 01|\rho_{ij}|10\rangle &= \langle S_i^+ S_j^- \rangle, \\ \langle 01|\rho_{ij}|11\rangle &= \left\langle S_i^+ \left(\frac{1}{2} + S_j^z \right) \right\rangle, \\ \langle 10|\rho_{ij}|10\rangle &= \left\langle \left(\frac{1}{2} + S_i^z \right) \left(\frac{1}{2} - S_j^z \right) \right\rangle, \\ \langle 10|\rho_{ij}|11\rangle &= \left\langle \left(\frac{1}{2} + S_i^z \right) S_j^+ \right\rangle, \\ \langle 11|\rho_{ij}|11\rangle &= \left\langle \left(\frac{1}{2} + S_i^z \right) \left(\frac{1}{2} + S_j^z \right) \right\rangle. \end{aligned} \quad (23)$$

The rest of the matrix elements follow from Hermiticity of the density matrix. For the class of states given by $|\psi_L(t)\rangle$ we can calculate these expectation values in a straightforward manner, exploiting the translational symmetry of the states. We get that if $j \neq i \pm 1$ the density matrix is diagonal, in fact

$$\rho_{ij} = \rho_i \otimes \rho_j, \quad j \neq i \pm 1. \quad (24)$$

Here ρ_i and ρ_j are the single spin-density matrices as given in Eq. (17). Thus there is no concurrence between spins that are not nearest neighbors. For the case when $j = i \pm 1$ we get that

$$\begin{aligned} \langle 00|\rho_{i,i\pm 1}|00\rangle &= \langle 01|\rho_{i,i\pm 1}|01\rangle = \langle 10|\rho_{i,i\pm 1}|10\rangle \\ &= \frac{1}{4} \sin^2(J_x t/2), \quad \langle 11|\rho_{i,i\pm 1}|11\rangle = 1 - \frac{3}{4} \sin^2(J_x t/2). \end{aligned} \quad (25)$$

The only nonzero off-diagonal matrix element is

$$\langle 00|\rho_{i,i\pm 1}|11\rangle = \frac{-i}{4} \sin(J_x t). \quad (26)$$

For density matrices such as we have, with all vanishing off-diagonal elements except the corner ones, it is easy to find the concurrence in terms of the matrix elements of the density matrix itself. The positive square roots of the eigenvalues of the matrix $\rho_{i,i\pm 1} \tilde{\rho}_{i,i\pm 1}$, arranged in nonincreasing order, are $|b| + \sqrt{a(1-3a)}$, a , a , $-|b| + \sqrt{a(1-3a)}$, where $a = \sin^2(J_x t/2)/4$ and $b = |\sin(J_x t)/4|$. Thus in this case we get

$$C_{i,i\pm 1}(t) = \max\left(0, \frac{1}{2} [|\sin(J_x t)| - \sin^2(J_x t/2)]\right). \quad (27)$$

Thus we can explicitly calculate the concurrences at all times, as we see that times at which Q is a maximum, namely at $t = \pi/J_x, 2\pi/J_x, \dots$, all the concurrences vanish, including nearest-neighbor ones. In fact there is a period of time around when Q reaches its maximum that there is no two-body entanglement at all. We get that $C_{i,i\pm 1} = 0$ if $|\tan(J_x t/2)| > 2$ or if $t = 2\pi k/J_x, k = 0, \pm 1, \pm 2, \dots$

In Fig. 1 is shown the entanglement measure Q and the nearest-neighbor concurrence as a function of time. This figure is independent of the number of spins in the chain, as long as $L > 2$. The concurrence is dominated by other types of entanglement. It has been conjectured that [22]

$$\tau_i - \sum_{j \neq i} C_{i,j}^2 \geq 0. \quad (28)$$

As we have shown earlier, the average of τ_i is nothing but the entanglement measure Q , and from translational invariance of the states under discussion, this is also equal to any τ_i . For this class of states the inequality is easily seen to be rigorously true. This difference is interpreted as the generalization of the residual tangle, entanglement not present in the form of two-body correlations. In Fig. 2 is plotted this residual tangle, which is dominated by the tangle of individual spins with the others, and not by the concurrence.

Although both Q and the residual tangle are maximum for states such as $|\psi_4(\pi/J_x)\rangle$ the n tangle measure, as previously stated, *vanishes* for these states. We may calculate explicitly

