Coherent control of trapped ions using off-resonant lasers

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In this paper we develop a unified framework to study the coherent control of trapped ions subject to state-dependent forces. Taking different limits in our theory, we can reproduce previous designs of quantum gates and propose a different design of fast gates based on continuous laser beams. We demonstrate how to simulate Ising Hamiltonians in a many ions setup, and how to create highly entangled states and induce squeezing. Finally, in a detailed analysis we identify the physical limits of this technique and study the dependence of errors on the temperature.

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I. INTRODUCTION

Trapped ions constitute one of the most promising systems to implement a scalable quantum computer [1]. In such a computer, information is stored in long-lived atomic states, and a universal set of gates is obtained by manipulating these states with lasers and entangling the ions via the vibrational modes [2]. During the last years we have seen experimental demonstrations of various two-qubit gates [3–6], and it remains to implement a robust scheme for scalability. Current visions of a scalable computer are based on the idea of moving the ions out of their storage area (or quantum memory) and make them interact in pairs, performing two-qubit gates [2,7]. Basic steps toward the experimental implementation of this idea have been already demonstrated [8].

However, outside quantum computing there are other promising areas in which trapped ions may be of use. On the one hand, there is a great interest in preparing highly entangled or squeezed states, which can be used both for metrology [9,10], or—in a more fundamental fashion—to characterize their decoherence properties. The entanglement of ions is covered in a variety of theoretical papers [11–14], and it has been experimentally demonstrated for small systems [15–17]. On the other hand, it has also been suggested that ion traps can be used to simulate a various spin Hamiltonians, ranging from local to long-range interactions [18,19].

In all of these tasks—quantum computing, creation of entanglement, and quantum simulation—the goal is to induce some unitary evolution on the internal state of the ions, which is used to store the information. For instance, in the case of quantum computing it suffices to realize a phase gate between two ions,

$$U_{\mathrm{ph}}(T) = e^{i\pi\sigma_1^z\sigma_2^z/4}, \tag{1}$$

while in the case of quantum simulation we want a more general evolution,

$$U(T) = e^{i(\Sigma_{i,j}\vec{\sigma}_i J_{ij}\vec{\sigma}_j + \Sigma_i \vec{B}_i\vec{\sigma}_i)T}, \tag{2}$$

where the matrices J_{ij} and the vectors $\vec{B}_i$ determine the spin Hamiltonian that we want to simulate. Since the real interaction between the ions is described by more complicated Hamiltonians [see Eq. (3) below], any of these transformations is an effective one, realized after influencing the dynamics of the ions with external fields.

This process, in which we dynamically modify the parameters of the system—Rabi frequencies, detunings, magnetic fields, etc.—in order to achieve a well-defined target operation, is called *coherent quantum control*. The development of coherent control has been tightly connected to the fields of nuclear magnetic resonance (NMR) [20] and atomic and molecular spectroscopy [21]. While in the last field we have seen a strong development of optimal control, the theoretical and experimental achievements have been oriented toward controlling the evolution of particular states, so as to optimize atomic or molecular transitions, or certain chemical processes. Instead, for applications to quantum computing, we need to be able to control the evolution of a linear subspace of our Hilbert space as a whole. This requirement has been identified in NMR, where pulse sequences have been developed to simulate arbitrary Hamiltonians and reduce decoherence [20].

Coherent control has been also implicit in any proposal for quantum computing [2,13,22–30] and quantum simulation [18,19] with trapped ions. While in these works the design of the control relied always on the intuition of the researcher and on a clever choice of approximations, in this paper we show that many of these schemes can be translated into a unified framework based on state dependent forces and tunable traps. As characteristic examples, we will show how to implement the pushing gate [2] and the fast gates based on laser pulses from Ref. [28]. We will also propose an alternative and more general model of fast gate based on continuous off-resonant lasers. As further applications, we demonstrate how to induce collective $H=(\Sigma_i\sigma_i^z)^2$ and nearest-neighbor $H=\Sigma_i J\sigma_i^z\sigma_{i+1}^z$ Ising interactions, and use them to produce squeezing, and generate cluster and Greenberger-Home-

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Zeilinger (GHZ) states of up to 30 atoms within a extremely short time, $T=\mathcal{O}(1/\omega)$, where ω is the frequency of the ion trap.

Furthermore, within this unified framework we can address important requirements of all these coherent processes. Namely, they should: (i) be independent of temperature (so that one does not need to cool the ions to their ground state after they are moved to or from their storage area); (ii) require no addressability (to allow the ions to be as close as possible during the gate so as to strengthen their interaction), and (iii) be fast (in order to minimize the effects of decoherence during the gate, and to speed up the computation). All of these requirements can be formulated as constraints of the control problem, and as we will see below, they can be easily solved. Last but not least, we study the scaling of resources as we try to make our control faster and answer the question of what is the ultimate limit for the speed of our quantum gates or entangling processes, which is shown to be determined both by dissipation and nonharmonic contributions to the restoring forces.

The paper is organized as follows. In Sec. II we develop the formalism to study our system. First of all, we introduce the Hamiltonian for any number of ions subject to state dependent forces and quasi-1D confinement, and derive the harmonic approximation and the normal modes. Then, we show how to implement a unitary transformation, $U=\exp[i\phi(\{\sigma_i^z\})]$, made of robust geometric gates, while leaving the motional state unchanged. In Sec. III we apply our methods to quantum computing in two-qubit setups. We demonstrate how to recover previous designs of a phase gate, including the adiabatic pushing gate [2] and the fast gate based on $\pi/2$ laser pulses that kick the ions [28]. Most important, since the generation of perfect and very short laser pulses is a difficult task, we design an arbitrarily fast quantum gate based on continuous laser sources. In Sec. IV we study the coherent control of many-ion setups. We prove that arbitrary spin interactions of the type (2) can be simulated with the appropriate time-dependent forces, and develop a numerical method to find them. As applications, we demonstrate numerically the creation of squeezed, cluster, and GHZ states of up to 30 ions in a very short time. Section V focuses on the study of errors. First of all we introduce a model for dissipation on the vibrational degrees of freedom. Next, we solve this model exactly and analyze the fidelity of the effective unitary operations (1), (2). We prove that for a perfect control and no dissipation our scheme is insensitive to temperature. Furthermore, the errors due to interaction with the environment can be computed and optimized using the same tools of quantum control as in Secs. III and IV. Finally, we study the errors due to anharmonic terms in the interaction and show that both this and dissipation set an upper bound on the speed of the gate. In Sec. VI we summarize our results and offer perspectives for future work.

II. THEORY

A. Normal modes

As mentioned before, this paper studies a set of N ions, in an essentially one-dimensional confinement [45] and subject to some external forces. Our model Hamiltonian is

$$H=\sum_i\left[\frac{p_i^2}{2m}+V_{e,i}(x_i)-F_i(t)x_i\right]+\sum_{i<j}\frac{e^2}{4\pi\epsilon_0}\frac{1}{|x_i-x_j|}. \quad (3)$$

In this equation, $V_{e,i}$ is the trapping potential that confines the kth ion, and it may be the same for all of them or may change from ion to ion as in the case of microtraps [2]. The time-dependent external forces are denoted by $F_i(t)$ and, as we will see below, they depend on the internal state of the ion, σ_i^z.

If we expand the previous energy around the equilibrium configuration without forces, given by $(\partial H/\partial x_i)(x_1^{(0)},\ldots,x_N^{(0)})=0$ and $F_i=0$, we obtain a set of coupled harmonic oscillators [31],

$$H=\sum_i\left[\frac{p_i^2}{2m}-F_i(t)x_i\right]+\frac{1}{2}mV_{ij}x_ix_j+E_0. \quad (4)$$

The constant E_0 is the energy of the ions at their equilibrium positions. The matrix V describes the restoring forces: it is symmetric, positive definite, and it can decomposed as $V=M\Omega^2M^t$, with a positive diagonal matrix of frequencies, $\Omega_{kl}=\omega_k\delta_{kl}$, and an orthogonal transformation ($MM^t=M^tM=\mathbb{I}$). Using this canonical transformation to define the normal modes $x_i=\Sigma_kM_{ik}Q_k$, $p_i=\Sigma_kM_{ik}P_k$, we arrive at the Hamiltonian

$$H=\sum_k\left[\frac{P_k^2}{2m}+\frac{1}{2}m\omega_k^2Q_k^2-\sum_i F_iM_{ik}Q_k\right]. \quad (5)$$

