Robustness of the adiabatic quantum search

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The robustness of the local adiabatic quantum search to decoherence in the instantaneous eigenbasis of the search Hamiltonian is examined. We demonstrate that the asymptotic time complexity of the ideal closed case is preserved, as long as the Hamiltonian dynamics is present. In the special case of pure decoherence where the environment monitors the search Hamiltonian, it is shown that the local adiabatic quantum search performs as the classical search.

Although the adiabatic approach to quantum computation [1,2] seems to differ significantly from the traditional circuit model, it has been proved that these two models are, in a certain sense, equivalent [3]. However, this equivalence does not concern the robustness to noise, relaxation, or decoherence. Since the adiabatic schemes operate close to the energy ground state it seems natural to guess that the adiabatic quantum computer should be robust against relaxation effects [2]. The alleged resistance to noise has been examined by analytic means in Ref. [4] and it has been argued that adiabatic quantum computers should be robust to decoherence [2,5]. Unitary control errors and resistance to decoherence have been numerically investigated in Ref. [6].

In this paper, we examine the local adiabatic search algorithm [7,8] in the presence of decoherence in the instantaneous energy eigenbasis [9]. We demonstrate analytically a robustness to this particular form of decoherence in the sense that the asymptotic time complexity of the ideal closed case is preserved, no matter how small the Hamiltonian contribution is to the dynamics. Only in the wide-open case [10], where the Hamiltonian part is completely absent, there is a difference in the time complexity.

Adiabatic quantum computation works by keeping the system close to the ground state of a time-dependent Hamiltonian. This feature is in contrast with, e.g., holonomic implementations of quantum gates [11], which share the feature of adiabatic evolution, but where it is essential that the gate can operate on arbitrary superpositions without too large errors. For the functioning of the adiabatic quantum computer in the presence of decoherence, on the other hand, it is sufficient to require that the probability of finding the system in the instantaneous ground state of $H(s)$ is conserved. This can be seen as one possible generalization of the concept of adiabaticity to open systems. In this generalized sense the wide-open case has an adiabatic limit, although the Hamiltonian dynamics is absent. One may note that the wide-open case can be seen as a quantum computational scheme in its own right, a "wide-open adiabatic quantum computer," where the dynamics is governed by pure decoherence. A dif-

ferent approach to the concept of adiabaticity for open system has been put forward in Ref. [12], and applied to adiabatic quantum computing in Ref. [13].

The N-element search problem consists of finding a single marked element in a disordered N-element list. The search problem is associated with an N-dimensional Hilbert space with orthonormal basis $\{|k\rangle\}_{k=1}^{N}$, where the marked item corresponds to $|\mu\rangle \in \{|k\rangle\}_{k=1}^{N}$. Following Refs. [1,7,8], we consider the family of Hamiltonians

$$H(s) = -(1-s)|\psi\rangle\langle\psi| - s|\mu\rangle\langle\mu|, \tag{1}$$

where

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}|k\rangle \tag{2}$$

and $s = t/T \in [0,1]$, T being the run time of the search. If the evolution is adiabatic and we start in the energy ground state $|\psi\rangle$, this family of Hamiltonians takes us to the marked state $|\mu\rangle$ and thus solves the search problem. The only relevant subspace is spanned by $|\psi\rangle$ and $|\mu\rangle$. We denote the instantaneous eigenvalues and orthonormal eigenvectors of $H(s)$ restricted to the relevant subspace by $E_n(s)$ and $|E_n(s)\rangle$, respectively, where $n = 0, 1$. We further define

$$\Delta(s) = E_1(s) - E_0(s) = \sqrt{\frac{1 + (N-1)(2s-1)^2}{N}} \tag{3}$$

and

$$Z(s) = |\langle \dot{E}_0(s)|E_1(s)\rangle| = \frac{\sqrt{N-1}}{1 + (N-1)(2s-1)^2}. \tag{4}$$

A useful property of Z is

$$\int_0^1 Z(s)ds \leq \frac{\pi}{2}, \tag{5}$$

for all N.