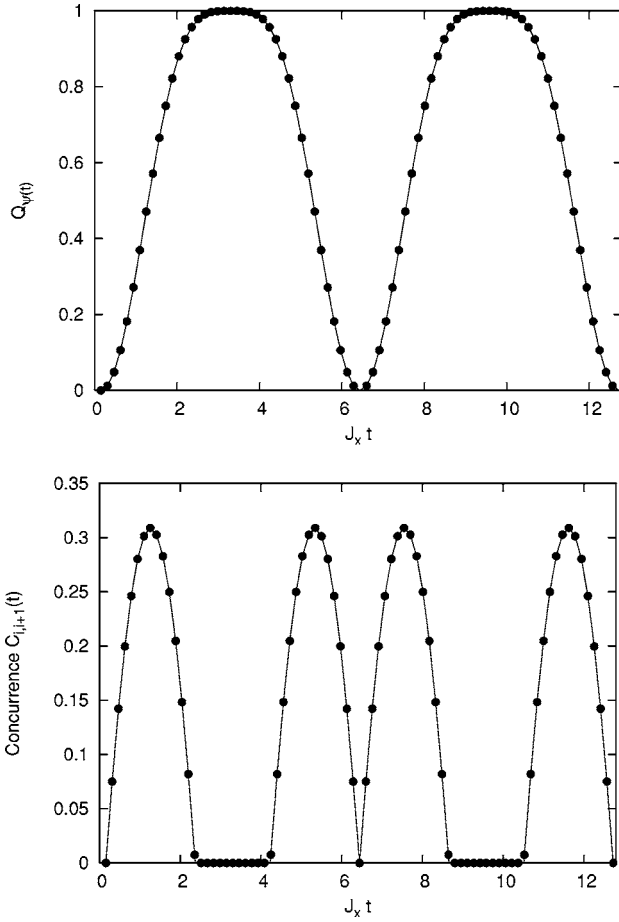


FIG. 1. The Meyer and Wallach measure of entanglement Q and the nearest-neighbor concurrence for the state $|\psi_4(t)\rangle$ as functions of (scaled) time. Plotted are the numerical (points) and the formula (solid line). Periodic boundary conditions were used.

this measure for the states in Eq. (13), and we find that

$$\tau_N(\psi_L) = \frac{1}{2^{L-2}} \sin^L(J_x t). \quad (29)$$

Thus the n tangle decreases exponentially with the number of qubits for the cluster state, it seems to be a rare entanglement

feature, and in particular for the states at $t/J_x = \pi, 2\pi, \dots$, such as those in Eq. (16), the n tangle vanishes. We note in passing that one class of states for which both the n tangle and the Q measures are high are easily obtained from the states discussed here so far by symmetrizing with respect to the “up” and “down” spins. Thus we consider initial states that are N -GHZ states, with the dynamics of nearest-neighbor coupling,

$$|\phi_L(t)\rangle = U_{xx}^t \frac{1}{\sqrt{2}} (|0\rangle^{\otimes L} + |1\rangle^{\otimes L}) = \frac{1}{\sqrt{2}} (1 + \otimes_{k=1}^L \sigma^x) |\psi_L(t)\rangle. \quad (30)$$

The last equality follows since the time evolution commutes with the operator σ^x that flips spins in the standard basis. For these states $\langle S_k^z \rangle = \langle S_k^+ \rangle = 0$ for all k , implying that the single spin reduced density matrix is maximally mixed, and the measure Q is unity for all time t . The n tangle though changes from the maximal value of unity at zero time (the N -GHZ states) and oscillates with exact returns to unity at multiples of π/J_x .

$$\tau_N(\phi_L) = |\cos^{L/2}(J_x t/2) + i^{L/2} \sin^{L/2}(J_x t/2)|^4. \quad (31)$$

Thus for this class of symmetrized cluster states, the n tangle does not decrease exponentially with the number qubits and can have the maximal value at nonzero times. We remind the reader that this measure requires that the number of qubits L be even.

For $L=3$ the state ϕ is

$$|\phi_3(\tau)\rangle = \frac{1}{2\sqrt{2}} \{ [e^{i\pi/2} + \cos(\pi/2)] (|000\rangle + |111\rangle) - i \sin(\pi/2) [\hat{\pi}(001 + 110)] \}, \quad (32)$$

where we have written the scaled time $\tau = J_x t$, that may be simply viewed as a real parameter. For three qubits the residual tangle provides a global entanglement measure [22]. For the state $|\phi_3(\tau)\rangle$, the one- and two-spin reduced density matrices are simply $\rho_1 = I_2/2$, where I_2 is the two-dimensional identity operator and

$$\rho_{12} = \frac{1}{4} \begin{pmatrix} 1 + \cos^2(\tau) & i \sin(2\tau)/2 & i \sin(2\tau)/2 & -\sin^2(\tau) \\ & \sin^2(\tau) & \sin^2(\tau) & -i \sin(2\tau)/2 \\ & & \sin^2(\tau) & -i \sin(2\tau)/2 \\ & & & 1 + \cos^2(\tau) \end{pmatrix}. \quad (33)$$

The other matrix elements of ρ_{12} follow from Hermiticity of this matrix. Due to translational invariance these are the only relevant operators. The spectrum of ρ_{12} is $\{0, 0, 1/2, 1/2\}$, independent of the parameter τ , and the spectrum of $\rho_{12} \tilde{\rho}_{12}$ is

similarly $\{0, 0, 1/4, 1/4\}$. Thus the concurrence vanishes between any two qubits for all values of the parameter (time) τ . The tangle between one qubit and the other two is $\tau_1 = 4 \det \rho_1 = 1$, thus the residual tangle is $\tau_1 - C_{1,2}^2 - C_{1,3}^2 = 1$.

Thus we have a continuous one-parameter family of states, of which the GHZ state is a special case, that have maximal entanglement, as measured by both Q and the residual tangle. Note that in the case of three qubits the residual tangle is also maximized for all time, a feature that generalizes to a higher number of qubits, while the n tangle oscillates as indicated above.

