

## Multiphoton entanglement through a Bell-multiport beam splitter

Yuan Liang Lim<sup>1,\*</sup> and Almut Beige<sup>1,2,†</sup>

<sup>1</sup>*Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BZ, United Kingdom*

<sup>2</sup>*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*

(Received 12 August 2004; revised manuscript received 27 January 2005; published 13 June 2005)

Multiphoton entanglement is an important resource for linear optics quantum computing. Here we show that a wide range of highly entangled multiphoton states, including  $W$ -states, can be prepared by interfering *single* photons inside a Bell multiport beam splitter and using postselection. A successful state preparation is indicated by the collection of one photon per output port. An advantage of the Bell multiport beam splitter is that it redirects the photons without changing their inner degrees of freedom. The described setup can therefore be used to generate polarization, time-bin, and frequency multiphoton entanglement, even when using only a single photon source.

DOI: 10.1103/PhysRevA.71.062311

PACS number(s): 03.67.Mn, 42.50.Dv

### I. INTRODUCTION

Entanglement spurs a great deal of interest in quantum information processing [1,2], quantum cryptography [3,4], and for fundamental tests of quantum mechanics [5,6]. For many practical purposes, photons provide the most favored qubits as they possess very long lifetimes and an ease in distribution. However, it is very difficult to simulate a direct interaction between the photons, which either requires non-linear couplings between optical modes or the presence of a large number of ancilla photons [2]. Hence photons are difficult to entangle. One way to overcome this problem is to create polarization or time-bin entanglement via photon pair creation within the same source as in atomic cascade and parametric down-conversion experiments. This has already been demonstrated experimentally by many groups [6–8]. Other, still theoretical proposals employ certain features of the combined level structure of atom-cavity systems [9–11], photon emission from atoms in free space [12], or accordingly initialized distant single photon sources [13,14].

Alternatively, highly entangled multiphoton states can be prepared using independently generated single photons with no entanglement in the initial state, linear optics, and postselection. In general, the photons should enter the linear optics network such that the information about the origin of each photon is erased. Afterwards postselective measurements are performed in the output ports of the network [15]. Using this approach, Shih and Alley verified the generation of maximally entangled photon pairs in 1988 by passing two photons simultaneously through a 50:50 beam splitter and detecting them in different output ports of the setup [16]. For a recent experiment based on this idea using quantum dot technology, see Ref. [17].

Currently, many groups experimenting with single photons favor parametric down conversion because of the quality of the produced output states. However, these experi-

ments cannot be scaled up easily, since they do not provide efficient control over the arrival times of the emitted photons. It is therefore experimentally challenging to interfere more than two photons successfully. Interesting experiments involving four photons have nevertheless been performed [18,19] but going to higher photon numbers might require different technologies. To find alternatives to parametric down conversion, a lot of effort has been made over the last years to develop sources for the generation of single photons on demand [20–23]. Following these proposals, a variety of experiments has already been performed, demonstrating the feasibility and characterizing the quality of these sources based on atom-cavity systems [24–26], quantum dots [27], and nitrogen-vacancy color centers [28,29].

Motivated by these recent developments, several authors studied the creation of multiphoton entanglement by passing photons generated by a single photon source through a linear optics network [30–36]. A variety of setups has been considered. Zukowski *et al.* showed that the  $N \times N$  Bell multiport beam splitter can be used to produce higher dimensional Einstein-Podolsky-Rosen (EPR) correlations [30]. Special attention has been paid to the optimization of schemes for the generation of a  $N$ -photon entangled (so-called NOON) state with special applications in lithography [31–34]. Wang studied the event-ready generation of maximally entangled photon pairs without number resolving detectors [35] and Sagi proposed a scheme for the generation of  $N$ -photon polarization entangled Greenberger-Horne-Zeilinger (GHZ) states [36]. It is possible to prepare arbitrary multiphoton states [37] using, for example, probabilistic but universal linear optics quantum gates, like the one described in Ref. [38], but this approach is not always the most favorable.

Here we are interested in the generation of highly entangled qubit states of  $N$  photons using only a single photon source and a symmetric  $N \times N$  Bell multiport beam splitter, which can be realized by combining single beam splitters into a symmetric linear optics network with  $N$  input and  $N$  output ports [30,39]. In the two-photon case, the described scheme simplifies to the experiment by Shih and Alley [16]. To entangle  $N$  photons, every input port  $i$  of the Bell multiport should be entered by a single photon prepared in a state

\*Email address: yuan.lim@imperial.ac.uk

†Email address: a.beige@imperial.ac.uk

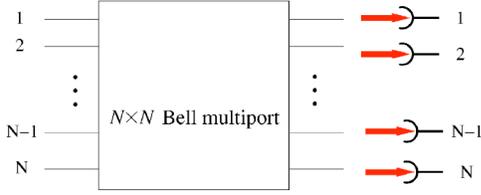


FIG. 1. Experimental setup for the generation of multiphoton entanglement by passing  $N$  single photons through an  $N \times N$  Bell multiport beam splitter. The state preparation is considered successful under the condition of the collection of one photon per output.

$|\lambda_i\rangle$ . The photons then interfere with each other before leaving the setup (see Fig. 1). We consider the state preparation as successful under the condition of the collection of one photon per output port, which can be relatively easily distinguished from cases with at least one empty output port.

Processing the multiphoton state prepared with the above described preselection process further would require quantum nondemolition photon measurements. Performing such measurements is not possible without nonlinearities and without using relatively sophisticated experimental setups [42,43]. In general, the detection of a successful state preparation destroys the prepared multiphoton state. This is known as state preparation via postselection, which is nevertheless a useful technique, if the detection accomplishes a nontrivial task. Entangled state preparation with postselection can be used, for example, for secure quantum key distribution and quantum secret sharing [44], teleportation [45], and in testing entanglement with witnesses and observing a violation of a Bell's inequality [46]. Moreover, combining this technique with single photon sources could be used to prepare them in a highly entangled state, which can then be used for the generation of multiphoton entanglement on demand [13].