It is now useful to write this Hamiltonian in dimensionless form, using the characteristic length of the oscillators, $\alpha_k=(\hbar/m\omega_k)^{1/2}$, so that $P_k=\hbar\tilde{P}_k/\alpha_k$ and $Q_k=\alpha_k\tilde{Q}_k$, $\tilde{M}_{ik}=M_{ik}\alpha_k$. With this we arrive to

$$H=\sum_k\left[\frac{1}{2}\hbar\omega_k(\tilde{P}_k^2+\tilde{Q}_k^2)-\sum_i F_i\tilde{M}_{ik}\tilde{Q}_k\right], \quad (6)$$

or, using Fock operators, $a_k\equiv(\tilde{Q}_k+i\tilde{P}_k)/\sqrt{2}$,

$$H=\sum_k\left[\hbar\omega_k\left(a_k^\dagger a_k+\frac{1}{2}\right)-\sum_i\frac{F_i\tilde{M}_{ik}}{\sqrt{2}}(a_k^\dagger+a_k)\right]. \quad (7)$$

B. Robust phases

We will now demonstrate how to obtain a robust phase by applying forces to a harmonic oscillator. Let us begin with the toy model

$$H=\hbar\omega a^\dagger a-f(t)\frac{1}{\sqrt{2}}(a^\dagger+a). \quad (8)$$

When integrating the Schrödinger equation associated with this Hamiltonian, it will convenient to use the overcomplete basis of coherent states [32,33],

$$|z\rangle := e^{-|z|^2/2}\sum_n \frac{z^n}{\sqrt{n!}}|n\rangle =: |Q+iP\rangle. \tag{9}$$

Under the Hamiltonian (8), the coherent states behave somehow like classical particles in phase space, because their center, given by $\langle Q\rangle$ and $\langle P\rangle$, follows a classical trajectory, while the width of these wave packets, given by the uncertainty of Q and P, remains fixed.

More precisely, for an initial coherent state, $|\psi(0)\rangle=|z_0\rangle$, the solution to the Schrödinger equation $i\hbar\dot{\psi}=H(t)\psi$ is given also by a coherent state $|\psi(t)\rangle=e^{i\phi(t)}|z(t)\rangle$, whose phase and position satisfy

$$\frac{dz}{dt} = -i\omega z + i\frac{1}{\sqrt{2}\hbar}f(t), \tag{10a}$$

$$\frac{d\phi}{dt} = \frac{1}{2\sqrt{2}\hbar}f(t)(\bar{z}+z). \tag{10b}$$

The first equation has a solution,

$$z(t) = e^{-i\omega t}\left[z_0 + \frac{i}{\sqrt{2}\hbar}\int_0^t d\tau\, e^{i\omega\tau}f(\tau)\right], \tag{11}$$

that results from composing a displacement with a rotation of angular speed ω. Using the rotating phase-space coordinates, $z_r := e^{i\omega t}z =: Q_r + iP_r$, we get rid of the motion due to the unforced harmonic oscillators and find

$$\frac{dz_r}{dt} = ie^{i\omega\tau}f(\tau)/\hbar\sqrt{2}, \tag{12a}$$

$$\frac{d\phi}{dt} = \mathrm{Im}\frac{dz_r}{dt}\bar{z}_r = \frac{dP_r}{dt}Q_r - \frac{dQ_r}{dt}P_r = 2\frac{dA}{dt}. \tag{12b}$$

The last equation means indeed that the growth of the phase $\dot{\phi}$ is proportional to the area A spanned by the coherent state as it moves through the phase space. The phase is not only geometric in this sense, but also in the extended definition of geometric phase given in Ref. [34]. Applying the formulas in the previous reference one finds that if the total phase is $\phi(t)$, and the dynamical phase is always twice the geometric one, $\phi_d(t)=-2\phi_g(t)$, so that in the end $\phi(t)=-\phi_g(t)$.

In this work we are interested in this phase and on making it depend on the internal state of the particles governed by the oscillator. We want, however, neither to influence the motional state of the particles, nor to entangle internal and motional degrees of freedom. For this reason we set a time limit on the duration of the force and impose that after a fixed time T the coherent wave packet is restored to its original state,

$$\int_0^T d\tau\, e^{i\omega\tau}f(\tau) = 0. \tag{13}$$

Using this condition we derive a simple formula for the total phase,

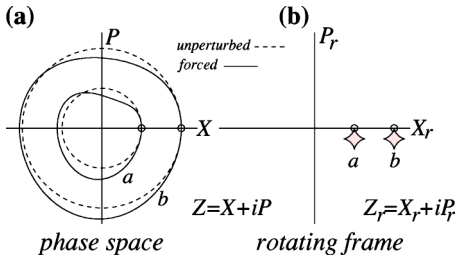

FIG. 1. Trajectories on phase space of a coherent wave packet subject to a single forced harmonic oscillator, $H=\omega a^\dagger a + F\sin(2t)(a+a^\dagger)$. In (a) we plot the usual phase-space trajectories, $\langle a\rangle=\langle X+iP\rangle/\sqrt{2}$, without forcing (dashed) and with $F=0.1$ (solid), for two initial conditions (empty circles). In (b) we plot the same, but on a rotating frame of reference, $\langle a\rangle=\langle e^{-i\omega t}(X_r+iP_r)\rangle/\sqrt{2}$.

$$\phi(T) = \mathrm{Im}\int_0^T d\tau\frac{i}{\sqrt{2}\hbar}e^{i\omega\tau}f(\tau)\overline{z_r}(\tau)$$
$$= \frac{1}{2\hbar^2}\mathrm{Im}\int_0^T d\tau_1\int_0^{\tau_1} d\tau_2 e^{i\omega(\tau_1-\tau_2)}f(\tau_1)f(\tau_2). \tag{14}$$

As a simple application, in Fig. 1 we show the phase-space trajectories obtained by forcing two coherent states with a sinusoidal force, $F(t)\propto\sin(2\omega t)$, where ω is the frequency of the Harmonic oscillator. Even though looking at Fig. 1(a) the orbits of different initial conditions also seem very different, on the rotating frame of reference the enclosed area A is always the same [Fig. 1(b)]. In other words, the phase ϕ is insensitive to the initial motional state of the system and it is thus robust. This property is of crucial importance when we seek applications to real systems that cannot be cooled to the zero-phonon limit, but which thanks to Eq. (13) will pick up the same phase regardless of the temperature.

C. Phase of two ions

We will now apply the results from Sec. II B to a pair of ions. In this case there are two normal modes: the center of mass, $x_c=(x_1+x_2)/2$, which oscillates with frequency ω_c, and the stretch mode, $x_s=x_2-x_1$, which oscillates with frequency ω_s. If the ions are stored in the same harmonic trap, $V_{e,k}(x_k)=\frac{1}{2}m\omega^2x_k^2$, these frequencies are found to be incommensurate, $\omega_c=\omega$ and $\omega_s=\omega\sqrt{3}$. If we store the ions in two microtraps (or in a more complicated potential), the value of ω_s can be tuned from $\omega\sqrt{3}$ down to ω.

If we exert a similar state-dependent force on both ions, for instance by means of an off-resonance laser that induces an AC Stark shift on one of the internal state of the ions, the Hamiltonian (3) will look as follows:

$$H = \hbar\omega a_c^\dagger a + \hbar\omega_s a_s^\dagger a_s - F(t)\sigma_1^z x_1 - F(t)\sigma_2^z x_2$$
$$= \hbar\omega a_c^\dagger a + \hbar\omega_s a_s^\dagger a_s + F(t)(\sigma_2^z-\sigma_1^z)d - F(t)(\sigma_1^z+\sigma_2^z)\frac{\alpha_c}{\sqrt{2}}$$
$$\times(a_c+a_c^\dagger) - F(t)(\sigma_2^z-\sigma_1^z)\frac{\alpha_s}{2\sqrt{2}}(a_s+a_s^\dagger), \tag{15}$$

where d is the equilibrium distance between the ions, $\alpha_{c,r}^2 = \hbar/m\omega_{c,r}$ are the characteristic length scales of the oscillators, and σ_i^z is an operator that takes value +1 or −1 depending on whether the ion is on internal state $|+1\rangle$ or $|-1\rangle$.

Since the modes are now decoupled, we can apply the formulas of Sec. II B almost directly. We first obtain a pair of commensurability relations on the force,

$$\int_0^T d\tau\, e^{i\omega_{c,r}\tau} F(\tau) = 0, \tag{16}$$

which are just a generalization of Eq. (13). Next we obtain the total phase, which up to local and global contributions is

$$\phi = \sigma_1^z \sigma_2^z \int_0^T d\tau_1 \int_0^T d\tau_2 \mathcal{G}(\tau_1 - \tau_2) F(\tau_1) F(\tau_2), \tag{17}$$

where $\mathcal{G}(t) = [(1/\omega_c)\sin(\omega_c|t|) - (1/\omega_s)\sin(\omega_s|t|)]/2m\hbar$.

The goal in Sec. III will be to tune the forces so that Eq. (16) is satisfied and the phase becomes $\phi = \pi\sigma_1^z\sigma_2^z/4$, the required value for a two-qubit quantum phase (1). We will then show two optimal solutions to this problem, which use either pulsed or continous forces.