For $L=4$ the state is

$$\begin{aligned}
 |\phi_4(\tau)\rangle = & \frac{1}{\sqrt{2}} \cos^2(\pi/2) (|0000\rangle + |1111\rangle) \\
 & - \frac{i}{2\sqrt{2}} \sin(\tau) \hat{\pi}(|1100\rangle) \\
 & - \frac{1}{\sqrt{2}} \sin^2(\pi/2) (|1010\rangle + |0101\rangle). \quad (34)
 \end{aligned}$$

While $Q(\phi_4)=1$ for all τ , the n tangle is maximized for $\tau = \pi$, in which case the state becomes proportional to $|1010\rangle + |0101\rangle$, which is local unitarily equivalent to the 4-GHZ state, by say flipping the first and third spins. However, for a larger number of qubits, the state that maximized the n tangle is apparently not the N -GHZ state. For instance, for $L=6$ and 8 we get

$$\begin{aligned}
 |\phi_6(\pi)\rangle = & \frac{1}{4\sqrt{2}} [000000 + \hat{\pi}(101000 + 100100 - 110000) \\
 & + 1 \leftrightarrow 0], \quad (35)
 \end{aligned}$$

$$\begin{aligned}
 |\phi_8(\pi)\rangle = & \frac{1}{4\sqrt{2}} [00000000 + \hat{\pi}(00010001 - 01100110 \\
 & + 10101010 - 00001111 + 01000100) + 1 \leftrightarrow 0]. \quad (36)
 \end{aligned}$$

There is a total of 32 terms in each state and we have temporarily dispensed with the ket notation.

B. Transverse field

We now turn on an external field in the transverse direction. This model, the kicked transverse Ising model, has been studied recently as noted above and is also an integrable case [27,28], and the Jordan-Wigner transformation can be used to diagonalize it. In this case we have

$$|\psi_L(t)\rangle = [U_{xx}(J_x)U_{x,z}(B, \pi/2)]^t |\psi_L(0)\rangle, \quad (37)$$

where t is an integer time, the number of kicks. We now proceed to diagonalize the operator, indicating the key steps. It maybe noted that unlike the treatment in Ref. [28] we do not assume the thermodynamic limit, and in this sense the way we solve this problem is also different, though the technique is the same as that for the usual Ising model in a transverse field.

In the kicked transverse Ising spin chain treated here, the Ising interaction is in the x -direction and the magnetic field is switched on at integer times along the z direction. The first step is to replace the spin variables by Jordan-Wigner fermions through a nonlocal transformation [9]:

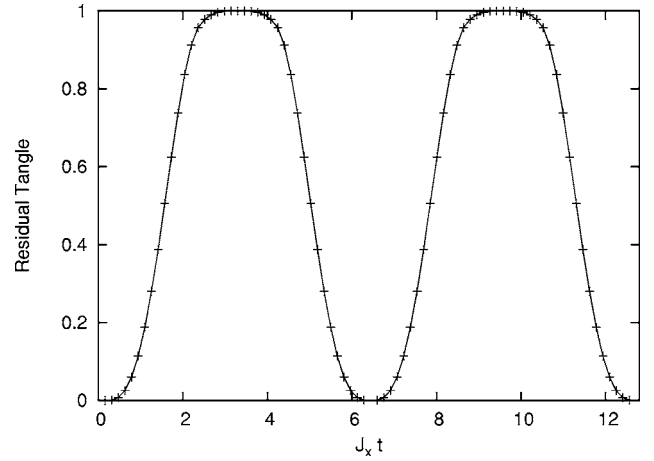


FIG. 2. The residual tangle for the state $|\psi_4(t)\rangle$ as a function of scaled time. Plotted are the numerical (points) and the formula (solid line). Periodic boundary conditions were used.

$$S_l^\dagger = \exp\left(i \sum_{n=1}^{l-1} c_n^\dagger c_n\right) c_l^\dagger, \quad S_l^z = c_l^\dagger c_l - \frac{1}{2}. \quad (38)$$

The operators c_l and c_l^\dagger obey the usual fermion anticommutation rules. The interaction term in U_{xx} reduces to a combination of nearest-neighbor fermion hopping, pair-fermion annihilation, and creation terms on a lattice,

$$\begin{aligned}
 U_{xx} = \exp \left[-\frac{iJ_x}{4} \left(\sum_{l=1}^{L-1} (c_l^\dagger - c_l)(c_{l+1}^\dagger + c_{l+1}) \right. \right. \\
 \left. \left. - (-1)^{N_F} (c_L^\dagger - c_L)(c_1^\dagger + c_1) \right) \right], \quad (39)
 \end{aligned}$$

where $N_F = \sum_{i=1}^L c_i^\dagger c_i$ is the total number of fermions. The last term is due to the periodic boundary condition. The magnetic field term in $U_{x,z}(B, \pi/2)$ becomes a chemical potential term for the total number of fermions. The eigenstates of U will have a definite even or odd fermion number, since N_f commutes with U , and we can find the eigenstates in the two sectors separately.