One advantage of using a Bell multiport beam splitter for the generation of multiphoton entanglement is that it redirects the photons without changing their inner degrees of freedom, like polarization, arrival time, and frequency. The described setup can therefore be used to generate polarization, time-bin, and frequency entanglement. Especially, time-bin entanglement can be very robust against decoherence and has, for example, applications in long-distance fiber communication [40]. Moreover, the preparation of the input product state does not require control over the relative phases of the incoming photons, since the phase factor of each photon contributes at most to a global phase of the combined state with no physical consequences.

This paper is organized as follows. In Sec. II we introduce the notation for the description of photon scattering through a linear optics setup. Sec. III shows that a wide range of highly entangled photon states can be obtained for  $N=4$ , including the  $W$ -state, the GHZ-state, and a double singlet state. Afterwards we discuss the generation of  $W$ -states for arbitrary photon numbers  $N$  and calculate the corresponding probabilities for a successful state preparation. Finally we conclude our results in Sec. V.

## II. PHOTON SCATTERING THROUGH A LINEAR OPTICS SETUP

Let us first introduce the notation for the description of the transition of the photons through the  $N \times N$  Bell multiport

beam splitter. In the following,  $|+\rangle$  and  $|-\rangle$  are the states of a photon with polarization “+” or “-,” respectively. Alternatively,  $|+\rangle$  could describe a single photon with an earlier arrival time or a higher frequency than a photon prepared in  $|-\rangle$ . As long as the states  $|\pm\rangle$  are orthogonal and the incoming photons are in the same state with respect to all other degrees of freedom, the calculations presented in this paper apply throughout. Moreover, we assume that each input port  $i$  is entered by one independently generated photon prepared in  $|\lambda_i\rangle = \alpha_{i+}|+\rangle_i + \alpha_{i-}|-\rangle_i$ , where  $\alpha_{i\pm}$  are complex coefficients with  $|\alpha_{i+}|^2 + |\alpha_{i-}|^2 = 1$ . If  $a_{i\mu}^\dagger$  denotes the creation operator for one photon with parameter  $\mu$  in input port  $i$ , the  $N$ -photon input state can be written as

$$|\phi_{\text{in}}\rangle = \prod_{i=1}^N \left( \sum_{\mu=+,-} \alpha_{i\mu} a_{i\mu}^\dagger \right) |0\rangle \quad (1)$$

with  $|0\rangle$  being the vacuum state with no photons in the setup.

We now introduce the unitary  $N \times N$ -multiport transformation operator, namely the scattering matrix  $S$ , that relates the input state of the system to the corresponding output state

$$|\phi_{\text{out}}\rangle = S|\phi_{\text{in}}\rangle. \quad (2)$$

Using Eq. (1) and the relation  $S^\dagger S = 1$  yields

$$\begin{aligned} |\phi_{\text{out}}\rangle &= S \left( \sum_{\mu=+,-} \alpha_{1\mu} a_{1\mu}^\dagger \right) S^\dagger S \left( \sum_{\mu=+,-} \alpha_{2\mu} a_{2\mu}^\dagger \right) \cdots \\ &\times S^\dagger S \left( \sum_{\mu=+,-} \alpha_{N\mu} a_{N\mu}^\dagger \right) S^\dagger S |0\rangle \\ &= \prod_{i=1}^N \left( \sum_{\mu=+,-} \alpha_{i\mu} S a_{i\mu}^\dagger S^\dagger \right) |0\rangle. \end{aligned} \quad (3)$$

In the following, the matrix elements  $U_{ji}$  of the unitary transformation matrix  $U$  denote the amplitudes for the redirection of a photon in input  $i$  to output  $j$ . Since the multiport beam splitter does not contain any elements that change the inner degrees of freedom of the incoming photons, the transition matrix  $U$  does not depend on  $\mu$ . Denoting the creation operator for a single photon with parameter  $\mu$  in output port  $j$  by  $b_{j\mu}^\dagger$  therefore yields

$$S a_{i\mu}^\dagger S^\dagger = \sum_j U_{ji} b_{j\mu}^\dagger. \quad (4)$$

Inserting this into Eq. (2) we can now calculate the output state of the  $N \times N$  multiport given the initial state (1) and obtain

$$|\phi_{\text{out}}\rangle = \prod_{i=1}^N \left[ \sum_{j=1}^N U_{ji} \left( \sum_{\mu=+,-} \alpha_{i\mu} b_{j\mu}^\dagger \right) \right] |0\rangle. \quad (5)$$

This equation describes the independent redirection of all photons to their potential output ports. Conservation of the norm of the state vector is guaranteed by the unitarity of the transition matrix  $U$ .

The state preparation is considered successful under the condition of the collection of one photon per output port. To calculate the final state, we apply the corresponding projector to the output state (5) and find that the thereby postselected  $N$ -photon state equals, up to normalization,

$$|\phi_{\text{pro}}\rangle = \sum_{\sigma} \left[ \prod_{i=1}^N U_{\sigma(i)i} \left( \sum_{\mu=\pm,-} \alpha_{i\mu} b_{\sigma(i)\mu}^{\dagger} \right) \right] |0\rangle. \quad (6)$$

Here the  $\sigma$  are the  $N!$  possible permutations of the  $N$  items  $\{1, 2, \dots, N\}$  and  $\sigma(i)$  denotes the  $i$ th element of the list obtained after applying a certain permutation  $\sigma$  onto the list  $\{1, 2, \dots, N\}$ . Moreover, the norm of the state (6) squared, namely

$$P_{\text{suc}} = \|\phi_{\text{pro}}\|^2, \quad (7)$$

is the success rate of the scheme and probability for the collection of one photon in each output  $j$ .