D. Phase for any number of ions

The case of $N > 2$ ions exhibits a richer structure, due to the fact that the phase depends on all pair products $\sigma_i^z\sigma_j^z$ of the polarizations of the atoms. If we apply the formula for the phase of one harmonic oscillator (14) to each of the modes in the effective Hamiltonian for the chain (7), we obtain a total phase

$$\phi = \sum_{ij} \sigma_i^z \sigma_j^z \int_0^T \int_0^T d\tau_1 d\tau_2 F_i(\tau_1) \mathcal{G}_{ij}(\tau_1 - \tau_2) F_j(\tau_2), \tag{18}$$

with a Hermitian kernel

$$G_{ij}(t) = \sum_k \frac{M_{ik}M_{jk}}{2m\omega_k\hbar} \sin(\omega_k|t|), \tag{19}$$

plus a generalization of Eq. (16)

$$\sum_i \int_0^T d\tau e^{i\omega_k\tau} \alpha_k M_{ik} F_i(\tau) = 0, \quad \forall\, k. \tag{20}$$

In Sec. IV we will show how Eq. (18) can be related to an effective Ising interaction, $H = \Sigma_{ij} J_{ij} \sigma_i^z \sigma_j^z$, whose precise shape can be engineered and which can produce interesting entangled states. Remarkably, the interaction is always pairwise due to the harmonic model with which we are working.

III. FAST PHASE GATE FOR TWO IONS

A very important application of the techniques studied so far is to design a two-qubit quantum gate that is robust enough to be included in a scalable ion-trap quantum computer. This task has been pursued in a previous work [28] using a slightly more complicated method, in which the force was achieved by kicking the ion with $\pi/2$ laser pulses, and the distribution and number of these pulses had to be designed "by hand." In this section we review this work in the light of our formalism and rephrase it as an optimal control problem. This allows us to consider more general forces, and to find, for instance, a design of a gate that involves the shortest time, and the weakest and smoothest varying forces.

A. Kicking forces

We will consider two ions in a one-dimensional harmonic trap, interacting with a laser beam on resonance. The Hamiltonian modeling this system can be written as $H = H_0 + H_1$ where $H_0 = \hbar\omega_c a_c^\dagger a_c + \hbar\omega_s a_s^\dagger a_s$ describes the normal modes of the ions and

$$H_l = \frac{\Omega(t)}{2}[\sigma_1^+ e^{i\hbar k x_1} + \sigma_2^+ e^{i\hbar k x_2} + \text{H.c.}] \tag{21}$$

$$= \frac{\Omega(t)}{2}[\sigma_1^x e^{-i\hbar k x_1 \sigma_1^z} + \sigma_2^x e^{-i\hbar k x_2 \sigma_2^z}]. \tag{22}$$

This term describes processes in which the internal state of an ion is changed and, as a consequence of the absorption and emission of photons, the atom gains momentum, $\hbar k$. The Rabi frequency $\Omega(t)$ is a function of the intensity of the lasers that induce these internal transitions, and looking for the simplest setup we assume that it is the same for both ions.

In Ref. [28] we explained how to use the Hamiltonian H_l to kick the ions. The process consists of applying very fast laser pulses, in which the Rabi frequency $\Omega(t)$ and the duration of the pulse δt satisfy $\int_0^{\delta t}\Omega(\tau)d\tau = \pi$ and $\delta t \ll 2\pi/\omega$. Let us study the evolution of the ions under a single laser pulse. Since the pulse is much shorter than a period of the trap, we can neglect the influence of H_0. We then use the formula

$$\mathcal{T}\exp\left[i\int_0^{\delta t} d\tau \frac{\Omega(\tau)}{2} \vec{n}\vec{\sigma} \right] = \cos(\theta) + i\sin(\theta)\vec{n}\vec{\sigma}, \tag{23}$$

where $\mathcal{T}$ is the time-ordering operator; $\|\vec{n}\| = 1$ is a unitary vector and we impose that a $\pi/2$ pulse is produced: $\theta = \int_0^{\delta t}\Omega(\tau)/2 = \pi/2$. Under these conditions the unitary evolution is described by

$$U_{\text{kick}} = \sigma_1^x \sigma_2^x e^{i\hbar k(x_1\sigma_1^z + x_2\sigma_2^z)}. \tag{24}$$

If at times $\{t_1, t_2, \ldots, t_{N_p}\}$ we send groups of $\{2n_1, 2n_2, \ldots, 2n_{N_p}\}$ very short laser pulses with alternating momenta, $+k, -k, +k$, …, the total operation can be written as $U = \Pi_l \exp[n_l \times i\hbar k(x_1\sigma_1^z + x_2\sigma_2^z)]$, and can be thought of as induced by the effective force

$$F_i(t) = \sum_{l=1}^{N_p} 2n_l \times \hbar k \sigma_i^z \delta(t - t_l). \tag{25}$$

When the number of pulses is odd, similar considerations can be done, but now a total spin flip $\sigma_1^x\sigma_2^x$ has to be included by hand at the end of the process.

B. Phase gate based on kicks

The parametrization of Eq. (25) is a means to simplify the problem of finding the optimal forces that produce the phase

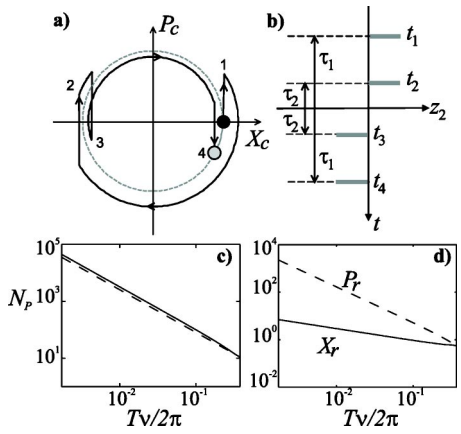

FIG. 2. (a) Trajectory in phase space of the center-of-mass state of the ions $(\tilde{Q}_c,\tilde{P}_c)$ during the two-qubit gate (solid line), connecting a possible initial state (black filled circle) to its final state (grey filled circle), for the protocol designed in Sec. III B. Part (b) shows how the laser pulses (bars) distribute in time for this scheme. Below we plot (c) the number of pairs of pulses, and (d) maximum displacement in phase space required to produce a phase gate using this scheme.

gate (1). Using the previous notation, the conditions for restoring the motional state of the ions become

$$\sum_{l=1}^{N_p} n_l e^{-i\omega_{c,r}t_k} = 0. \tag{26}$$

If these equations are satisfied, the accumulated phase will be

$$\phi = \sum_{l,m=1}^{N_p} \left[\frac{\sin(\omega_c t_{lm})}{\omega_c} - \frac{\sin(\omega_s t_{lm})}{\omega_s} \right] \frac{2\hbar k^2 n_l n_m}{m} = \pi/4, \tag{27}$$

where $t_{lm}=|t_l-t_m|$ is the time between the lth and the mth kicks.

In Ref. [28] we have found two possible solutions for these equations. The first protocol that we proposed, performs the phase gate in a time $T=1.08/\nu$, using about four pulses, while the second protocol allows for an arbitrarily short gating time $T\sim N_p^{-2/3}/\nu$ at the expense of using more pulses or kicks.

The method for the first protocol is illustrated in Figs. 2(a) and 2(b), where we plot the phase space trajectories followed by the center-of-mass mode. This sequence consists basically on four groups of pulses given by $(n_l/n,t_l)=\{(\gamma,-\tau_1),(1,-\tau_2),(-1,\tau_2),(-\gamma,\tau_1)\}$. The number n tells us how many pulses are sent within each group. The parameter $0<\gamma=\cos(\theta)<1.0$ is a real number that describes how much the kicking lasers are tilted with respect to the axis of the trap. It is always possible to find a solution to Eq. (26) with $\tau_1\simeq 0.538(4)(2\pi/\nu)>\tau_2>0$. The results for the performance of the gate are summarized in Figs. 2(c) and 2(d): for realistic values of the Lamb-Dicke parameter [4] we only need to apply the sequence of pulses one or two times to implement a phase gate. A minor tilting angle will be, however, required to compensate for small deviations from the value of the phase $\pi/4$ for different Lamb-Dicke parameters.

The second protocol performs the gate in an arbitrarily short time T. For shortening the time we now require six groups of pulses, distributed according to $(n_l/n,t_n)=\{(-2,-\tau_1),(3,-\tau_2),(-2,-\tau_3),(2,\tau_3),(-3,\tau_2),(2,\tau_1)\}$, where the times τ_1, τ_2, and τ_3 are found numerically by solving the transcendental equations (26), with the constraint that the whole process takes a time $T=2\tau_1$. As Fig. 2 shows, the number of pulses, $N=14n$, increases with decreasing time as $N_p\propto T^{-3/2}$. This is just a consequence of a more general result that is shown later.