Now, the second step is to Fourier transform through

$$c_q = \frac{\exp(i\pi/4)}{\sqrt{L}} \sum_{l=1}^L \exp(-iq l) c_l, \quad (40)$$

where the allowed values for q are (taking L to be even)

$$q = \pm \frac{\pi}{L}, \pm \frac{3\pi}{L}, \dots, \pm \frac{(L-1)\pi}{L}, \quad N_F \text{ even}, \quad (41)$$

$$q = 0, \pm \frac{2\pi}{L}, \pm \frac{4\pi}{L}, \dots, \pm \frac{(L-2)\pi}{L}, \pi, \quad N_F \text{ odd}. \quad (42)$$

The lattice momentum q labels the momentum creation and annihilation operators that also obey the fermion anticommutation rules. The unitary operator U has a direct product structure in terms of these fermion variables:

$$U = e^{-i(BL/2)} \prod_{q>0} V_q, \quad N_F \text{ even}, \quad (43)$$

$$= e^{-i(BL/2)} V_0 V_\pi \prod_{q>0} V_q, \quad N_F \text{ odd}, \quad (44)$$

where

$$V_q = \exp\left(-i\frac{J_x}{2}[\cos(q)(c_q^\dagger c_q + c_{-q}^\dagger c_{-q}) + \sin(q) \times (c_q c_{-q} + c_{-q}^\dagger c_q^\dagger)]\right) \exp[-iB(c_q^\dagger c_q + c_{-q}^\dagger c_{-q})], \quad (45)$$

and

$$b_\pm(q) = a_\pm(q) \frac{\pm \sin(\theta_q) + \cos(J_x/2) \sin B - \cos q \cos B \sin(J_x/2)}{\sin(q) \sin(J_x/2)} e^{-i2B}. \quad (49)$$

This then completely solves the kicked transverse Ising model. Let us consider an initial state with m (even) fermions $|\psi(t=0)\rangle = |l_1, l_2, \dots, l_m\rangle$ where l_i denote the sites occupied by fermions (corresponding to $S_{l_i}^z = 1/2$ in terms of the original spin variables). The off-diagonal matrix element of ρ_l through time evolution with U is

$$\langle S_l^+(t) \rangle \equiv \langle \psi(t) | e^{i\pi \sum_n c_n^\dagger c_n} | \psi(t) \rangle = 0, \quad (50)$$

as the time evolution mixes only states with even number of fermions. The diagonal matrix elements of ρ_l depend on $\langle S_l^z \rangle \equiv \langle \psi(t) | c_l^\dagger c_l | \psi(t) \rangle - 1/2$. This can be calculated from the time-evolved operator,

$$c_q(t) = V_q^\dagger c_q V_q = \zeta_q c_q - \text{sgn}(q) \eta_q c_{-q}^\dagger, \quad (51)$$

where the expansion coefficients are given as

$$\zeta_q = |a_+(q)|^2 e^{-it\theta_q} + |a_-(q)|^2 e^{it\theta_q}, \quad (52)$$

$$\eta_q = a_+(q)^* b_+(q) e^{-it\theta_q} + a_-(q)^* b_-(q) e^{it\theta_q}. \quad (53)$$

The diagonal matrix element can be expressed in terms of the Fourier transforms of the above functions; after some manipulations, we have

$$V_0 = \exp\left[-i\left(B + \frac{J_x}{2}\right)c_0^\dagger c_0\right], \quad V_\pi = \exp\left[-i\left(B - \frac{J_x}{2}\right)c_\pi^\dagger c_\pi\right]. \quad (46)$$

The eigenstates of U are direct products of eigenstates of V_q . The operators V_0 and V_π are diagonal in the number basis states. For V_q , the four basis states are $|0\rangle, |\pm q\rangle = c_{\pm q}^\dagger |0\rangle, |-q\rangle = c_{-q}^\dagger c_q^\dagger |0\rangle$. The eigenstates of V_q , for $q \neq 0, \pi$ are given by

$$V_q |\pm q\rangle = e^{-i(J_x/2+B)} |\pm q\rangle, \quad V_q |\pm\rangle = e^{-i(J_x/2+B)} e^{\pm i\theta_q} |\pm\rangle. \quad (47)$$

Here the eigenstates $|\pm\rangle$ are given by $|\pm\rangle \equiv a_\pm(q)|0\rangle + b_\pm(q)|-qq\rangle$. Using

$$\cos(\theta_q) = \cos(B)\cos(J_x/2) - \cos(q)\sin(B)\sin(J_x/2),$$

we have

$$a_\pm(q)^{-1} = \sqrt{1 + \left(\frac{\cos(J_x/2) - \cos(\theta_q \pm B)}{\sin q \sin B \sin(J_x/2)}\right)^2}, \quad (48)$$

$$\langle S_l^z(t) \rangle = -\frac{1}{2} + \frac{1}{L} \sum_q |\eta_q|^2 + \sum_{i=1}^m |\zeta(l-l_i)|^2 - |\eta(l-l_i)|^2. \quad (54)$$

In the above we used two more auxiliary functions defined by

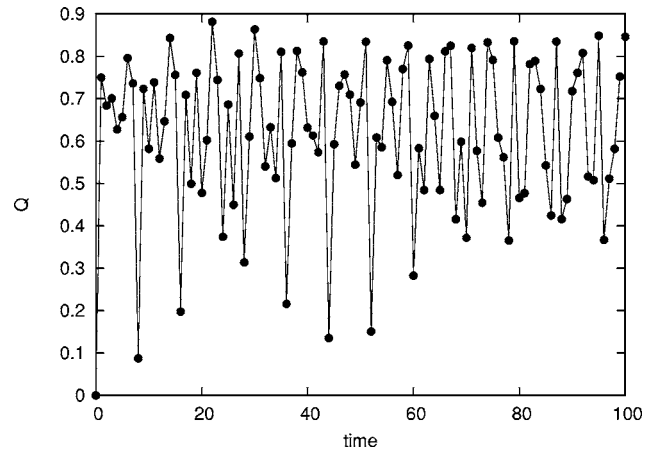


FIG. 3. The measure Q for the kicked transverse Ising interaction, when the initial state is the vacuum state and $L=10$, and the parameters are $J_x = \pi/2, B = \pi/3$. Shown are the results of the numerical calculations (points) and using the formula (solid line). Periodic boundary conditions are assumed.