### A. The Bell multiport beam splitter

Motivated by a great variety of applications, we are particularly interested in the generation of highly entangled photon states of a high symmetry, an example being  $W$ -states. This suggests to consider symmetric multiports, which redirect each incoming photon with equal probability to all potential output ports. A special example for such an  $N \times N$  multiport is the Bell multiport beam splitter. Its transformation matrix

$$U_{ji} = \frac{1}{\sqrt{N}} \omega_N^{(j-1)(i-1)} \quad (8)$$

is also known as a discrete fourier transform matrix and has been widely considered in the literature [30,39,41]. Here  $\omega_N$  denotes the  $N$ th root of unity,

$$\omega_N \equiv \exp(2i\pi/N). \quad (9)$$

Proceeding as in Sec. II D of Ref. [30], it can easily be verified that  $U$  is unitary as well as symmetric. Especially for  $N=2$ , the transition matrix (8) describes a single 50:50 beam splitter.

## III. THE GENERATION OF 4-PHOTON STATES

Before we discuss the  $N$ -photon case, we investigate the possibility to prepare highly entangled 4-photon states using specially prepared photons and a  $4 \times 4$  Bell multiport beam splitter. For  $N=4$ , the transition matrix (8) becomes

$$U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \quad (10)$$

where we used the abbreviation

$$\omega \equiv \omega_4. \quad (11)$$

The following analysis illustrates the richness of the problem as well as it motivates possible generalizations for the case of arbitrary photon numbers.

### A. Impossible output states

Let us first look at the seemingly trivial situation, where every input port of the multiport beam splitter is entered by

one photon in the same state, let us say in  $|+\rangle$ , so that

$$|\phi_{\text{in}}\rangle = a_{1+}^{\dagger} a_{2+}^{\dagger} a_{3+}^{\dagger} a_{4+}^{\dagger} |0\rangle. \quad (12)$$

Using Eqs. (6) and (10), we then find that the collection of one photon per output port prepares the system in the postselected state

$$|\phi_{\text{pro}}\rangle = \sum_{\sigma} \left[ \prod_{i=1}^4 U_{\sigma(i)i} b_{i+}^{\dagger} \right] |0\rangle = 0. \quad (13)$$

This means that it is impossible to pass four photons in the same state through the considered setup with each of them leaving the multiport in a different output port. More generally speaking, the state with four photons in the same state does not contribute to the event of collecting one photon per output port. It is therefore impossible to prepare any superposition containing the states  $b_{1+}^{\dagger} b_{2+}^{\dagger} b_{3+}^{\dagger} b_{4+}^{\dagger} |0\rangle$  and  $b_{1-}^{\dagger} b_{2-}^{\dagger} b_{3-}^{\dagger} b_{4-}^{\dagger} |0\rangle$ , respectively. The reason is destructive interference of certain photon states within the linear optics setup, which plays a crucial role in the generation of multiphoton entanglement via postselection.

### B. The 4-photon $W$ -state

We now focus our attention on the case where input port 1 is entered by a photon prepared in  $|+\rangle$  while all other input ports are entered by a photon in  $|-\rangle$ , i.e.,

$$|\phi_{\text{in}}^W\rangle = a_{1+}^{\dagger} a_{2-}^{\dagger} a_{3-}^{\dagger} a_{4-}^{\dagger} |0\rangle. \quad (14)$$

Using again Eqs. (6) and (10), we find that the collection of one photon per output port corresponds to the postselected 4-photon state

$$|\phi_{\text{pro}}^W\rangle = \sum_{j=1}^4 U_{j1} b_{j+}^{\dagger} \sum_{\sigma_j} \left[ \prod_{i=2}^4 U_{\sigma_j(i)i} b_{\sigma_j(i)-}^{\dagger} \right] |0\rangle, \quad (15)$$

where the  $\sigma_j$  are the  $3!$  permutations that map the list  $\{2, 3, 4\}$  onto the list  $\{1, \dots, (j-1), (j+1), \dots, 4\}$ . If  $|j_{\text{out}}\rangle$  denotes the state with one photon in  $|+\rangle$  in output port  $j$  and one photon in  $|-\rangle$  everywhere else,

$$|j_{\text{out}}\rangle \equiv b_{N-}^{\dagger} \cdots b_{(j+1)-}^{\dagger} b_{j+}^{\dagger} b_{(j-1)-}^{\dagger} \cdots b_{1-}^{\dagger} |0\rangle, \quad (16)$$

and  $\beta_j$  is a complex probability amplitude, then the output state (15) can be written as

$$|\phi_{\text{pro}}^W\rangle = \sum_{j=1}^4 \beta_j |j_{\text{out}}\rangle. \quad (17)$$

Furthermore, we introduce the reduced transition matrices  $U_{\text{red}}^{(j)}$ , which are obtained by deleting the first column and the  $j$ th row of the transition matrix  $U$ . Then one can express each  $\beta_j$  as the permanent [47,48] of a matrix,

$$\beta_j = U_{j1} \sum_{\sigma_j} \prod_{i=2}^4 U_{\sigma_j(i)i} = U_{j1} \text{perm}(U_{\text{red}}^{(j)\text{T}}). \quad (18)$$

The output state (17) equals a  $W$ -state if the coefficients  $\beta_j$  are all of the same size and differ from each other at most by a phase factor.