C. Phase gate based on continuous forces

The use of $\pi/2$ pulses to introduce momentum in the ions has some inconveniences. First of all, each of the pulses has to be perfect, and induce a complete population transfer from one internal state to the other one. If this is not the case, systematic errors on each of the pulses can lead to an exponential decrease of the fidelity. Furthermore, as we increase the gating speed, the pulses may become too long to be considered as instantaneous kicks, and the previous formalism fails.

What we have found, and what is also one of the main results of this paper, is that the phase gate may be produced also by applying continuous forces. The search for this forces is then no more difficult than solving an eigenvalue equation, where one may add restrictions such as smoothness of the force, and minimal total work.

Let us take the real vector space $L^2([0,T])$ of space of square integrable real forces in the $[0,T]$ interval, with the usual scalar product $(f,g)=\int_0^t f(t)g(t)dt$. From this Hilbert space, we choose a subspace $\mathcal{H}$ of functions which are orthogonal to the modes $e^{i\omega_{c,r}t}$,

$$\int_0^T d\tau\, e^{i\omega_c\tau}F(\tau) = \int_0^T d\tau\, e^{i\omega_s\tau}F(\tau) = 0. \tag{28}$$

Within $\mathcal{H}$, the phase and the smoothness of the gate are given by $\phi[F]=(F,\mathcal{G}F)$ and $S[F]:=(F,-(d^2/dt^2)F)$, respectively. We will prove that the optimal (i.e., smoothest) force that produces a phase gate $\phi=\pi/4$, is simply proportional to F_μ, where F_μ is the eigenstate,

$$-\mu\frac{d^2}{dt^2}F_\mu = \mathcal{G}F_\mu, \tag{29}$$

with largest eigenvalue $|\mu|$. If rather than measuring the optimality with $S[F]$ we use just the norm, $N[F]:=\|F\|_2=(F,F)$, then the eigenvalue problem is simpler,

$$\mu F_\mu = \mathcal{G}F_\mu. \tag{30}$$

Let us prove this useful result. We have to work with four functionals, which are the $S[F]$ and $\phi[F]$ defined above plus two more, which measure the displacements originated by the force: $D_{r,c}[F]=\int_0^T d\tau\, e^{i\omega_{c,r}\tau}F(\tau)$. By choosing the space of real periodic functions which are orthogonal to the Fourier modes $e^{i\omega_{c,r}\tau}$ we ensure that everything is well defined and

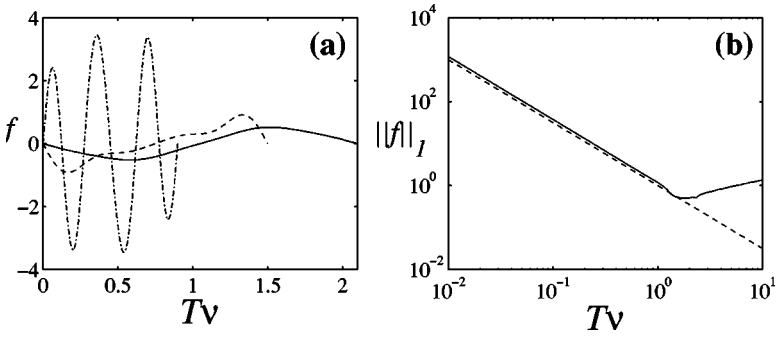

FIG. 3. (a) Optimal forcing for a gating time $T\nu=2.1$ (solid), 1.5 (dashed), and 0.9 (dash-dot), where $\omega=2\pi\times\nu$ is the frequency of the ion trap. (b) Intensity of the force vs duration of the gate (solid) and visual aid $(T\nu)^{-3/2}$ (dashed). All magnitudes have been adimensionalized following the text.

also that the constraints $D_{c,r}[F]=0$ are satisfied. This leaves us with the problem of finding a force which minimizes $S[F]$, while satisfying the last constraint $\phi[F]=\pi/4$. There exists, however, a much easier dual problem which is formulated as finding the maximum of $\phi[F]$ subject to the quadratic constraint $S[F]=1$. Using the theory of Lagrange multipliers, this amounts to finding the maximum of

$$\mathcal{L}[F]=\phi[F]-\mu(S[F]-1), \tag{31}$$

where μ is the Lagrange multiplier. Differentiating the Lagrangian we obtain Eq. (29), from where it follows that $\mu\|F\|_2^2=\phi$ is the maximal phase to be achieved, and the associated eigenstate $F_\mu\sqrt{\pi/(4\mu)}/\|F\|_2$ is the force we were looking for.

One might wonder about the need for solving the dual problem instead of the original one and whether indeed both problems are equivalent. The answer to the first question is that the resulting generalized eigenvalue equation behaves better. Thanks to the operator $-d^2/dt^2$ being semidefinite positive, its presence on the left-hand side of the eigenvalue equation (30) can be handled by redefining the scalar product. To answer the second question let us assume that F is a solution of the original problem. We define $F_d:=F/\sqrt{S[F]}$, a solution which has $S[F_d]=1$ and $\phi[F_d]=\pi/4S[F]=:\phi_d$. Not let us assume that there exists a function g with $S[g]=1$ and a larger phase, $\phi[g]=:\phi_g>\phi_d$. We can define $g_o:=g\sqrt{\pi/4\phi_g}$ which now has $\phi[g_o]=\pi/4$ and $S[g_o]=\pi/4\phi_g<\pi/4\phi_d=S[F]$, which contradicts the fact that F was the optimal solution of the original problem. A similar reasoning can be done the other way around, thus proving the equivalence of both problems.

Even though we have been able to relate the control problem to an eigenvalue equation (29), there exists no simple analytical solution to this problem and we have to resort to some simple numerics. However, a very nice feature of the two-ion problem is that, by scaling quantities with respect to the trap strength ω and the wave-packet size $\alpha=(\hbar/m\omega)^{-1/2}$, we can compute the optimal force independent of the setup. Using these units and expanding the force in terms of Fourier modes,

$$f(t)=\sum_{m=-N_m}^{N_m} c_n e^{i2m\pi t/T}=\frac{F(t)\alpha}{\hbar\omega}, \tag{32}$$

we can express Eq. (29) as an eigenvalue equation for the vector $\vec{c}$, that is to be solved numerically. The number of modes N_m can in principle be any number above 3, because some degrees of freedom are lost when satisfying the constraint (16). However, the numerical experiments show that indeed $N_m=4$ provides very good solutions. As an example, in Fig. 3(a) we show three possible forces for a duration of the gate $T\omega/2\pi=2.1$, 1.5, and 0.9. We have computed other solutions for a wider range of gate speeds. In Fig. 3(b) we plot the mean intensity $\|f\|=\int_0^T|f(t)|dt$ vs the total time T, and demonstrate the law $T^{-3/2}$ obtained above.

An interesting question is how much energy we have to put into the system in order to produce faster and faster gates. With our current formalism, this can be answered very quickly. Let us assume that we have the optimal force that produces a phase gate in time $T\ll 2\pi/\omega_{c,r}$. Since the time is very short, we can perform a Taylor expansion of the function $\mathcal{G}(\tau)$ obtaining

$$\begin{aligned}|\phi| &\simeq \left|\int_0^T\int_0^T d\tau_1 d\tau_2 F(\tau_1)F(\tau_2)\frac{(\omega_c^2-\omega_s^2)}{12m\hbar}|\tau_1-\tau_2|^3\right| \\ &\leqslant \frac{\omega_c^2-\omega_s^2}{12m\hbar}T^3\|F\|_1^2,\end{aligned} \tag{33}$$

where $\|F\|_1=\int_0^T d\tau|F(\tau)|$ is just a measure of the force applied. From here we see that

$$\|F\|_1\simeq T^{-3/2} \tag{34}$$

or, in the case of the kicked ions, $N_p\simeq T^{-3/2}$, the scaling that the numerical simulations already showed.