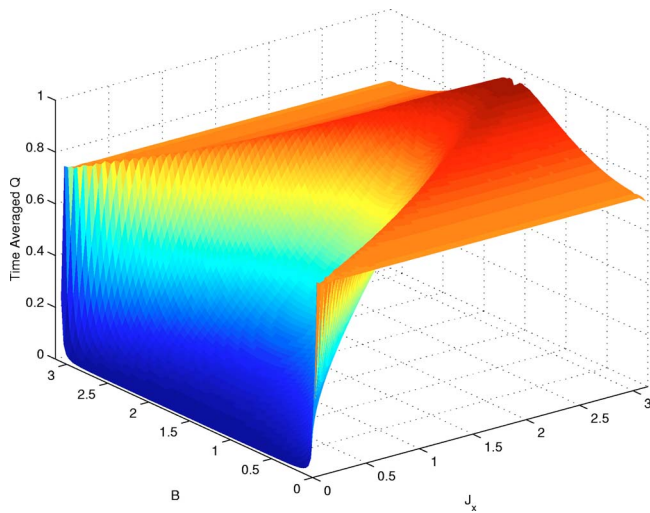


FIG. 4. (Color online) The time-averaged Q as a function of system parameters for the kicked transverse Ising model. $L=20$ in this case, and the averaging is done over a thousand kicks, by which time the average is stationary.

$$\eta(l) = \frac{2}{L} \sum_{q>0} \eta_q \cos(ql), \quad (55)$$

$$\zeta(l) = \frac{2}{L} \sum_{q>0} \zeta_q \cos(ql). \quad (56)$$

In particular for the initial unentangled state $|\psi_L(0)\rangle = |0\rangle^{\otimes L}$, as a special case we can calculate $\langle S_l^z(t) \rangle$ at any site using the above,

$$\langle S_l^z(t) \rangle = \langle \psi_L(0) | S_l^z(t) | \psi_L(0) \rangle = \frac{1}{L} \sum_q |\eta_q|^2 - 1/2. \quad (57)$$

Here the q summation extends to both positive and negative allowed values. Hence using translational symmetry the entanglement measure Q is given in this case by

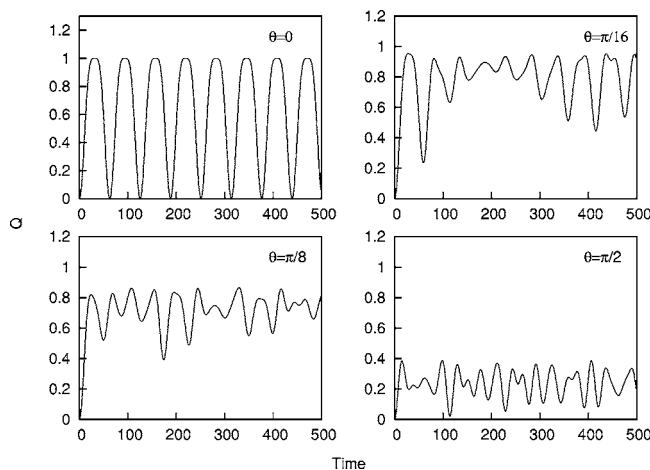


FIG. 5. The entanglement measure Q as a function of time, for different tilt angles of the external field. The parameters are $J_x = 0.1, B = 0.1, L = 10$.

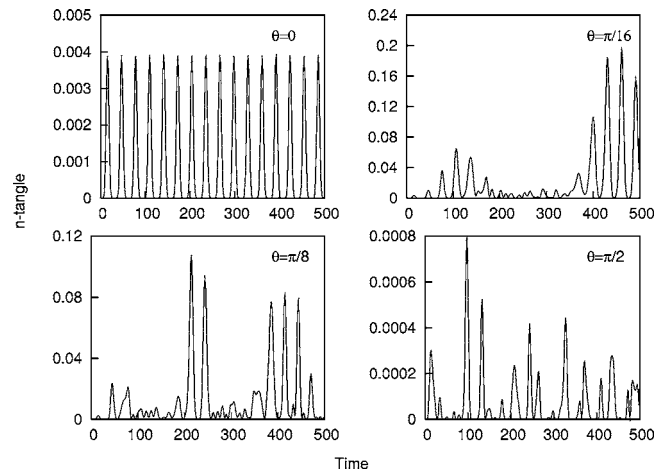


FIG. 6. The n tangle as a function of time. The parameters and tilt angles of the external field are the same as in the previous figure.

$$\begin{aligned} Q[\psi_L(t)] &= 4x(1-x), \quad x = \frac{1}{L} \sum_q |\eta_q|^2 \\ &= \frac{4}{L} \sum_q |a_+(q)a_-(q)\sin(\theta_q t)|^2. \end{aligned} \quad (58)$$

As illustrated in the example (Fig. 3) the oscillations of Q are now much more complicated. The advantage of having an easily computable formula such as Eq. (58) is that we can study the entanglement measures as a function of the interaction strength and transverse magnetic fields more comprehensively. In order to do that we time average Q over sufficiently long scales and plot this as a function of J_x and B in Fig. 4. This figure shows some interesting features, especially the large Q parts, which correspond to $J_x = \pi$. Note that both the lines $B=0$ and $J_x=0$ have been discussed previously, the latter case turns off the interaction and produces no entanglement, while the former is the zero-field case for which the cluster states were realized.