To show that this is indeed the case, we calculate the reduced matrices  $U_{\text{red}}^{(j)}$  explicitly [49] and obtain

$$U_{\text{red}}^{(1)} = \frac{1}{2} \begin{pmatrix} \omega & \omega^2 & \omega^3 \\ \omega^2 & \omega^4 & \omega^6 \\ \omega^3 & \omega^6 & \omega^9 \end{pmatrix}, \quad U_{\text{red}}^{(2)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & \omega^4 & \omega^6 \\ \omega^3 & \omega^6 & \omega^9 \end{pmatrix},$$

$$U_{\text{red}}^{(3)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & \omega^3 \\ \omega^3 & \omega^6 & \omega^9 \end{pmatrix}, \quad U_{\text{red}}^{(4)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & \omega^3 \\ \omega^2 & \omega^4 & \omega^6 \end{pmatrix}. \quad (19)$$

The coefficients  $\beta_j$  differ at most by a phase factor if the norm of the permanents of the transpose of these reduced matrices is for all  $j$  the same. To show that this is the case we now define the vector

$$\mathbf{v} = (\omega, \omega^2, \omega^3), \quad (20)$$

multiply each row of the matrix  $U_{\text{red}}^{(1)}$  exactly  $(j-1)$  times with  $\mathbf{v}$ , and obtain the new matrices

$$\tilde{U}_{\text{red}}^{(1)} = U_{\text{red}}^{(1)}, \quad \tilde{U}_{\text{red}}^{(2)} = \frac{1}{2} \begin{pmatrix} \omega^2 & \omega^4 & \omega^6 \\ \omega^3 & \omega^6 & \omega^9 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\tilde{U}_{\text{red}}^{(3)} = \frac{1}{2} \begin{pmatrix} \omega^3 & \omega^6 & \omega^9 \\ 1 & 1 & 1 \\ \omega & \omega^2 & \omega^3 \end{pmatrix}, \quad \tilde{U}_{\text{red}}^{(4)} = U_{\text{red}}^{(4)}. \quad (21)$$

The above described multiplication amounts physically to the multiplication of the photon input state with an overall phase factor and

$$|\text{perm}(U_{\text{red}}^{(j)\text{T}})| = |\text{perm}(\tilde{U}_{\text{red}}^{(j)\text{T}})|. \quad (22)$$

Moreover, using the cyclic symmetry of permanents [47], we see that

$$\text{perm}(U_{\text{red}}^{(j)\text{T}}) = \text{perm}(\tilde{U}_{\text{red}}^{(j)\text{T}}). \quad (23)$$

This implies together with Eq. (18) that the norm of the coefficients  $\beta_j$  is indeed the same for all  $j$ . Furthermore, using the above argument based on the multiplication of phase factors to the photon input state, one can show that

$$\beta_j = \beta_1 \left( \prod_{k=0}^3 \omega^k \right)^{j-1}. \quad (24)$$

Inserting this into Eq. (15), we find that the postselected state with one photon per output port equals, after normalization [50], the  $W$ -state

$$|\hat{\phi}_{\text{pro}}^W\rangle = \frac{1}{2} [b_{1+}^\dagger b_{2-}^\dagger b_{3-}^\dagger b_{4-}^\dagger - b_{1-}^\dagger b_{2+}^\dagger b_{3-}^\dagger b_{4-}^\dagger + b_{1-}^\dagger b_{2-}^\dagger b_{3+}^\dagger b_{4-}^\dagger - b_{1+}^\dagger b_{2+}^\dagger b_{3+}^\dagger b_{4+}^\dagger] |0\rangle. \quad (25)$$

In analogy, we conclude that an input state with one photon in  $|-\rangle$  in input port 1 and a photon in  $|+\rangle$  in each of the other input ports, results in the preparation of the  $W$ -state

$$|\hat{\phi}_{\text{pro}}^{W'}\rangle = \frac{1}{2} [b_{1-}^\dagger b_{2+}^\dagger b_{3+}^\dagger b_{4+}^\dagger - b_{1+}^\dagger b_{2-}^\dagger b_{3+}^\dagger b_{4+}^\dagger + b_{1+}^\dagger b_{2+}^\dagger b_{3-}^\dagger b_{4+}^\dagger - b_{1+}^\dagger b_{2+}^\dagger b_{3+}^\dagger b_{4-}^\dagger] |0\rangle \quad (26)$$

under the condition of the collection of one photon per output port. Both states, (25) and (26), can be generated with probability

$$P_{\text{suc}} = \frac{1}{16}. \quad (27)$$

Transforming them into the usual form of a  $W$ -state with equal coefficients of all amplitudes [51] only requires further implementation of a Pauli  $\sigma_z$  operation (i.e., a state dependent sign flip) on either the first and the third or the second and the fourth output photon, respectively.

### C. The 4-photon GHZ-state

Besides generating  $W$ -states, the proposed setup can also be used to prepare 4-photon GHZ-states. This requires feeding each of the input ports 1 and 3 with one photon in  $|+\rangle$  while the input ports 2 and 4 should each be entered by a photon in  $|-\rangle$  such that

$$|\phi_{\text{in}}^{\text{GHZ}}\rangle = a_{1+}^\dagger a_{2-}^\dagger a_{3+}^\dagger a_{4-}^\dagger |0\rangle. \quad (28)$$

Calculating again the output state under the condition of collecting one photon per output port, we obtain

$$|\phi_{\text{pro}}^{\text{GHZ}}\rangle = \sum_{\sigma} U_{\sigma(1)1} U_{\sigma(2)2} U_{\sigma(3)3} U_{\sigma(4)4} \times b_{\sigma(1)+}^\dagger b_{\sigma(2)-}^\dagger b_{\sigma(3)+}^\dagger b_{\sigma(4)-}^\dagger |0\rangle, \quad (29)$$

where the  $\sigma$  are the  $4!$  permutations that map the list  $\{1, 2, 3, 4\}$  onto itself. On simplification, one finds that there are only two constituent states with nonzero coefficients and  $|\phi_{\text{pro}}^{\text{GHZ}}\rangle$  becomes after normalization

$$|\hat{\phi}_{\text{pro}}^{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} [b_{1+}^\dagger b_{2-}^\dagger b_{3+}^\dagger b_{4-}^\dagger - b_{1-}^\dagger b_{2+}^\dagger b_{3-}^\dagger b_{4+}^\dagger] |0\rangle, \quad (30)$$

which equals the GHZ-state up to local operations. Transforming Eq. (30) into the usual form of the GHZ-state requires changing the state of two of the photons, for example, from  $|+\rangle$  into  $|-\rangle$ . This can be realized by applying a Pauli  $\sigma_x$  operation to the first output port as well as a  $\sigma_y$  operation to the third output.