Decoherence in the instantaneous energy eigenbasis is modeled by the master equation

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$$\frac{d}{ds}\rho(s) = -iAT[H(s),\rho(s)] - BT[W(s),[W(s),\rho(s)]], \quad (6)$$

where $A \geq 0$ and $B \geq 0$ are constants independent of N. Here, $W(s)$ is assumed to be Hermitian, nondegenerate, and fulfill $[W(s),H(s)]=0$. Furthermore, let $w_n(s)$ be the eigenvalues of $W(s)$ corresponding to the eigenvectors $|E_n(s)\rangle$ and define $\Gamma(s) = w_1(s) - w_0(s)$.

Next, we implement the idea of local adiabatic search [7] by making a monotone, sufficiently smooth reparametrization $s \in [0,1] \rightarrow r = f(s) \in [0,1]$ of $H(s)$ and $W(s)$ in such a way that more time is spent near the minimum energy gap. In the closed case ($B=0$), it was shown in Ref. [7] that the optimal choice

$$f^{-1}(r) = \frac{1}{L}\int_0^r \frac{1}{\Delta^2(r')}dr',$$

$$L = \int_0^1 \frac{1}{\Delta^2(r')}dr' = \frac{N}{\sqrt{N-1}}\arctan(\sqrt{N-1}) \leq \frac{\pi}{2}\frac{N}{\sqrt{N-1}}$$

$$(7)$$

yields the criterion $T \gg \sqrt{N}$ for the run time, in analogy with the Grover search [14]. Applying this reparametrization results in the transformation $s \rightarrow r$ as well as in multiplication by $(df^{-1}/dr)(r)$ of the right-hand side of Eq. (6). Assume that $\rho(0) = |E_0(0)\rangle\langle E_0(0)|$ and let

$$Y(r) = \langle E_0(r)|\rho(r)|E_0(r)\rangle - \langle E_1(r)|\rho(r)|E_1(r)\rangle$$
$$\equiv \rho_{00}(r) - \rho_{11}(r). \quad (8)$$

We now address the main objective of this paper, which is to determine how the probability to remain in the ground state depends on the run-time T and the parameters A and B. The strategy is to express this probability, indirectly in terms of $Y(r)$, as an integral equation. Thereafter, we apply appropriate estimates to obtain a lower bound for the probability.

We may rewrite Eq. (6) as an integral equation that takes the form

$$1 - Y(r) = 4I(r), \quad (9)$$

where

$$I(r) = \frac{1}{2}I_+(r) + \frac{1}{2}I_-(r),$$

$$I_\pm(r) = \int_0^r e^{-T[BQ(r')\pm iAR(r')]}Z(r')u_\pm(r')dr',$$

$$u_\pm(r') = \int_0^{r'} e^{T[BQ(r'')\pm iAR(r'')]}Z(r'')Y(r'')dr'', \quad (10)$$

and

$$Q(r) = \int_0^r \Gamma^2(r')\frac{df^{-1}(r')}{dr'}dr' = \frac{1}{L}\int_0^r \frac{\Gamma^2(r')}{\Delta^2(r')}dr',$$

$$R(r) = \int_0^r \Delta(r')\frac{df^{-1}(r')}{dr'}dr' = \frac{1}{L}\int_0^r \frac{1}{\Delta(r')}dr'. \quad (11)$$

We further define

$$\zeta = \min_{r \in [0,1]} \frac{\Gamma^2(r)}{\Delta(r)}. \quad (12)$$

In the case where $A > 0$, we wish to estimate $|1-Y(r)|$. This can be done by calculating an upper bound for $|I_\pm(r)|$, using $Z(r)\Delta(r) \leq \sqrt{(N-1)/N}, \exp[-TBQ(r)] \leq 1$, $\exp\{-TB[Q(r)-Q(r')]\} \leq 1$ if $r \geq r'$, as well as Eqs. (5) and (12), which result in

$$|I_\pm(r)| \leq \frac{\pi L}{T}\sqrt{\frac{N-1}{N}}\frac{1}{\sqrt{B^2\zeta^2 + A^2}}$$
$$+ \frac{L}{T}\frac{\pi}{2}\int_0^r \left|\frac{d}{dr'}\left(\frac{Z(r')\Delta(r')}{B\frac{\Gamma^2(r')}{\Delta(r')} \pm iA}\right)\right|dr'. \quad (13)$$