The case when $J_x = \pi, B = \pi/2$ simplifies considerably, as in this case $a_{\pm}(q) = 1/\sqrt{2}$ and $\theta_q = \pi - q$. Thus Q can be calculated more explicitly and results in

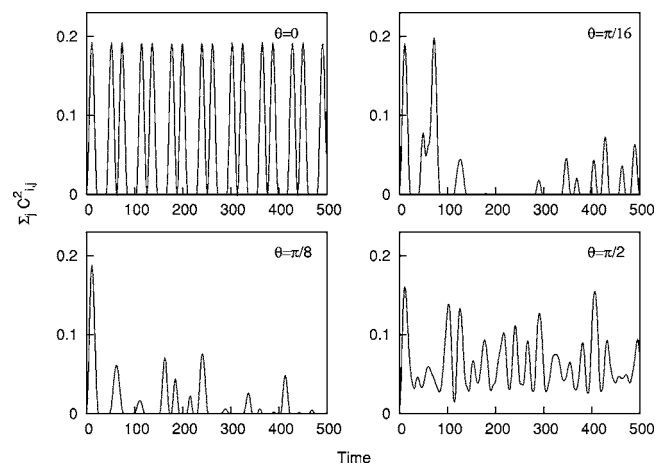


FIG. 7. The sum of the two-body tangles as a function of time, for various tilt angles of the external field. The parameters are $J_x = 0.1, B = 0.1, L = 10$.

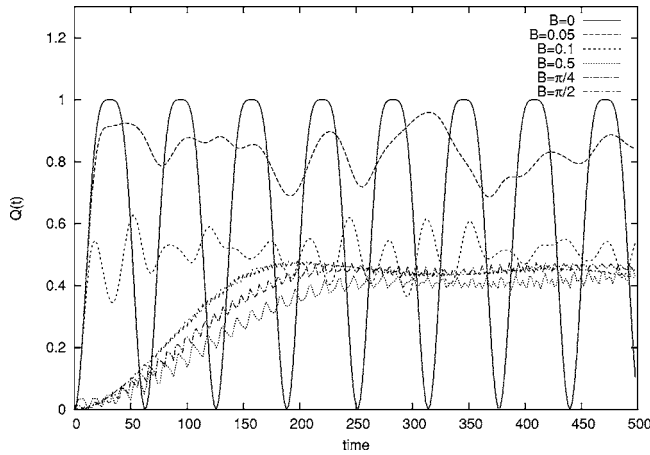


FIG. 8. The entanglement measure Q as a function of time, for various magnitudes of the external field. The parameters are $J_x = 0.1, \theta = \pi/4, L = 10$.

$$Q(t) = \begin{cases} 1 & \text{if } t \neq kL/2 \\ 0 & \text{if } t = kL/2 \end{cases} \quad k = 0, 1, 2, \dots \quad (59)$$

Thus either the state is maximally entangled by the measure Q or is not at all entangled. As in the zero field case, if the initial state is an N -GHZ state itself, according to the Q measure it remains maximally entangled, as in this case also $\langle S_k^- \rangle = \langle S_k^+ \rangle = 0$ for all times. As in that case the n -tangle measure is now significant, although not maximal in general. In the case $B = \pi/2, J_x = \pi$ both the Q measure and the n tangle are unity and represent highly entangled states, which appear to be in the nature of cluster states discussed previously for the zero-field case. Incidentally, this point is also on the critical line $J_x = 2B$ of the (unkicked) transverse Ising model.

IV. ENTANGLEMENT IN THE NONINTEGRABLE CASE

We now consider the case when the field is tilted in the x - z plane, that is, the unitary operator is a slight modification of the transverse Ising case:

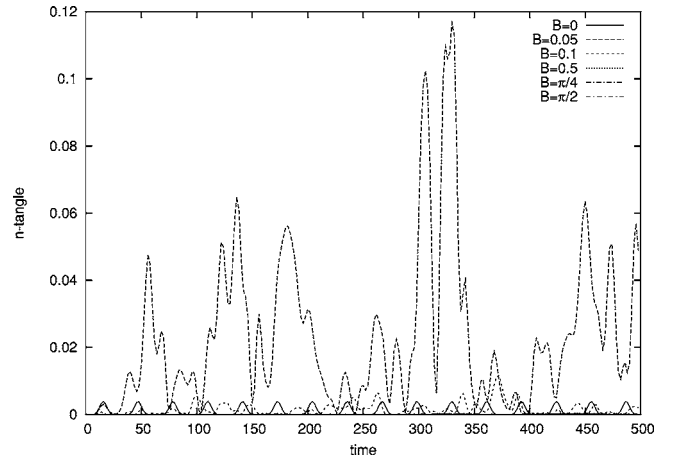
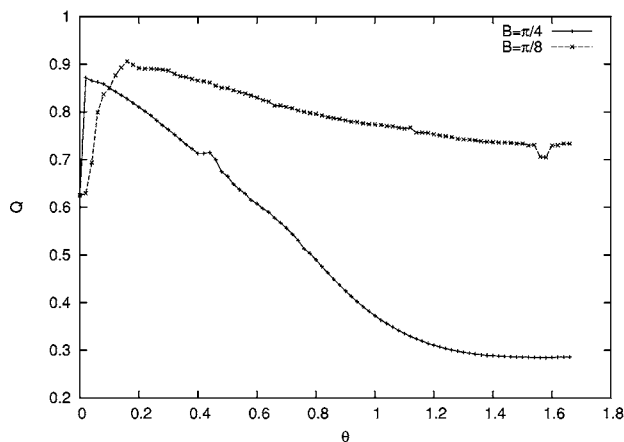