Finally, we remark that the probability for the creation of the GHZ-state (31) is twice as high as the probability for the generation of a  $W$ -state (27),

$$P_{\text{suc}} = \frac{1}{8}. \quad (31)$$

Unfortunately, the experimental setup shown in Fig. 1 does not allow for the preparation of GHZ-states for arbitrary photon numbers  $N$ . For a detailed description of polarization entangled GHZ-states using a different network of 50:50 and polarizing beam splitters see Ref. [36].

#### D. The 4-photon double singlet state

For completeness, we now ask for the output of the proposed state preparation scheme, given that the input state equals

$$|\phi_{\text{in}}\rangle = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger |0\rangle. \quad (32)$$

Proceeding as above, we find that this results in the preparation of the state

$$|\phi_{\text{pro}}^{\text{DS}}\rangle = \sum_{\sigma} U_{\sigma(1)1} U_{\sigma(2)2} U_{\sigma(3)3} U_{\sigma(4)4} \times b_{\sigma(1)+}^\dagger b_{\sigma(2)+}^\dagger b_{\sigma(3)-}^\dagger b_{\sigma(4)-}^\dagger |0\rangle \quad (33)$$

under the condition of the collection of one photon per output port. Here the permutation operators  $\sigma$  are defined as in Sec. III C, which yields

$$|\phi_{\text{pro}}^{\text{DS}}\rangle = \frac{1}{2} [b_{1+}^\dagger b_{2+}^\dagger b_{3+}^\dagger b_{4-}^\dagger + b_{1+}^\dagger b_{2-}^\dagger b_{3+}^\dagger b_{4+}^\dagger - b_{1+}^\dagger b_{2-}^\dagger b_{3-}^\dagger b_{4+}^\dagger - b_{1-}^\dagger b_{2+}^\dagger b_{3+}^\dagger b_{4-}^\dagger] |0\rangle. \quad (34)$$

This state can be prepared with probability

$$P_{\text{suc}} = \frac{1}{16}. \quad (35)$$

The state (34) is a double singlet state, i.e., a tensor product of two 2-photon singlet states, with a high robustness against decoherence [18].

#### E. The general 4-photon case

Finally, we consider the situation where the input state is of the general form (1). Calculating Eq. (6), we find that the unnormalized output state under the condition of one photon per output port equals in this case

$$|\phi_{\text{pro}}\rangle = \frac{i}{4} (\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4) |\hat{\phi}_{\text{pro}}^{\text{DS}}\rangle + \frac{1}{2\sqrt{2}} (\gamma_6 - \gamma_5) |\hat{\phi}_{\text{pro}}^{\text{GHZ}}\rangle + \frac{1}{4} (\gamma_8 + \gamma_{10} - \gamma_7 - \gamma_9) |\hat{\phi}_{\text{pro}}^{\text{W}}\rangle + \frac{1}{4} (\gamma_{12} + \gamma_{14} - \gamma_{11} - \gamma_{13}) |\hat{\phi}_{\text{pro}}^{\text{W}'}\rangle \quad (36)$$

with the states  $|\hat{\phi}_{\text{pro}}^{\text{W}}\rangle$ ,  $|\hat{\phi}_{\text{pro}}^{\text{W}'}\rangle$ ,  $|\hat{\phi}_{\text{pro}}^{\text{GHZ}}\rangle$ , and  $|\hat{\phi}_{\text{pro}}^{\text{DS}}\rangle$  as defined in Eqs. (25), (26), (30), and (34) and with the coefficients

$$\gamma_1 = \alpha_{1+} \alpha_{2+} \alpha_{3-} \alpha_{4-}, \quad \gamma_2 = \alpha_{1-} \alpha_{2-} \alpha_{3+} \alpha_{4+},$$

$$\gamma_3 = \alpha_{1-} \alpha_{2+} \alpha_{3+} \alpha_{4-}, \quad \gamma_4 = \alpha_{1+} \alpha_{2-} \alpha_{3-} \alpha_{4+},$$

$$\gamma_5 = \alpha_{1+} \alpha_{2-} \alpha_{3+} \alpha_{4-}, \quad \gamma_6 = \alpha_{1-} \alpha_{2+} \alpha_{3-} \alpha_{4+},$$

$$\gamma_7 = \alpha_{1+} \alpha_{2-} \alpha_{3-} \alpha_{4-}, \quad \gamma_8 = \alpha_{1-} \alpha_{2+} \alpha_{3-} \alpha_{4-},$$

$$\gamma_9 = \alpha_{1-} \alpha_{2-} \alpha_{3+} \alpha_{4-}, \quad \gamma_{10} = \alpha_{1-} \alpha_{2-} \alpha_{3-} \alpha_{4+},$$

$$\gamma_{11} = \alpha_{1-} \alpha_{2+} \alpha_{3+} \alpha_{4+}, \quad \gamma_{12} = \alpha_{1+} \alpha_{2-} \alpha_{3+} \alpha_{4+},$$

$$\gamma_{13} = \alpha_{1+} \alpha_{2+} \alpha_{3-} \alpha_{4+}, \quad \gamma_{14} = \alpha_{1+} \alpha_{2+} \alpha_{3+} \alpha_{4-}. \quad (37)$$

The form of the coefficients (37) reflects the full symmetry of the transformation of the input state. Each of the entangled states  $|\hat{\phi}_{\text{pro}}^{\text{DS}}\rangle$ ,  $|\hat{\phi}_{\text{pro}}^{\text{GHZ}}\rangle$ ,  $|\hat{\phi}_{\text{pro}}^{\text{W}}\rangle$ , and  $|\hat{\phi}_{\text{pro}}^{\text{W}'}\rangle$  are generated independently from the different constituent parts of the input (1). Besides, Eq. (36) shows that the output state is constrained to be of a certain symmetry, namely the symmetry introduced by the symmetry of the  $N \times N$  Bell multiphot beam splitter and the postselection criteria of finding one photon per output port. As before, the norm squared of  $|\phi_{\text{pro}}\rangle$  equals the probability for a successful state preparation.