By use of $Z(r)\Delta^2(r) = \sqrt{N-1}/N$ and Eq. (12), the integral on the right-hand side of Eq. (13) can be estimated as

$$\int_0^r \left|\frac{d}{dr'}\left(\frac{Z(r')\Delta(r')}{B\frac{\Gamma^2(r')}{\Delta(r')} \pm iA}\right)\right|dr'$$
$$\leq \frac{A}{B^2\zeta^2 + A^2}\int_0^1 Z(r')\left|\frac{d}{dr'}\Delta(r')\right|dr'$$
$$+ \frac{B}{B^2\zeta^2 + A^2}\int_0^1 Z(r')\left|\frac{d}{dr'}\Gamma^2(r')\right|dr'. \quad (14)$$

Note that we have extended the integration interval from $[0,r]$ to $[0,1]$. Since both $Z(r)$ and $\Delta(r)$ are symmetric around $r = \frac{1}{2}$, it follows that $Z(r)|(d/dr)\Delta(r)|$ has the same symmetry. Moreover, $\Delta(r)$ is increasing on the interval $[\frac{1}{2},1]$. Hence, $Z(r)|(d/dr)\Delta(r)| = Z(r)(d/dr)\Delta(r)$ on $[\frac{1}{2},1]$, which leads to

$$\int_0^1 Z(r)\left|\frac{d}{dr}\Delta(r)\right|dr = 2\int_{1/2}^1 Z(r)\frac{d}{dr}\Delta(r)dr \leq 2\sqrt{\frac{N-1}{N}} \leq 2. \quad (15)$$

To deal with the second term on the right-hand side of Eq. (14), we introduce the following condition:

$$\int_0^1 Z(r)\left|\frac{d}{dr}\Gamma^2(r)\right|dr \leq K, \quad (16)$$

where K is a constant independent of N [15]. Since $B\Gamma^2(r)$ can be seen as the instantaneous strength of the decoherence, the condition in Eq. (16) essentially states that the fluctuations in strength are not allowed to grow with N. If one assumes that $\Gamma(r) = \eta(\Delta(r))$, where $\eta:(0,\infty) \rightarrow (0,\infty)$ is an increasing, sufficiently smooth function, it can be shown that it is sufficient that $\eta(x) \leq Cx^\sigma$, where C and $\sigma \geq \frac{1}{2}$ are constants, to fulfill the condition in Eq. (16). This means that the condition is fulfilled for the particular case where $W(r)$

$=H(r)$. By combining Eqs. (7) and (13)–(16), and using that $B^2\zeta^2+A^2\geqslant A^2$ and $B^2\Gamma^4(0)+A^2\geqslant A^2$, we obtain

$$\rho_{00}(r)\geqslant 1-2\pi^2\frac{\sqrt{N}}{T}\left(\frac{1}{A}+\sqrt{\frac{N}{N-1}}\frac{KB}{A^2}\right). \qquad (17)$$

Hence, it is a sufficient condition for local adiabaticity that $T\gg\sqrt{N}$. In conclusion, an increased degree of eigenbasis decoherence does not change the asymptotic behavior of the run-time of the adiabatic search. This result is independent of the explicit form of $W(r)$ as long as Eq. (16) is fulfilled and $[W(r),H(r)]=0$.

In the wide-open case [10] ($A=0$), the protective effect of the Hamiltonian dynamics is absent and one may expect that the asymptotic behavior depends on the explicit choice of $W(r)$. To verify this point, we let $W(r)$ be such that $\Gamma(r)=\Delta^\sigma(r)$, $\sigma\geqslant 1$ [17]. We further put $B=1$ for convenience. Notice that the choice $W(r)=H(r)$ corresponds to $\Gamma(r)=\Delta(r)$. We prove that in the wide-open case with $\Gamma(r)=\Delta^\sigma(r)$, a sufficient and necessary condition for adiabaticity is $T\gg N^\sigma$. Note that we have to show that the sufficient condition is also necessary, as we wish to prove that the wide-open case is essentially different from the $A\neq 0$ case.