D. Adiabatic pushing

As a final remark, we want to relate the methods presented here with the *pushing gate* introduced in Ref. [2]. That work proposed to trap the ions in separate microtraps, $V_{e,i}(x_i)=\frac{1}{2}m\omega^2(x_i-x_i^{(0)})^2$, and to apply a state-dependent force on two neighboring ions. This force should be switched on and off adiabatically with respect to the period of the traps, $2\pi/\omega$, in order to approach the ions to each other and later on return them to their equilibrium positions. The adiabaticity condition ensures that the ions remain at all times in the ground state of the effective Hamiltonian (4), which is now time dependent, because the equilibrium positions $x_k^{(0)}$ and the equilibrium energy E_0 both depend on the instantaneous value of the forces. After restoring the ions to their original positions, the only effect on the quantum state of the ions is a state-dependent phase,

$$\phi = \int_0^T E_0[x_1^{(0)}(t),\ldots,x_N^{(0)}(t)]dt. \tag{35}$$

The previous analysis is found in Refs. [2,35,36]. A very important point is that, in order not to excite the ions and regard the process as truly adiabatic, the forces have to be weak and change very slowly, and no cubic contributions to the energy should appear. In other words, we should be able to describe the change of E_0 using at most quadratic terms, $E_0[\vec{x}^{(0)}(t)] \simeq E_0[\vec{x}^{(0)}(0)] + \frac{1}{2}mV_{ij}\delta x_i^{(0)}\delta x_j^{(0)}$. Hence rather than using the adiabatic theorem we can integrate the problem exactly. For a single harmonic oscillator we get

$$z(t) = e^{-i\omega t}z_0 + \frac{1}{\sqrt{2}}f(t) - \frac{1}{\sqrt{2}}\int_0^t d\tau\, e^{i\omega\tau}\frac{1}{\omega}f'(\tau), \tag{36}$$

where the adiabatic condition corresponds to neglecting the last term $f'(\tau)/\omega$, and the force only has to satisfy $f(T) = f(0) = 0$. Repeating the arguments of previous sections, for two ions in neighboring traps the total phase becomes

$$\phi = \int_0^T d\tau \frac{\omega_c^{-1} - \omega_s^{-1}}{2m\hbar}F(\tau)^2\sigma_1^z\sigma_2^z. \tag{37}$$

Here $\omega_c = \omega$ and ω_s now depend slightly on the separation of the microtraps, but the same formula applies for the case in which both ions coexist in the same trap—a situation that could not be considered with the formalism of previous papers.

IV. QUANTUM CONTROL OF SEVERAL IONS

We will now study one-dimensional (1D) setups with more than two ions. As we showed before, we can still control the geometric phases and use them to simulate a variety of spin Hamiltonians (Sec. IV A). The design of the forces for these simulations is still a control problem, but a much more difficult one. For instance, a crucial difference is that in setups with more than two ions either addressability or a spatial modulation of the forces are required. As a possible application of this result we study how to optimally generate entangled states and squeezing. In particular, we show that this can be done for a large number of ions (up to 30) in a very short time (Sec. IV B).

A. Simulation of spin Hamiltonians

Given any Ising Hamiltonian

$$H_J = \sum_{ij} J_{ij}\sigma_i^z\sigma_j^z, \quad J = J^t \in \mathbb{R}^{N\times N},$$

and a time T, it is possible to design a set of state-dependent forces $F_i(t;J)$ such that after applying these forces for a time T, the dynamics of the ions simulates this spin Hamiltonian. In other words,

$$\mathcal{T}[e^{i\int_0^t H(\tau)d\tau}] = e^{-iH_JT},$$

where $\mathcal{T}$ denotes the time-ordered product and H is the true Hamiltonian of the ions (4).

The proof is very simple. Let us slice the time interval $[0,T]$ into N^2 subintervals, $0 < t_{11} = t_{12} < \cdots < t_{N,N-1} < T$. In a given time interval, $I = [t_{ij}, t_{i,j+1}]$, we will switch on two forces, and leave all other ions on their own,

$$F_k(t) = 0, \quad t \in [t_{ij}, t_{i,j+1}], \quad \forall\, k \neq i,j.$$

The active forces $F_i(t)$ and $F_j(t)$ must satisfy several equations,

$$0 = \int_I e^{i\omega_k\tau}M_{k\alpha}F_\alpha(\tau)d\tau, \quad \alpha = i,j, \quad k = 1\cdots N,$$

$$J_{ij} = \frac{1}{t_{i,j+1} - t_{ij}}\int_I\int_I d\tau_1 d\tau_2 F_i(\tau_1)\mathcal{G}_{ij}(\tau_1 - \tau_2)F_j(\tau_2). \tag{38}$$

It is not difficult to convince oneself that these equations always have a solution, and that by repeating this procedure we will get an effective total phase ϕ that resembles the one produced by the Ising model during a time T. □

We have to make several remarks here. The first one is that since the operator that we want to simulate is symmetric, $J_{ij} = J_{ji}$, and since the diagonal terms only contribute to a global phase, the number of intervals can be actually decreased to $N(N-1)/2$.

However, more important is the fact that we can use coherent control to find optimal forces $\vec{F}$, which instead of piecewise continuous are the smoothest possible and have the optimal norm, while giving rise to the same effective Hamiltonian. This task has been performed numerically for some models, and the results will be shown in the following section.

From the point of view of quantum simulation, we would like to be able to model more than just an Ising model, which is essentially classical. For instance, one would like to be able to introduce transverse magnetic fields $\Sigma_i h_i\sigma_i^x$ or to simulate a Heisenberg interaction $\vec{\sigma}_i\vec{\sigma}_j$ and in general, a unitary operation of the form (2) would be desirable. The answer to this problem is once more the stroboscopic evolution, or a Trotter expansion of the operator (2),

$$U \simeq \left\{\prod_{\alpha=x,y,z} e^{i(T/N)t(\Sigma_{ij}J_{ij}^\alpha\sigma_i^\alpha\sigma_j^\alpha + \Sigma_i h_i^\alpha\sigma_i^\alpha)}\right\}^N. \tag{39}$$

In this expansion, we decompose the total unitary as a product of phase gates, that are originated by forces that depend on σ_i^x, σ_i^y, and σ_i^z. In practice, one would switch on a magnetic field h_i^z and perform a phase gate with coefficients J_{ij}^z for a time T/N, rotate the spins so that σ_y becomes σ_z, apply the phases with J_{ij}^y, etc.

This technique is equivalent to the sequences of pulses developed in NMR quantum computing [20] to induce a given unitary by combining the evolution under a fixed interaction Hamiltonian (i.e., the one that describes the interaction of the atom) with external magnetic fields that rotate the interacting spins.

It is also worth noticing that if we switch on the state-dependent forces acting on different ions, and make them oscillate with a single frequency Ω around a constant value, $F_i(t)=f_i \sin(\Omega t)\sigma_i^z$, for a long time, the effective interaction is a particular spin Hamiltonian,

$$H=\sum_{ijk} f_i \frac{M_{ik}M_{jk}[1+\delta_{\Omega,0}]}{4m\hbar(\omega_k^2-\Omega^2)} f_j \sigma_i^z \sigma_j^z + \mathcal{O}\left(\frac{f_i^2}{\omega_k^2}\right). \quad (40)$$

In the limit $\Omega \to 0$, this model corresponds to the one found in Ref. [18]. As it was shown there, depending on whether the forces operate longitudinally or transversely to the ion trap, this continuous force will give rise to long-range or short-range interactions.

B. Coherent control and design of entanglers

The simulation of an Ising interaction is by itself interesting, and has important applications such as creation of many-qubit quantum gates, quantum simulation, and adiabatic quantum computing. However, a most straightforward and useful application of an Ising Hamiltonian is the generation of many-body highly entangled states called *graph states* [37]. Roughly speaking, let us imagine that we have a set of N spins, which we represent by points or vertices, and let us connect these points by lines or edges. The resulting graph can be described by an adjacency matrix which takes value $J_{ij}=1$ only if the spins i and j are connected [38]. To each graph we can thus associate a Hamiltonian of the form $H_J=\Sigma_{i,j}J_{ij}\sigma_i^z\sigma_j^z$. It has been shown that after applying this interaction over a certain time on a transversely polarized state, the outcome is a highly entangled state called a graph state:

$$|\psi_G\rangle = \frac{1}{2^{N/2}} e^{-i(\pi/8)H_{\mathcal{G}}}(|0\rangle+|1\rangle)^{\otimes N}. \quad (41)$$

When the graph has a lattice geometry, these states are also known as cluster states [39], and form the basic ingredient of the one-way quantum computer [40]. However, a particularly important case without lattice geometry is the GHZ state,

$$|\mathrm{GHZ}\rangle \sim |0\rangle^{\otimes N}+|1\rangle^{\otimes N}, \quad (42)$$

which is essentially generated by the interaction $J_{ij}=1$ or $H_J=(\Sigma_i\sigma_i^z)^2$. The GHZ state is one of the best studied entangled states; it constitutes a canonical example of Schrödinger cat state, and it could have important applications in the field of precision frequency measurements, providing a $1/\sqrt{N}$ precision increase for N entangled ions [9,10], a point already demonstrated experimentally in Ref. [17].