#### IV. THE GENERATION OF $N$ -PHOTON $W$ -STATES

Using the same arguments as in Sec. III B, we now show that the  $N \times N$  Bell multiphot beam splitter can be used for the generation of  $W$ -states for arbitrary photon numbers  $N$ . Like Bell states,  $W$ -states are highly entangled but their entanglement is more robust [51]. Moreover, as  $N$  increases,  $W$ -states perform better than the corresponding GHZ-states against noise admixture in experiments to violate local realism [52] and are important for optimal cloning protocols [53].

In analogy to Eq. (14), we assume that the initial state contains one photon in  $|+\rangle$  in the first input port, while every other input port is entered by a photon prepared in  $|-\rangle$  so that

$$|\phi_{\text{in}}\rangle = a_{1+}^\dagger \prod_{i=2}^N a_{i-}^\dagger |0\rangle. \quad (38)$$

Using Eq. (6), we find that the state of the system under the condition of the collection of one photon per output port equals

$$|\phi_{\text{pro}}\rangle = \sum_{j=1}^N U_{j1} b_{j+}^\dagger \sum_{\sigma_j} \left[ \prod_{i=2}^N U_{\sigma_j(i)i} b_{\sigma_j(i)-}^\dagger \right] |0\rangle, \quad (39)$$

where the  $\sigma_j$  are the  $(N-1)!$  permutations that map the list  $\{2, 3, \dots, N\}$  onto the list  $\{1, 2, \dots, (j-1), (j+1), \dots, N\}$ . As expected, the output is a superposition of all states with one photon in  $|+\rangle$  and all other photons prepared in  $|-\rangle$ .

To prove that Eq. (39) describes indeed a  $W$ -state, we use again the notation introduced in Eqs. (16) and (17) and write

$$|\phi_{\text{pro}}\rangle \equiv \sum_j \beta_j |j_{\text{out}}\rangle. \quad (40)$$

To show that the coefficients  $\beta_j$  differ from  $\beta_1$  at most by a phase factor, we express the amplitudes  $\beta_j$  as in Eq. (18) using the permanents of the reduced transition matrices and find

$$\beta_j = U_{j1} \sum_{\sigma_j} \prod_{i=2}^N U_{\sigma_j(i)i} = U_{j1} \text{perm}[U_{\text{red}}^{\text{T}}(j)]. \quad (41)$$

Inserting the concrete form of the transition matrix  $U$ , this yields

$$\beta_j = \frac{1}{\sqrt{N^N}} \sum_{\sigma_j} \prod_{i=2}^N \omega_N^{[\sigma_j(i-1)](i-1)}. \quad (42)$$

Proceeding as in Sec. III B, we now multiply  $\beta_j$  with the phase factor

$$v_j \equiv \left( \prod_{k=0}^{N-1} \omega_N^k \right)^{-(j-1)} \quad (43)$$

and obtain

$$\begin{aligned} v_j \beta_j &= \frac{1}{\sqrt{N^N}} \sum_{\sigma_j} \prod_{i=2}^N \omega_N^{(\sigma_j(i-j))(i-1)} \\ &= \frac{1}{\sqrt{N^N}} \sum_{\sigma_j} \prod_{i=2}^N \omega_N^{(\text{mod}_N(\sigma_j(i-j)))(i-1)}. \end{aligned} \quad (44)$$

The expression  $\text{mod}_N(\sigma_j(i-j)+1)$  represents a set of  $(N-1)!$  permutations that map  $\{2, 3, \dots, N\}$  onto the list  $\{2, 3, \dots, N\}$ . It is therefore equivalent to the permutations  $\sigma_1(i)$ , which allows us to simplify Eq. (44) even further and to show that

$$v_j \beta_j = \frac{1}{\sqrt{N^N}} \sum_{\sigma_1} \prod_{i=2}^N \omega_N^{[\sigma_1(i-1)](i-1)} = \beta_1. \quad (45)$$

From this and the fact that  $1+2+\dots+(N-1) = \frac{1}{2}N(N-1)$  we finally arrive at the relation

$$\beta_j = \left( \prod_{k=0}^{N-1} \omega_N^k \right)^{j-1} \beta_1 = \begin{cases} \beta_1, & \text{if } N \text{ is odd} \\ (-1)^{j-1} \beta_1, & \text{if } N \text{ is even.} \end{cases} \quad (46)$$

This shows that the amplitudes  $\beta_j$  are all of the same size and the Bell multiport can indeed generate  $N$ -photon  $W$ -states. If one wants the coefficients  $\beta_j$  to be exactly the same, one can remove unwanted minus signs in the case of even photon numbers by applying a  $\sigma_z$  operation in each output port with an even number  $j$ .

In the case  $N=2$ , the above described state preparation scheme reduces to the familiar example, where two photons prepared in the two orthogonal states  $|+\rangle$  and  $|-\rangle$  pass through a 50:50 beam splitter. The collection of one photon in each output port prepares the system in this case in the state  $1/\sqrt{2}[b_{1+}^\dagger b_{2-}^\dagger - b_{1-}^\dagger b_{2+}^\dagger]|0\rangle$ , which can be transformed into  $1/\sqrt{2}[b_{1+}^\dagger b_{2-}^\dagger + b_{1-}^\dagger b_{2+}^\dagger]|0\rangle$  by performing a conditional sign flip, i.e., depending on whether the photon is in  $|+\rangle$  or  $|-\rangle$ , in one of the output ports.