To prove the sufficiency, we insert $\Gamma(r)=\Delta^\sigma(r)$ into Eq. (11), and use that $\Delta(r)\geqslant 1/\sqrt{N}$ and $\sigma\geqslant 1$, to obtain

$$Q(r)=\frac{1}{L}\int_0^r\Delta^{2\sigma-2}(r')dr'\geqslant\frac{1}{LN^{\sigma-1}}r. \qquad (18)$$

Inserting Eq. (18) into Eq. (10) gives

$$1-Y(r)\leqslant 4\int_0^r\int_0^{r'}e^{-T(r'-r'')/LN^{\sigma-1}}Z(r')Z(r'')dr''dr'. \qquad (19)$$

Finally, we use $Z(r)\leqslant\sqrt{N-1}$ and Eq. (5) to obtain

$$\rho_{00}(r)\geqslant 1-\frac{\pi^2}{2}\frac{N^\sigma}{T}. \qquad (20)$$

Thus, a sufficient condition for adiabaticity is $T\gg N^\sigma$.

Now we show that this condition is also necessary. We let Y_T denote the solution of Eq. (9) for a given run-time T. It can be proved that [16]

$$Y_T(r)\geqslant Y_0(r), \qquad (21)$$

which means that an evolution with nonzero run time remains closer to the instantaneous ground state than the evolution with zero run time. Insert Eq. (21) into Eq. (10) and combine with Eq. (9) to obtain

$$1-Y_T(1)\geqslant I_0, \qquad (22)$$

where

$$I_0=4\int_0^1\int_0^r e^{-T[Q(r)-Q(r')]}Z(r)Z(r')Y_0(r')dr'dr \qquad (23)$$

and

$$Y_0(r)=\frac{1-(N-1)(2r-1)}{\sqrt{N}\sqrt{1+(N-1)(2r-1)^2}}$$

$$\geqslant-\frac{(N-1)(2r-1)}{\sqrt{N}\sqrt{1+(N-1)(2r-1)^2}}. \qquad (24)$$

In order for $Y_T(1)\to 1$, I_0 has to go to zero, since $Y_T(r)\leqslant 1$. Hence, we have found a necessary condition for the system to approach adiabaticity.

In order to express this necessary condition in terms of the runtime T, let us use Eq. (24) in Eq. (23) and make the change of variables $x=\sqrt{N-1}(2r-1)$ and $y=\sqrt{N-1}(2r'-1)$. This yields

$$I_0\geqslant\sqrt{\frac{N-1}{N}}I(\alpha,\sqrt{N-1}), \qquad (25)$$

where

$$I(\alpha,\beta)=\int_{-\beta}^\beta\frac{e^{-\alpha\Phi(x)}}{1+x^2}\int_{-\beta}^x\frac{(-y)e^{\alpha\Phi(y)}}{(1+y^2)^{3/2}}dydx \qquad (26)$$

with

$$\alpha=\frac{T}{2LN^{\sigma-1}\sqrt{N-1}}=\frac{T}{2N^\sigma\arctan(\sqrt{N-1})},$$

$$\Phi(x)=\int_0^x(1+x'^2)^{\sigma-1}dx'. \qquad (27)$$

Furthermore

$$\frac{d}{d\beta}I(\alpha,\beta)=-2\frac{e^{-\alpha\Phi(\beta)}}{1+\beta^2}\int_0^\beta\frac{y\sinh(\alpha\Phi(y))}{(1+y^2)^{3/2}}dy$$

$$+2\frac{\beta e^{-\alpha\Phi(\beta)}}{(1+\beta^2)^{3/2}}\int_0^\beta\frac{\cosh(\alpha\Phi(x))}{1+x^2}dx$$

$$\geqslant 2\frac{\beta e^{-\alpha\Phi(\beta)}}{(1+\beta^2)^{3/2}}\int_0^\beta\frac{e^{-\alpha\Phi(x)}}{1+x^2}dx\equiv F(\alpha,\beta)>0. \qquad (28)$$