FIG. 9. The n tangle as a function of time for various magnitudes of the external field. The parameters are the same as that of the previous figure.

$$|\psi_L(t)\rangle = [U_{xx}(J_x)U_{x,z}(B, \theta)]^t |\psi_L(0)\rangle. \quad (60)$$

The case when θ is different from both zero and $\pi/2$, as has been noted earlier, constitutes a nonintegrable model. The Jordan-Wigner transformation no longer renders the problem into one of noninteracting fermions. Here we study the influence of the entanglement content of the states $|\psi_L(t)\rangle$, again when the initial state is the “vacuum” state. Once more the time t takes integer values. Since the Jordan-Wigner transformation does not help, many of the results in this section are done purely numerically, with the help of the fast Hadamard transform.

We start with a given exchange coupling, and strength of the external field, while varying the angle of tilt of this field from zero to 90° , both these extremes being integrable. In Figs. 5 and 6 we see the result of this for a particular case. We note that the $\theta=0$ case is integrable and is essentially the zero-field case we have discussed earlier. In this case the Q measure of entanglement reaches the maximum value of unity and drops back to zero periodically. With a nonzero tilt angle we see that while the maximum drops from unity, the

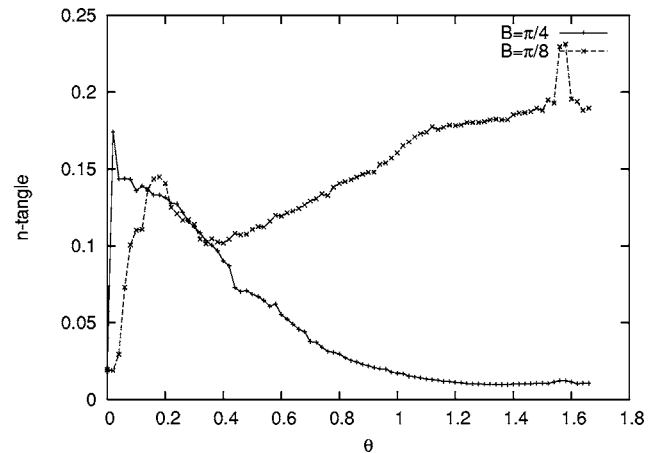


FIG. 10. The time-averaged entanglement Q and n tangle as a function of the tilt angle, for two magnitudes of the external field. The parameters are $J_x = \pi/4, L = 10$.

propensity to untangle also decreases considerably, thereby providing on the average larger entanglement than for the zero-tilt case. Increasing the angle of tilt further decreases the typical value of entanglement produced. The n -tangle measure shows more complicated behavior, with an intermediate angle producing states that have a large n tangle.

The increase in the average multipartite entanglement as measured by Q is accompanied by decreasing overall two-body correlations, as captured by the pairwise concurrence among the qubits. This is illustrated in Fig. 7 where we show the sum of the two-body tangles (or the square of the concurrences), between a given qubit and the rest of them. Due to translational symmetry the sum is independent of the focus qubit. This figure shows the rather substantial concurrences that are present in the integrable cases (both $\theta=0$ and $\theta=\pi/2$), compared to the nonintegrable ones. Thus like the GHZ state that has no two-body correlations, such as the concurrence, these appear to be highly entangled states with small concurrences. The entanglement present in the state appears to be predominantly not of the two-body type. In fact we noted this previously for the cluster states that when Q was the maximum possible the concurrences identically vanished.

Thus it appears that both the Q and n -tangle measures are sensitive to the nonintegrability of the spin chain and from this preliminary data it is plausible that entanglement is enhanced on the average. We have found this to be the case for other values of the parameters, not shown here. We can hold the angle fixed and vary the magnitude of the external field. Both Figs. 8 and 9 are of this kind. In this case it is seen that small values of the magnetic field are enough to prevent the states from completely disentangling. Larger fields also bring down the average along with the fluctuations, until for sufficiently large fields the chain seems to reach smoothly an entanglement plateau. The n tangle again shows more complicated behavior, and can be substantially large in comparison with the integrable cases.

In order to see the effect of the angle and field strength more comprehensively, we again time average the entanglement measures. This averaging is done over a large number (1000) of kicks such that the average is stationary. The results of this are shown in Fig. 10, where it is seen that the Q measure increases sharply with the angle for a fixed magnitude B of the field, and then decreases smoothly until the transverse field is reached. The sharp increase is observed in the case $B=J_x$, while smoother behavior is seen otherwise. The n -tangle measure shows similar characteristics, except that in one case the transverse field case also has a high average entanglement value.