### A. Success probabilities

Let us finally comment on the success rate of the proposed  $W$ -state preparation scheme. Computing the probability (7) can be done by finding the amplitude  $\beta_1$  with the help of Eq. (41). Although the definition of the permanent of a matrix resembles the definition of the determinant, there exist only few theorems that can be used to simplify their calculation [47]. In fact, the computation of the permanent is an

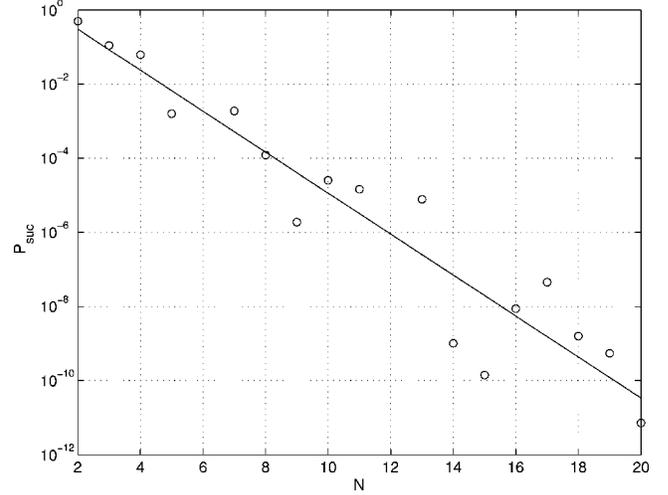


FIG. 2. The success rate for the generation of  $N$ -photon  $W$ -states  $P_{\text{suc}}$  as a function of  $N$ . The solid line approximates the exact results via the equation  $P_{\text{suc}} = e^{a-bN}$  with  $a = 1.35 \pm 1.32$  and  $b = 1.27 \pm 0.10$ .

NP-complete problem. We therefore calculated  $P_{\text{suc}}$  numerically (see Fig. 2).

As it applies to linear optics schemes in general, the success probability decreases unfavorably as the number of qubits involved increases. Here the probability for the collection drops on average exponentially. We observe the interesting effect of a nonmonotonic decreasing success probability as  $N$  increases. For example, the probability of success for  $N=13$  is higher than for  $N=9$ . Moreover, for  $N=6$  and  $N=12$ ,  $W$ -state generation is not permitted due to destructive interference. This does not apply to  $N=18$  which is also a multiple of 6.

## V. CONCLUSIONS

We analyzed the generation of multiphoton entanglement with the help of interference and postselection in a linear optics network consisting of an  $N \times N$  Bell multiport beam splitter. Each input port should be entered by a single photon prepared in a certain state  $|\lambda_i\rangle$ . As long as the photons are the same with respect to all other degrees of freedom and it can be guaranteed that photons prepared in the same state overlap within their coherence time inside the linear optics network, the described scheme can be implemented using only a single photon source [24–29]. We believe that the described approach allows one to entangle much higher photon numbers than what can be achieved in parametric down conversion experiments.

An, in general, highly entangled output state is obtained under the condition of the collection of one photon per output port. The motivation for this postselection criteria is that distinguishing this state from other output states does not require photon number resolving detectors.

First we analyzed the case  $N=4$  and showed that the  $4 \times 4$  Bell multiport allows for the creation of a variety of highly symmetric entangled states including the  $W$ -state, the GHZ-state, and double singlet states. It was found that some states are easier to prepare than others. A straightforward

generalization of the 4-photon case yields a scheme for the creation of  $N$ -photon  $W$ -states. We calculated the rates for successful state preparations and showed that they decrease in a nonmonotonic fashion and on average exponentially with  $N$ .

The motivation for considering a Bell multiport beam splitter was that it only redirects the photons without affecting their inner degrees of freedom. The proposed setup can therefore be used to produce polarization, time-bin, and frequency entanglement, respectively. To generate, for example, polarization entangled photons, the initial photon states may differ in polarization but should otherwise be exactly the same. The high symmetry of the Bell multiport beam splitter allows for the generation of a variety of highly entangled

symmetric states. To verify the generation of a certain type of entanglement one could, for example, use local measurements as it has recently been proposed by Toth and Gühne [46,54].

#### ACKNOWLEDGMENTS

Y.L.L. acknowledges financial support by the DSO National Laboratories in Singapore. A.B. thanks the Royal Society and the GCHQ for funding as a James Ellis University Research Fellow. This work was supported in part by the European Union and the UK Engineering and Physical Sciences Research Council.