This expression is obtained by separating the integrals "$\int_{-\beta}^\beta$" into "$\int_{-\beta}^0+\int_0^\beta$" and making the change of variables $x\to-x$ and $y\to-y$ in the $\int_{-\beta}^0$ integrals, as well as by using the inequality

$$\frac{y}{(1+y^2)^{(3/2)}}\leqslant\frac{\beta}{\sqrt{1+\beta^2}}\frac{1}{1+y^2}, \quad\forall\, y\in[0,\beta]. \qquad (29)$$

It follows from Eq. (28) that $I(\alpha,\beta)$ is increasing in β, which together with Eqs. (25) and (28), gives

$$I_0\geqslant\frac{1}{\sqrt{2}}I(\alpha,\sqrt{N-1})\geqslant\frac{1}{\sqrt{2}}I(\alpha,1)\geqslant\frac{1}{\sqrt{2}}\int_0^1 F(\alpha,\beta')d\beta'>0, \qquad (30)$$

where we have assumed that $N\geqslant 2$. Thus, if $I_0\to 0$ then $\int_0^1 F(\alpha,\beta')d\beta'\to 0$ necessarily. Furthermore, we have

$$\frac{d}{d\alpha}\int_0^1 F(\alpha,\beta)d\beta = -2\int_0^1\left[\frac{\beta\Phi(\beta)e^{-\alpha\Phi(\beta)}}{(1+\beta^2)^{3/2}}\int_0^\beta\frac{e^{-\alpha\Phi(x)}}{1+x^2}dx\right.$$
$$\left.+\frac{\beta e^{-\alpha\Phi(\beta)}}{(1+\beta^2)^{3/2}}\int_0^\beta\frac{\Phi(x)e^{-\alpha\Phi(x)}}{1+x^2}dx\right]d\beta < 0$$

$$(31)$$

for $\beta > 0$. It follows that $\int_0^1 F(\alpha,\beta)d\beta$ is a strictly decreasing function in α. Hence, a necessary condition for this expression to go to zero is that $\alpha \to \infty$. For large N it follows from the expression for α in Eq. (27) that it is necessary for adiabaticity that $T \gg N^\sigma$.

In Fig. 1, we supplement the above analytic results with numerical simulations of the dynamics of Eq. (6) with the choice $W(r) = H(r)$ and initial condition $\rho_{00}(0) = 1$. We interpolate between the closed and the wide-open case, by letting $A = \cos(\omega\pi/2)$ and $B = \sin(\omega\pi/2)$, where ω goes from 0 to 1. Furthermore, we have assumed the success probability $\rho_{00}(1) = 0.5$. These simulations confirm the predictions concerning the asymptotic behavior, viz., that the evolution of the local adiabatic quantum search stays near the instantaneous ground state if $T \gg \sqrt{N}$ for all cases except the wide-open one, where $T \gg N$.

In conclusion, we have demonstrated that the local adiabatic search is robust to decoherence in the instantaneous eigenbasis of the search Hamiltonian, as long as the Hamiltonian dynamics is present. Up to a condition on the fluctuations, this result is independent of the explicit form of the decoherence term. This independence does no longer hold in absence of the Hamiltonian part, in which case the asymptotic behavior of the run time of the local search changes. The protective effect of the Hamiltonian dynamics is an indication of robustness of the quantum adiabatic search, which may be of importance in physical implementations of a working search scheme that outperforms any known classical search algorithm.
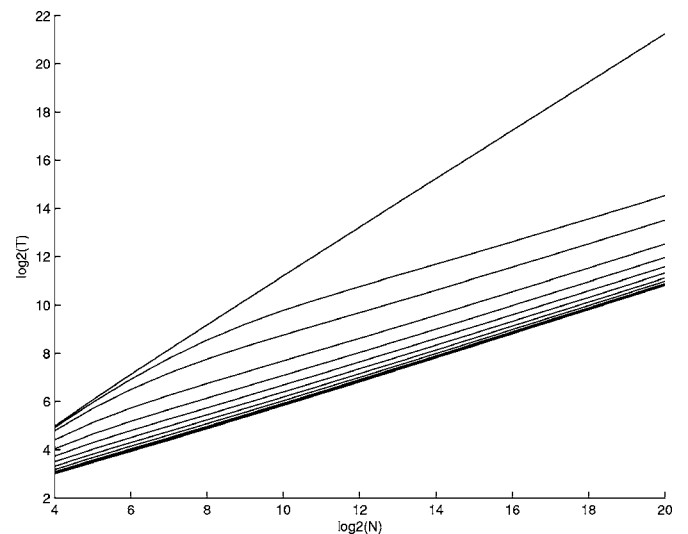