We next study the time averaged entanglement measures as a function of field strength and tilt, for a fixed exchange coupling J_x . The averaging is done over large enough times to ensure stationarity of this quantity, and is shown in Fig. 11. Only six spins are considered here as for each field configuration time evolution is done one thousand times, before calculating the average. However, the case of larger number of spins is qualitatively similar. The principal features seen for the Q measure is that there is enhanced entanglement for both small, nonzero, field strengths and tilt angles. The sharp

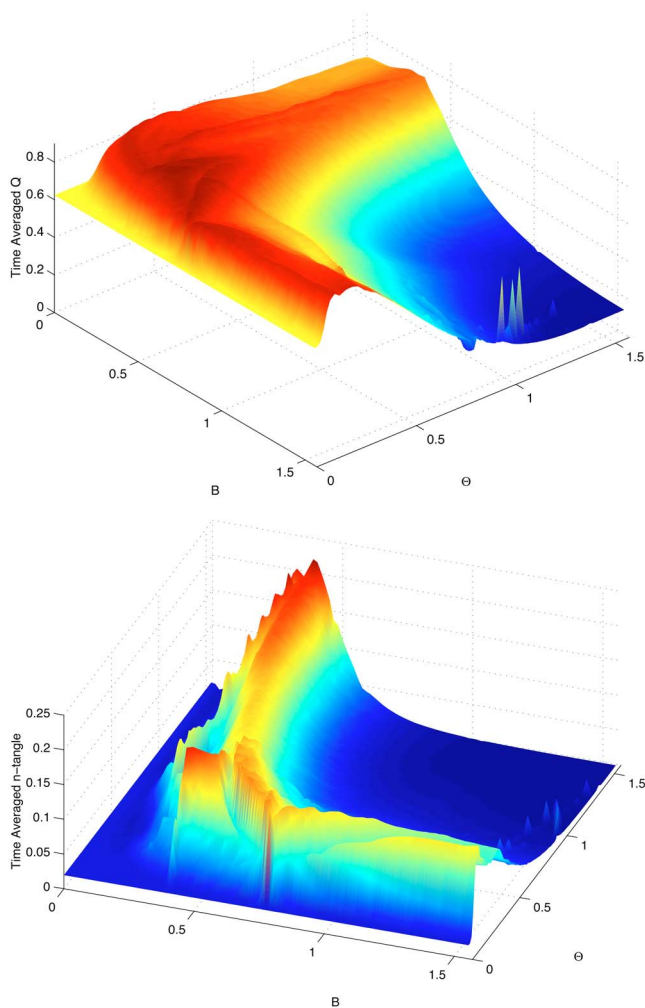


FIG. 11. (Color online) The time-averaged Q and n tangle as a function of external field parameters for the kicked transverse Ising model. $J_x=\pi/4$ and $L=6$ in this case.

transition at $B=J_x$ is seen as a fold in the surface plot of this figure. The high entanglement spots fall in approximate hyperbolas in the B - θ space. The time-averaged n tangle is also shown in Fig. 11, where the hyperbolic region of high entanglement is also visible, but not so close to the small field and tilt angle values as for the Q measure.

From the results presented so far it appears that entanglement can be enhanced in nonintegrable regions of the spin chains, but there could be integrable regions such as for the zero tilt angle case which could produce large entanglement. We have not shown results for the residual tangle in these cases, as this measure is *practically identical* to Q , this in turn implying that the sum of the concurrences is nearly vanishing. In other words, two-body correlations as measured by the concurrence are a rare commodity in these spins chains. More work needs to be done, especially with different initial states, for a better understanding of the implications of nonintegrability on the entanglement in spin chains. The kicked transverse Ising model in a tilted field is a natural example to explore this further.

V. SUMMARY

We have studied the Q , the n tangle, the residual tangle, and concurrence measures for a spin chain that is capable of showing both integrable and nonintegrable behaviors. The model is the kicked Ising model, kicked with a field that could be transverse or tilted to the exchange coupling direction. The integrable cases correspond to the zero, parallel, and transverse fields. In the zero or parallel cases the states generated from the vacuum state are essentially the “cluster” states, for which we have derived the entanglement measures and shown that while the Q measure is large, the n -tangle measure can be exponentially small and the concurrences can vanish. We also point out that symmetrization produces highly entangled states that are capable of both large Q and large n tangles.

In the case of the transverse field, we solve the time evolution by means of the Jordan-Wigner transformation exactly. This enables calculation of many quantities analytically, of which we have displayed the Q measure and pointed out the combinations of field strength and exchange couplings that lead to states with large entanglement. The Jordan-Wigner transformation does not help in the case of the tilted field and is a nonintegrable case, that has been previously studied from a fidelity point of view. We have studied this case numerically and shown that time-averaged

entanglement can be enhanced in the nonintegrable cases; however, it is quite likely that this entanglement is not in the form of two-body entanglements. A more detailed study of the nonintegrable case needs to be carried out to fully assess the impact of nonintegrability on multipartite entanglement.

The entanglement measures Q and the n tangle have been calculated for random states and it has been shown using quantized chaotic maps that these are realized for states evolving under conditions of quantum chaos [18,25]. The random state entanglement measure Q for instance is an overestimate for the kicked Ising model even with a tilted magnetic field, most likely indicating the effects of translation symmetry, placing strong constraints on the “randomness” of these states. Future directions are many, including a more detailed study of states that has been shown here to have both large Q and n tangles, especially from an information theoretic viewpoint. Another is the evaluation of the issues studied here with other multipartite entanglement measures, for instance, the distance to the nearest completely unentangled state [29].

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