We have investigated how to implement these highly entangled states using our quantum control techniques. The idea is very simple: we design a matrix J_{ij} for our graph state, and look for the time- and state-dependent forces that implement the phase transformation $\exp(-i\pi H_J\pi/4)$ within a fixed time T. For simplicity, even though it is not warranted to succeed, we look for forces that have a common time dependence $F_i(t)=\gamma_i f(t)\sigma_i^z$, $|\gamma_i|\leqslant 1$. These forces could be implemented with an appropriate intensity mask, which determines the relative amplitudes γ_i, and a global intensity modulation, which gives the function $f(t)$. Expanding this modulation in Fourier modes, we find $N(N-1)/2+N$ equations which define a possible entangling procedure. We have solved numerically these equations, both for the GHZ state and for the cluster state. While in the first case we always found exact solutions with a small number of modes (i.e., 50 modes for 30 ions), the generation of the cluster state was always approximate with high fidelity, $F\simeq 99.9\%$. The error in this case has its origin in our particular choice of forces.

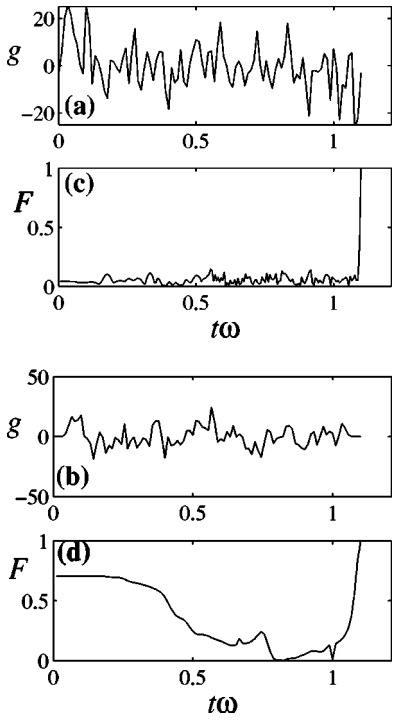

FIG. 4. Generation of a (a),(c) cluster state of $N=10$ ions and of (b),(d) a GHZ state of $N=20$ ions, using common forcing for a time $T=1.1/\omega$. (a),(b) Time dependence of the forces, $g(t)$. (c),(d) Fidelity F, with respect to the target state as a function of time.

In Fig. 4 we show the entangling procedure for a setup with 10 and 20 ions, even though chains of up to 30 ions have been considered. We measure the fidelity of the process as the overlap between the target state and the one achieved. If δJ is the difference between the desired interaction and the achieved one, then $F=(1/2^N)\Sigma_{\vec{s}}e^{-i\vec{s}^t\delta J T\vec{s}}$, where the sum is performed over all possible spin configurations, $s_k=\pm 1$. The time scales for the generation of the interaction are independent of the size of the system, and for instance we can produce a GHZ state of 20 ions in a time $T=1.1/\omega$, to be compared with the time $T=3000\pi/\omega$ required when individually addressing one of the vibrational modes [12]. The strength of the forces, though, grows moderately with the number of ions, which can be inconvenient. However, thanks to the periodicity of the forcing $f(t)$, if we divide the intensity of the forces by a factor of 2, $f(t)\to f(t)/2$, the same gate is produced in a longer time $4T$. Furthermore, the forces that we present in this paper are not optimal, and have been found with a straightforward Gauss-Newton method. If high fidelity is not required, one may find better solutions with fewer Fourier modes [N_m in Eq. (32)], but most important we expect significant improvements by the application of better

numerical algorithms to search for the optimal entanglers.

Using the Ising interaction we cannot only produce graph states, but also squeezed states: states in which the variance of one spin component ΔS_x has been decreased at the expense of increasing the other variances. As it was shown in Ref. [41], a Hamiltonian of the form $H=J_z^2$ (single axis squeezing) or $H=J_x^2-J_y^2$ (two-axis squeezing) can be used to produce squeezing. Both models can be simulated using our tools, either directly, as in the single-axis squeezing, or stroboscopically, for the *XY* interaction. Indeed, the stroboscopic simulation of the two-axis squeezing resembles the scheme of $\pi/2$ pulses used in Ref. [42] to effectively switch off the interaction in two-mode Bose-Einstein condensates [43].

V. OPTIMAL CONTROL OF ERRORS

Up to now, we have assumed that the motion of the ions is not disturbed during the time when the controlling forces are applied. In this section we will show how to take these effects into account for a realistic model of dissipation. The main result is that the fidelity of the process can still be computed and that there are two sources of error: one due to an imperfect control of the ions, which introduces some temperature dependence on the fidelity (Sec. V C), and another one due to the dissipation, that can be treated as another constraint for the control problem (Sec. V B). Finally, we will comment on possible extensions outside the harmonic regime (Sec. V D).

A. The model and an exact solution

We study the dissipation with a master equation that arises from coupling the phonon modes with a "classical" Bosonic bath in thermal equilibrium,

$$\frac{d}{dt}\rho = \frac{i}{\hbar}[H_I,\rho] + \sum_k \gamma N_k(2a_k^\dagger \rho a_k - a_k a_k^\dagger \rho - \rho a_k a_k^\dagger) + \sum_k \gamma_k(N_k + 1)(2a_k\rho a_k^\dagger - a_k^\dagger a_k\rho - \rho a_k^\dagger a_k). \quad (43)$$

Here γ_k describes the coupling to an external bath and N_k is the mean number of bosons in that bath and it is related to its temperature. The Hamiltonian in Eq. (43) is written in the interaction picture

$$H_I = \sum_{ki} \hbar[g_{ki}(t)a_k + g_{ki}^\star a_k^\dagger]\sigma_i^z. \quad (44)$$

In order not to obscure the discussion, we will assume that each phonon mode interacts with an independent bath. In that case, the forces in the rotating frame of reference become $\hbar g_{ki}(t)=(1/\sqrt{2})F_i(t)M_{ik}\alpha_k e^{-i\omega_k t}$. However, it is easy to generalize the following analysis to a more realistic model in which each ion couples independently to the environment, and the operators a_k and $a_k^\dagger$ do not represent the phonons, but the displacements of the ions.

To study the fidelity of a gate we only need to know the matrix elements of the reduced density matrix for the internal degrees of freedom. This matrix may be written as a collection of expectation values,

$$\rho_{\text{real}} = \sum_{\mathbf{s},\mathbf{r}} \langle \Sigma_{\mathbf{s},\mathbf{r}}\rangle \Sigma_{\mathbf{s},\mathbf{r}}, \quad (45)$$

where $\Sigma_{\mathbf{s},\mathbf{r}}=|s_1\cdots s_N\rangle\langle r_1\cdots r_N|$, r_k, $s_k=\pm 1$ form a complete basis for the space of $2^N\times 2^N$ complex matrices.

The calculations that provide us with the evolution of $\langle\Sigma_{\mathbf{s},\mathbf{r}}\rangle$ are detailed in the Appendix . Here we will only summarize the main result, which is that the reduced density matrix can be written as

$$\langle\Sigma_{\mathbf{s},\mathbf{r}}\rangle(t) = e^{-\kappa_{\mathbf{s},\mathbf{r}}}e^{-i\Sigma_{jk}J_{jk}s_j s_k}\langle\Sigma_{\mathbf{s},\mathbf{r}}\rangle(0)e^{i\Sigma_{jk}J_{jk}r_j r_k}. \quad (46)$$

In other words, the spin density matrix has the form

$$\rho_{\text{real}}(t) = e^{-\mathcal{L}}[U\rho_{\text{real}}(0)U^\dagger], \quad (47)$$

where $U=\exp(-i\Sigma J_{jk}\sigma_j^z\sigma_k^z)$ is the operation that we want to perform, and $\mathcal{L}(\rho)=\Sigma_{\mathbf{s},\mathbf{r}}\kappa_{\mathbf{s},\mathbf{r}}\Sigma_{\mathbf{s},\mathbf{r}}\rho\Sigma_{\mathbf{s},\mathbf{r}}$ is responsible for the decay of coherences.

In comparison with the previous part of the paper, the unperturbed orbits in phase space, that is, the evolution without external forcing, are now not circular orbits, but spirally decaying ones. This fact translates into new conditions for uncoupling internal and motional degrees of freedom (20),

$$\int_0^t d\tau e^{-(i\omega_k+\gamma_k)\tau}F_j(\tau) = 0, \quad \forall\ j,k, \quad (48)$$

which now depend on the exponential decay rate γ_k given by our dissipation model. This model dependence is also evident in the kernel that produces our unitary operation J_{ij}, which now reads

$$\mathcal{G}_{jl}(\tau) = \sum_k \frac{M_{jk}M_{lk}\alpha_k^2}{2m\omega_k\hbar}\sin(\omega_k|\tau|)e^{\gamma_k|\tau|}. \quad (49)$$

Finally, we would like to remark that the conditions for restoring the motional state, as well as the expressions for the phase and the kernel, are only slightly modified when we consider a local coupling to the environment.