- 
- [1] D. Gottesman and I. L. Chuang, *Nature (London)* **402**, 390 (1999).
  - [2] E. Knill, R. Laflamme, and G. J. Milburn, *Nature (London)* **409**, 46 (2001).
  - [3] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, New York, 1984), pp. 175–179.
  - [4] A. K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).
  - [5] J. S. Bell, *Physics (Long Island City, N.Y.)* **1**, 195 (1965).
  - [6] A. Aspect, P. Grangier, and G. Roger, *Phys. Rev. Lett.* **49**, 91 (1982).
  - [7] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, *Phys. Rev. Lett.* **75**, 4337 (1995).
  - [8] J. Brendel, N. Gisin, W. Tittel, and H. Zbinden, *Phys. Rev. Lett.* **82**, 2594 (1999); R. T. Thew, S. Tanzilli, W. Tittel, H. Zbinden, and N. Gisin, *Phys. Rev. A* **66**, 062304 (2002); H. de Riedmatten, I. Marcikic, W. Tittel, H. Zbinden, D. Collins, and N. Gisin, *Phys. Rev. Lett.* **92**, 047904 (2004).
  - [9] K. M. Gheri, C. Saavedra, P. Törmä, J. I. Cirac, and P. Zoller, *Phys. Rev. A* **58**, R2627 (1998).
  - [10] W. Lange and H. J. Kimble, *Phys. Rev. A* **61**, 063817 (2000).
  - [11] C. Schön, E. Solano, F. Verstraete, J. I. Cirac, and M. M. Wolf, e-print quant-ph/0501096.
  - [12] Y. L. Lim and A. Beige, *J. Phys. A* **38**, L7 (2005).
  - [13] Y. L. Lim and A. Beige, *Proc. SPIE* **5436**, 118 (2004).
  - [14] Y. L. Lim, A. Beige, and L. C. Kwek, e-print quant-ph/0408043.
  - [15] G. G. Lapaire, P. Kok, J. P. Dowling, and J. E. Sipe, *Phys. Rev. A* **68**, 042314 (2003).
  - [16] Y. H. Shih and C. O. Alley, *Phys. Rev. Lett.* **61**, 2921 (1988).
  - [17] D. Fattal, K. Inoue, J. Vuckovic, C. Santori, G. S. Solomon, and Y. Yamamoto, *Phys. Rev. Lett.* **92**, 037903 (2004).
  - [18] M. Eibl, S. Gaertner, M. Bourennane, C. Kurtsiefer, M. Zukowski, and H. Weinfurter, *Phys. Rev. Lett.* **90**, 200403 (2003); M. Bourennane, M. Eibl, S. Gaertner, C. Kurtsiefer, A. Cabello, and H. Weinfurter, *ibid.* **92**, 107901 (2004).
  - [19] Z. Zhao, T. Yang, Y.-A. Chen, A.-N. Zhang, M. Zukowski, and J.-W. Pan, *Phys. Rev. Lett.* **91**, 180401 (2003); Z. Zhao, A.-N. Zhang, Y.-A. Chen, H. Zhang, J.-F. Du, T. Yang, and J.-W. Pan, *ibid.* **94**, 030501 (2005).
  - [20] C. K. Law and H. J. Kimble, *J. Mod. Opt.* **44**, 2067 (1997).
  - [21] A. Kuhn, M. Hennrich, T. Bondo, and G. Rempe, *Appl. Phys. B: Lasers Opt.* **69**, 373 (1999).
  - [22] L.-M. Duan, A. Kuzmich, and H. J. Kimble, *Phys. Rev. A* **67**, 032305 (2003).
  - [23] E. Jeffrey, N. A. Peters, P. G. Kwiat, and G. Kwiat, *New J. Phys.* **6**, 100 (2004).
  - [24] M. Hennrich, T. Legero, A. Kuhn, and G. Rempe, *Phys. Rev. Lett.* **85**, 4872 (2000); A. Kuhn, M. Hennrich, and G. Rempe, *ibid.* **89**, 067901 (2002).
  - [25] J. McKeever, A. Boca, A. D. Boozer, R. Miller, J. R. Buck, A. Kuzmich, and H. J. Kimble, *Science* **303**, 1992 (2004).
  - [26] M. Keller, B. Lange, K. Hayasaka, W. Lange, and H. Walther, *Nature (London)* **431**, 1075 (2004).
  - [27] O. Benson, C. Santori, M. Pelton, and Y. Yamamoto, *Phys. Rev. Lett.* **84**, 2513 (2000); M. Pelton, C. Santori, J. Vuckovic, B. Zhang, G. S. Solomon, J. Plant, and Y. Yamamoto, *ibid.* **89**, 233602 (2002).
  - [28] C. Kurtsiefer, S. Mayer, P. Zarda, and H. Weinfurter, *Phys. Rev. Lett.* **85**, 290 (2000).
  - [29] A. Beveratos, R. Brouri, T. Gacoin, A. Villing, J.-P. Poizat, and P. Grangier, *Phys. Rev. Lett.* **89**, 187901 (2002); J. R. Rabeau, Y. L. Chin, S. Praver, F. Jelezko, T. Gaebel, and J. Wrachtrup, *Appl. Phys. Lett.* **86**, 131926 (2005).
  - [30] M. Zukowski, A. Zeilinger, and M. A. Horne, *Phys. Rev. A* **55**, 2564 (1997).
  - [31] H. Lee, P. Kok, N. J. Cerf, and J. P. Dowling, *Phys. Rev. A* **65**, 030101(R) (2002).
  - [32] J. Fiurasek, *Phys. Rev. A* **65**, 053818 (2002).
  - [33] X. B. Zou, K. Pahlke, and W. Mathis, *Phys. Rev. A* **66**, 014102 (2002).
  - [34] G. J. Pryde and A. G. White, *Phys. Rev. A* **68**, 052315 (2003).
  - [35] Xiang-Bin Wang, *Phys. Rev. A* **68**, 042304 (2003).
  - [36] Y. Sagi, *Phys. Rev. A* **68**, 042320 (2003).
  - [37] J. Fiurasek, S. Massar, and N. J. Cerf, *Phys. Rev. A* **68**, 042325 (2003).
  - [38] T. B. Pittman, B. C. Jacobs, and J. D. Franson, *Phys. Rev. Lett.* **88**, 257902 (2002).
  - [39] P. Törmä, S. Stenholm, and I. Jex, *Phys. Rev. A* **52**, 4853 (1995).
  - [40] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Rev. Mod.*

- Phys. **74**, 145 (2002).
- [41] P. Törmä, Phys. Rev. Lett. **81**, 2185 (1998).
- [42] J.-F. Roch, K. Vigneron, Ph. Grelu, A. Sinatra, J.-Ph. Poizat, and Ph. Grangier, Phys. Rev. Lett. **78**, 634 (1997).
- [43] W. J. Munro, Kae Nemoto, R. G. Beausoleil, and T. P. Spiller, Phys. Rev. A **71**, 033819 (2005).
- [44] Y.-A. Chen, A.-N. Zhang, Z. Zhao, X.-Q. Zhou, C.-Y. Lu, C.-Z. Peng, T. Yang, and J.-W. Pan, e-print quant-ph/0502131.
- [45] J. Joo, Y.-J. Park, S. Oh, and J. Kim, New J. Phys. **5**, 136 (2003).
- [46] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **92**, 087902 (2004).
- [47] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, England, 1985).
- [48] S. Scheel, e-print quant-ph/0406127.
- [49] The reason for not simplifying these matrices is that the following equations provide the motivation for the proof of the general case in Sec. IV.
- [50] In the following we denote normalized states by marking them with a caret.
- [51] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
- [52] A. Sen(De), U. Sen, M. Wieśniak, D. Kaszlikowski, and M. Zukowski, Phys. Rev. A **68**, 062306 (2003).
- [53] V. Buzek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
- [54] G. Toth and O. Gühne, e-print quant-ph/0405165.