FIG. 1. Local search with $W(r) = H(r)$ and success probability 0.5. The curves show $\log_2 T$ vs $\log_2 N$, where T is the run time needed to obtain the success probability 0.5, and where N is the list length. Each curve shows the result for a given degree of decoherence ω. Counted from below, the curves correspond to $\omega = 0, 0.1, \ldots,$ 0.9, 0.95, 1, interpolating between the closed ($\omega = 0$) and wide-open ($\omega = 1$) case. As seen, all curves tend to the slope $\frac{1}{2}$, except the uppermost wide-open case, which tends to the slope 1.

An interesting extension would be to apply the present analysis to adiabatic algorithms designed to solve other problems, such as, e.g., the NP-complete problems 3-SAT [1] and exact cover [2,18,19]. Although analytical results may not be achievable for these problems, numerical investigations could reveal whether or not the protective effect of the Hamiltonian dynamics is present. Moreover, one might consider whether there occurs a transition from the seemingly polynomial behavior found in [2], to an exponential time complexity, as the strength of decoherence increases.

[1] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, e-print quant-ph/0001106.

[2] E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, and D. Preda, Science **292**, 472 (2001).

[3] D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd, and O. Regev, e-print quant-ph/0405098; M. S. Siu, e-print quant-ph/0409024.

[4] J. Roland and N. J. Cerf, Phys. Rev. A **71**, 032330 (2005).

[5] W. M. Kaminsky and S. Lloyd, e-print quant-ph/0211152.

[6] A. M. Childs, E. Farhi, and J. Preskill, Phys. Rev. A **65**, 012322 (2001).

[7] J. Roland and N. J. Cerf, Phys. Rev. A **65**, 042308 (2002).

[8] W. van Dam, M. Mosca, and U. Vazirani, in *Proceedings of the 42nd Symposium on Foundations of Computer Science* (IEE Computer Society Press, New York, 2001), p. 279.

[9] Decoherence in the instantaneous energy eigenbasis is relevant, e.g., for a class of scenarios where the coupling to the environment is weak and dominated by the Hamiltonian of the system; see J. P. Paz and W. H. Zurek, Phys. Rev. Lett. **82**, 5181 (1999).

[10] I. C. Percival, J. Phys. A **27**, 1003 (1994).

[11] P. Zanardi and M. Rasetti, Phys. Lett. A **264**, 94 (1999).

[12] M. S. Sarandy and D. A. Lidar, Phys. Rev. A **71**, 012331 (2005).

[13] M. S. Sarandy and D. A. Lidar, e-print quant-ph/0502014.

[14] L. K. Grover, Phys. Rev. Lett. **79**, 325 (1997).

[15] As is evident from Eq. (17), improvement over the classical time complexities is obtained even for a weak N dependence in K, such as $K \propto N^a$, $a < \frac{1}{2}$.

[16] J. Åberg, D. Kult, and E. Sjöqvist (unpublished).

[17] The case where $\sigma < 1$ requires a different analysis.

[18] R. Orús and J. I. Latorre, Phys. Rev. A **69**, 052308 (2004).

[19] J. I. Latorre and R. Orús, Phys. Rev. A **69**, 062302 (2004).