B. Quantum control of errors due to dissipation

To understand better the implications of Eq. (47), let us study the dynamics of a single ion. In this case the reduced density matrix can be expressed uniquely in terms of the expectation values $\langle\sigma^z\rangle$ and $\langle\sigma^+\rangle$. Furthermore, since the magnetization is constant, we can compute the Uhlmann fidelity [46] exactly as a function of $\langle\sigma^+\rangle$. Let us assume that initially the system is in a pure state, and define $\langle\sigma^+\rangle_{\text{real}}=e^{i\delta\phi-\kappa}\langle\sigma^+\rangle_{\text{id}}$, where the subindex "id" denotes the ideal value obtained in absence of errors. The Uhlmann fidelity of the gate is

$$\mathcal{F}(\rho_{\text{real}},\rho_{\text{id}}) = \sqrt{1+|\langle\sigma^+\rangle_{\text{id}}|^2[2e^{-\kappa}\cos(\delta\phi)-1]}. \quad (50)$$

Two types of errors contribute to the decay of the fidelity. The first type is made of control errors. These errors contribute both to the spurious phases ($\delta\phi$) and to an effective decay due to not restoring the motional state of the ions [$\kappa\neq 0$ because $\beta(T)\neq 0$ and $U(T)\neq 1$]. These errors cause a smooth dependence on the temperature to appear, as shown later in Sec. V C.

The second type of errors is due to dissipation during the forcing of the ions. Their contribution to the exponential decay is

$$\kappa_{\text{dissip}} = \gamma\left(N+\frac{1}{2}\right)\int_0^t d\tau|\beta(\tau)|^2. \tag{51}$$

One would be led to think that if dissipation acts on a much larger scale than the time required to perform our gate we can neglect it completely. However, a simple argument shows that this is not the case. As we saw before in Sec. III C, the strength of the forces scales roughly as $F\simeq T^{-5}$. This scaling allows us to give a worst case estimate of $\beta(t)\simeq Ft$ and conclude that

$$\kappa_{\text{dissip}} \sim \gamma T^{-2}, \quad T \ll 1/\omega. \tag{52}$$

What this means is that slower gates will involve smaller displacements of the ions, which in turn translates into less dissipation. On the other hand, a too long application of a force also gives more time for the dissipation to operate and the optimal duration should be a compromise between both processes. It is thus possible and recommended to *optimize the forces* $F(t)$ taking not only into account the properties of the force (i.e., differentiablity and intensity), but also *trying to minimize the decay* κ *induced by the force*. From the numerical point of view, this new control problem is only slightly more complicated than the ones we have solved in Secs. III and IV, because κ_{dissip} is a nonlinear function of the forces.

C. Errors due to an imperfect control: Influence of temperature

Let us denote by $U_{\text{id}}=\exp(\Sigma_{ij}J_{ij}\sigma_i^z\sigma_j^z)$ the ideal operation that we want to perform, and by U_{real} the operation with errors. In this section, the only source of error that we consider is an imperfect control, denoted by a perturbation of the state-dependent force induced on the ions, $[F_i(t)+\delta F_i(t)]\sigma_i^z$. According to the previous analysis, the effect of this perturbation will be a residual state dependent displacement of the coherent wave packets at the end of the process, $\beta_k(T)$,

$$\beta_k = -i\int_0^T d\tau\, e^{i\omega_k\tau}\frac{\alpha_k M_{ik}\delta F_i(t)}{\hbar\sqrt{2}}\sigma_i^z =: \sum_{k,i}\beta_{ki}\sigma_i^z. \tag{53}$$

plus a perturbation of the phase

$$\delta\phi = \sum_{ij}\delta J_{ij}\sigma_i^z\sigma_j^z, \tag{54}$$

which can be interpreted as a change in the effective interaction between ions (Sec. IV),

$$\delta J_{ij} = \int_0^T\int_0^T d\tau_1 d\tau_2 \mathcal{G}_{ij}(\tau_2-\tau_1) \times[F_i(\tau_1)\delta F_j(\tau_2)+\delta F_i(\tau_1)F_j(\tau_2)].$$

We will assume as an initial condition a pure state of the internal degrees of freedom and a thermal state of the vibrational ones $\rho(0)=|\psi\rangle\langle\psi|\otimes\rho_{\text{vib}}(T)$. The Uhlmann fidelity at the end of the process is given by

$$\begin{aligned}\mathcal{F}(\rho_{\text{id}},\rho_{\text{real}}(t)) &= \text{Tr}_{\text{vib}}(\langle\psi|U_{\text{id}}^\dagger\rho(t)U_{\text{id}}|\psi\rangle)\\ &= \langle\psi|\text{Tr}_{\text{vib}}[U_{\text{id}}^\dagger\rho(t)U_{\text{id}}]|\psi\rangle.\end{aligned} \tag{55}$$

Expanding $|\psi\rangle=\Sigma c_{\mathbf{s}}|s_1\cdots s_N\rangle$, we obtain

$$\mathcal{F} = \sum_{\mathbf{s},\mathbf{r}} c_{\mathbf{s}}^\star c_{\mathbf{r}}\langle\Sigma_{\mathbf{s},\mathbf{r}}\rangle_{\rho(t)}. \tag{56}$$

When we neglect dissipation, the previous expectation values can be computed in terms of the final displacements $\beta_k(t)$ and the residual phases as follows:

$$\begin{aligned}\langle\Sigma_{\mathbf{s},\mathbf{r}}\rangle_{U_{\text{id}}^\dagger\rho(t)U_{\text{id}}} &= \text{Tr}\left\{\prod_k D[\beta_k(t)(s_k-r_k)]\rho_{\text{vib}}(T)\right\}\\ &\times e^{i(\mathbf{s}^t\delta J\mathbf{s}-\mathbf{r}^t\delta J\mathbf{r})}c_{\mathbf{s}}c_{\mathbf{r}}^\star.\end{aligned} \tag{57}$$

Here $D(z)=\exp(za^\dagger-z^\star a)$ is the displacement operator, $\rho_{\text{vib}}(T)$ is a thermal state,

$$\rho_{\text{vib}}(T) = \bigotimes_{k=1}^N\frac{\hbar\omega_k}{\pi k_B T}\iint dzd\bar{z}e^{-|z|^2\hbar\omega_k/k_BT}|z\rangle_k\langle z|, \tag{58}$$

and thus $\langle D(z)\rangle=\exp[-|z|^2(1/2+k_BT/\hbar\omega_k)]\equiv\exp(C_k)$ so that the total fidelity becomes

$$\mathcal{F} = \sum_{\mathbf{s},\mathbf{r}}|c_{\mathbf{s}}|^2\|c_{\mathbf{r}}|^2e^{i(\mathbf{s}^t\delta J\mathbf{s}-\mathbf{r}^t\delta J\mathbf{r})}e^{C_k(s_k-r_k)^2}. \tag{59}$$

D. Errors due to larger displacements

The previous studies can be generalized to arbitrary interactions and trapping potentials. Let us assume a complicated Hamiltonian,

$$H = \sum_i\left[\frac{p_i^2}{2m}-f_i(t)\sigma_i^z x_i\right]+V(x_1,\ldots,x_N), \tag{60}$$

describing the traps and the ion-ion interaction. The evolution equation for the position of the ions are of the form

$$\dot{x}_i = \frac{p_k}{m}, \quad \dot{p}_i = -\frac{\partial V}{\partial x_i}+f_i(t)\sigma_i^z. \tag{61}$$

Since the operators σ_i^z are conserved quantities, the previous equations can be thought of as a simple problem of Newtonian mechanics, even though in practice, both x_i and p_i are operators. We can thus represent a general solution as $\{x_i(t;\sigma),p_i(t;\sigma)\}$, where σ denotes the values of σ_i^z operators. The phase of the ions is then computed by analyzing the evolution of the σ_i^+ operators. These operators must undergo a unitary transformation $\sigma^+(t)=U(t)\sigma_i^+(0)U^\dagger(t)$ in which the dependence on the $\{\sigma_i^z\}$ operators must be of the form [47]

$$U(t) = \exp\left[\sum_i\theta_i(t)\sigma_i^z+\sum_{ij}\phi_{ij}(t)\sigma_i^z\sigma_j^z\right] = e^{i\phi}. \tag{62}$$

Using the commutation relation $[\sigma_i^+,\sigma_j^z]=-2\delta_{ij}\sigma_i^z\sigma_i^+$, we obtain

$$\sigma_i^+(t) = e^{2\theta_i\sigma_i^z+2\Sigma_{j\neq i}\phi_{ij}(t)\sigma_i^z\sigma_j^z}\sigma_i^+(0). \tag{63}$$

Combining this with the Heisenberg equation for σ_i^+,

$$i\hbar\frac{d\sigma_i^+}{dt}=2f_i(t)x_i\sigma_i^z\sigma_i^+=2\frac{\partial^2\phi}{\partial t\,\partial\,\sigma_i^z}\sigma_i^+, \tag{64}$$

we find, up to global phases,

$$\phi\sim\sum_i\int_0^T d\tau f_i(\tau)x_i(\tau)\sigma_i^z. \tag{65}$$

From this analysis we see that we must impose two conditions on the process. On one hand, the orbits of the ions must be periodic so as to disentangle the internal and motional degrees of freedom,

$$x_i(T;\sigma)\simeq x_i(0), p_i(T;\sigma)\simeq p_i(0). \tag{66}$$

On the other hand, the phase ϕ must be independent of the initial conditions, $\{x_i(0),p_i(0)\}$. Satisfying both conditions is impossible in general, but if we restrict ourselves to small displacements and harmonic restoring forces, $\partial V/\partial x_i\simeq\Sigma_j V_{ij}x_j$, it is possible to integrate Eq. (61) and recover our expressions for the phases (18).

If, however, the qubit and higher terms in $V(x_1,\ldots,x_N)$ become important, we will fail the restoring condition (66), and induce some entanglement between the motion and the spin of the ions. This is the case of very short gates requiring large displacements in phase space [Fig. 2(d)]. The errors due to the anharmonic terms are of the order

$$\left|\frac{\partial^3 V}{\partial x_i\,\partial x_j\,\partial x_l}x_ix_jx_l\right|\sim\frac{3\hbar\omega\alpha}{d}\left|\frac{x}{\alpha}\right|^3, \tag{67}$$

where $\alpha\simeq\sqrt{\hbar/m\omega}$ is the length associated to the harmonic oscillator, $d^3=e^2/2\pi\epsilon_0 m\omega^2$ is the equilibrium distance between two ions, and x is a typical displacement. Since a trivial analysis of these errors is not possible, we can only produce a pessimistic, first-order bound that restricts the error induced by this perturbation on the wave function. First we will give a worst case prediction for the maximum displacement of the ions as $x_{\max}<F_{\max}T^2/2m$, where $F_{\max}$ is the maximal force applied on the ions. Next we will use the scaling $\phi\sim\omega^2T^5F_{\max}/\hbar m$ to show that roughly $(x_{\max}/\alpha)\sim\phi/4\omega T$. With this, and first-order perturbation theory, we compute the error and estimate it as

$$E_{\text{anh}}=\left(\frac{\alpha}{d}\right)^2\frac{\phi^{3/2}}{4^{3/2}\omega T}. \tag{68}$$

If we want to apply our phase operations to build a quantum computer, we need $E<10^{-4}$ and there is a limit on the speed of the gate $T>10^{-3}/\omega$, which, nevertheless, gives gating rates of the order of 100 MHz.

VI. CONCLUSIONS

We have developed a unified framework to study the coherent control of trapped ions by means of state-dependent forces and robust geometric phases. Our techniques can be used to perform fast two-qubit gates between pairs of ions. For an adiabatic switching of the forces and for the case of pulsed lasers we are able to reproduce the proposals of Refs. [2,28], and with very little work we can design the optimal forces that produce a phase gate in a given time with the lowest intensity. Using the same tools and a larger number of ions, we can simulate either continuously or stroboscopically a number of spin $s=1/2$ Hamiltonians. Furthermore, we are also able to create highly entangled states and squeezing, and as prototypical examples we have shown how to obtain a GHZ state of 20 ions in a very short time, $T=1.1/\omega$. Finally we have studied the sources of error in the application of our gate, which are an imperfect control, dissipation, and anharmonic terms in the interaction. The first type of errors could be ideally corrected and introduce a smooth decay of the fidelity with the temperature. The second type of errors induces also a decay of the fidelity, but the amount of this error can be optimized using the tools of quantum control. Both dissipation and anharmonicity set upper limits on the speed of the gate. This limit is, however, very weak, since it allows theoretically a gating speed of hundreds of MHz, and it could be overcome by a numerical study of the role of anharmonic terms in the motion of the ions.

While concluding this paper we became aware of the work by P. Staanum, M. Drewsen, and K. Mølmer [30] on performing quantum gates using continuous laser beams. The ideas shown in Ref. [30] are equivalent to the development of a two-qubit gate done in Sec. II C, with the difference that we provide an optimal solution for the problem.

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APPENDIX: SOLUTION OF THE MASTER EQUATION

As we mentioned before, the density matrix is characterized by the expectation values $\langle\Sigma_{\mathbf{s},\mathbf{r}}\rangle$. However, it is much more difficult to work with these expectation values, than with

$$A:=\sum_{\mathbf{s},\mathbf{r}}V(t):=\sum_{\mathbf{s},\mathbf{r}}e^{\Sigma_k(\beta_k a_k^\dagger-\beta_k^\star a_k)}. \tag{A1}$$

By imposing that $V(0)=1$ and that $V(T)$ is at most a phase, we will be able to relate the reduced density matrices $\rho_{\text{real}}(0)$ and $\rho_{\text{real}}(T)$. It is easy to see that indeed the operator $V(t)$ is a displacement operator and that Eq. (A1) is essentially the solution of the nondissipative case, where β_k measures the separation in phase space between configurations with internal states $\mathbf{s}$ and $\mathbf{r}$.

The equation for the expectation value of an arbitrary operator A can be written as follows:

$$\frac{d}{dt}\langle A\rangle = \langle \partial_t A\rangle + \frac{i}{\hbar}\langle [H_I,A]\rangle - \gamma\langle (D_a A)a + a^\dagger (D_a)\rangle + \gamma N\langle D_a D_{a^\dagger} A + D_{a^\dagger} D_a A\rangle. \quad \text{(A2)}$$

Here, $D_a A := [A,a^\dagger]$ and $D_{a^\dagger}A := [a,A]$ are two superoperators which first of all commute, $[D_a, D_{a^\dagger}]=0$, and second they are related to the formal derivatives with respect to the operators a and $a^\dagger$. So, for instance, $D_a f(a,a^\dagger)=\partial_a f(a,a^\dagger)$ for any analytical function f.

If we substitute Eq. (A1) into Eq. (A2), and use

$$\partial_t A = \sum_k \frac{\partial A}{\partial \beta_k}\dot{\beta}_k + \sum_k \frac{\partial A}{\partial \beta_k^\star}\dot{\beta}_k^\star = U[\dot{\beta}_k(a_k^\dagger - \tfrac{1}{2}\beta_k^\star) + \dot{\beta}_k^\star(-a_k - \tfrac{1}{2}\beta_k)], \quad \text{(A3)}$$

we will obtain

$$\frac{d}{dt}\langle A\rangle = i\sum_k \langle A[(g_{k\mathbf{r}} - g_{k\mathbf{s}})a_k + g_{k\mathbf{r}}\beta_k + \text{H.c.}]\rangle + \sum_k \langle A[\dot{\beta}_k a_k^\dagger - \dot{\beta}_k^\star a_k + \tfrac{1}{2}(\dot{\beta}_k\beta_k^\star - \beta_k\dot{\beta}_k^\star)]\rangle + \sum_k \gamma_k N_k \langle A(-\beta_k^\star a_k + \beta_k a_k^\dagger)\rangle - \sum_k \gamma_k(N_k+\tfrac{1}{2})|\beta_k|^2\langle A\rangle, \quad \text{(A4)}$$

with the new parameters $g_{k\mathbf{r}} := \Sigma_i g_{ki}(t) r_i$.

Here is where we impose a particular evolution of the displacements on phase space, $\dot{\beta}_k + \gamma_k\beta_k + i(g_{k\mathbf{r}}^\star - g_{k\mathbf{s}}^\star)=0$. This equation has a trivial solution,

$$\beta_k(t) = i\int_0^t d\tau\, e^{-\gamma_k(t-\tau)}[g_{k\mathbf{r}}^\star(\tau) - g_{k\mathbf{s}}^\star(\tau)]. \quad \text{(A5)}$$

After substituting this value all terms containing Fock operators are cancelled and we are left with

$$\frac{d}{dt}\langle A\rangle = (-\dot{\kappa} + i\dot{\phi})\langle A\rangle, \quad \text{(A6)}$$

where the decay κ is

$$\kappa(t) = \sum_k \gamma_k\left(N_k + \tfrac{1}{2}\right)\int_0^t d\tau |\beta_k(\tau)|^2, \quad \text{(A7)}$$

and the total phase $\phi = \Sigma_{ij} J_{ij}(r_i + s_i)(r_j - s_j)$ is determined by the matrix

$$J_{ij} := \text{Im}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 g_{ki}(\tau_1) g_{kj}(\tau_2)^\star e^{-\gamma_k(\tau_2-\tau_1)}. \quad \text{(A8)}$$

Using the symmetry of this matrix, the formula for the phase can be rewritten as $\phi = \Sigma_{ij} J_{ij}(r_i r_j - s_i s_j)$, and the results mentioned in Sec. V A quickly follow.

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[47] We use the identity $(\sigma_j^z)^2=1$.