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Symmetry analyzer for nondestructive Bell-state detection using weak nonlinearities

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We describe a method to project photonic two-qubit states onto the symmetric and antisymmetric subspaces of their Hilbert space. This device utilizes an ancillary coherent state, together with a weak cross-Kerr nonlinearity, generated, for example, by electromagnetically induced transparency. The symmetry analyzer is nondestructive, and works for small values of the cross-Kerr coupling. Furthermore, this device can be used to construct a nondestructive Bell-state detector.

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Two-qubit measurements are an important resource in quantum information processing (QIP), enabling key applications such as the teleportation of states and gates, dense coding, and error correction. In particular, a measurement device that does not destroy the qubits is a very powerful tool, since it allows entanglement distillation [1] and efficient quantum computing based on measurements [2–4]. This is especially useful when the qubits interact weakly, and interaction-based quantum gates are hard to implement (for example, photonic qubits have negligible interaction). Furthermore, a nondestructive two-qubit measurement device can act as an deterministic source of entangled qubits.

Optical QIP is of special interest, because electromagnetic fields are ideal information carriers for long-distance quantum communication. Photonic quantum states generally suffer low decoherence rates compared to most massive qubit systems, but we need optical information processing devices that overcome the negligible interaction between the photons. Optical quantum computation and communication will therefore benefit greatly from nondestructive two-qubit measurements. Arguably the most important two-photon measurement is the measurement in the maximally entangled Bell basis. When the computational basis of a single-photon qubit is given by two orthogonal polarization states (H and V), then the Bell states can be written as $|\Psi^{\pm}\rangle = (|H,V\rangle \pm |V,H\rangle)/\sqrt{2}$ and $|\Phi^{\pm}\rangle = (|H,H\rangle \pm |V,V\rangle)/\sqrt{2}$. A nondestructive Bell measurement then projects the two photons onto one of the Bell states. This can be used in the teleportation of probabilistic gates into optical circuits [5,6], and consequently enables efficient linear optical quantum computing. In addition, a deterministic nondestructive Bell measurement would also act as a bright source of entangled photons.

Braunstein and Mann presented a linear optical method to distinguish two out of the four optical Bell states [7]. In 1999, it was shown independently by Vaidman and Yoran, and Lütkenhaus *et al.* that the Braunstein-Mann method is optimal [8,9]: When one is restricted to linear optics and photon counting (including feed-forward processing) at most half of the Bell states can be identified perfectly. This detection method is therefore *probabilistic*. Furthermore, it destroys the photons in the photon counting process, and is thus of limited use in efficient large-scale QIP.

One way to improve on this scheme is to move beyond linear optics, i.e., to induce an interaction between the photons. This can be achieved using a cross-Kerr medium, i.e., a nonlinear medium that can be described by an interaction Hamiltonian of the form

$$\hat{H}_K = \hbar\chi\hat{n}_a\hat{n}_c, \tag{1}$$

where $\hat{n}_k$ is the number operator for mode k, and $\hbar\chi$ is the coupling strength of the nonlinearity. A photon in mode c will then accumulate a phase shift $\theta = \chi t$ that is proportional to the number of photons in mode a. Such a medium can be used as an optical switch [10]. More to the point, when the nonlinearity is large (i.e., $\theta \approx \pi$), it naturally implements a controlled-phase gate at the single photon level. This inspired applications such as photon number quantum nondemolition (QND) measurements [11,12], Noon-state generation [13], a Fredkin gate [14], and culminated in a full-scale proposal for optical quantum computers [15]. In particular, with a large nonlinearity we can build a Bell state analyzer [16,17].

Unfortunately, natural Kerr media have extremely small nonlinearities, with a typical dimensionless magnitude of $\theta \approx 10^{-18}$ [18,19]. A large Kerr nonlinearity at the single-photon level is therefore practically impossible. However, there are ways to make nonlinearities of magnitude $\sim 10^{-2}$, for example with electromagnetically induced transparencies (EIT) [20–22], whispering-gallery microresonators [23], optical fibers [24], or cavity QED systems [25,26]. In this paper, we show how to build a nondestructive interferometric Bell-state analyzer with such small-but-not-tiny Kerr nonlinearities, and additional coherent state resources.

As a specific example of a very promising method for generating the form of nonlinearity required, we consider EIT in condensed matter systems. We have analyzed a model system at length [21], considering three photon modes interacting through dipole couplings to a four-level $\mathcal{N}$ atomic system [27]. Mode a generally describes a Fock state $|n_a\rangle$,

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and mode c is a coherent state $|\alpha_c\rangle$; these two fields interact through a third pump mode that is a sufficiently intense coherent state that both it and the internal atomic energy levels can be factored out of the evolution, creating an effective nonlinear Kerr interaction between modes a and c of the form given in Eq. (1). In general, it is difficult to achieve a substantial vacuum Rabi frequency using free-space fields [28], but encapsulating one or more atoms in a waveguide (such as a line defect in a photonic crystal structure) allows field transversality to be maintained at mode cross-sectional areas that have dimensions smaller than the optical wavelengths of the interacting fields. A two-dimensional photonic crystal waveguide constructed from diamond thin film with nitrogen-vacancy color centers fabricated in the center of the waveguide channel [29,30] could provide a sufficiently large nonlinearity to realize our method experimentally. For example, a cryogenic NV-diamond system with 2×10^4 color centers can generate a phase shift of more than 0.1 rad per signal photon with a probe photon number $n_c=\alpha_c^2=1.3\times 10^4$ and modest detunings.

We turn now to the application of such nonlinearities for Bell-state analysis. As mentioned above, it is well known that a beam splitter can be used to discriminate between the singlet and the remaining triplet Bell states [7]. If the two incoming modes are combined on a beam splitter, the Bell states are transformed as

$$|\Psi^-\rangle = |H,V\rangle - |V,H\rangle \rightarrow |H,V\rangle - |V,H\rangle,$$

$$|\Psi^+\rangle = |H,V\rangle + |V,H\rangle \rightarrow |HV,0\rangle - |0,HV\rangle, \qquad (2)$$

$$|\Phi^\pm\rangle = |H,H\rangle \pm |V,V\rangle \rightarrow |H^2,0\rangle - |0,H^2\rangle \pm |V^2,0\rangle \mp |0,V^2\rangle.$$

After the beam-splitter transformation, the singlet state, $|\Psi^-\rangle$, is *balanced*, i.e., it has only one photon in each spatial mode. On the other hand, the triplet states are *bunched*, i.e., they have coherent superpositions of either zero or two photons in each spatial mode. Our scheme proceeds by nondestructively distinguishing between these two cases, and subsequently transforming the states back to the Bell basis using a second beam splitter. This nondestructive symmetry analysis therefore allows the singlet state to be discriminated from the triplet states. As we discuss further below, a full nondestructive Bell measurement can be implemented by repeated applications of the symmetry analysis, interleaved with appropriate local operations.

It is important to note that the balanced and bunched states must be discriminated in such a way that no other information is discovered about the states. In particular, determining the number of photons in a particular spatial mode, even nondestructively, would destroy the coherence of the bunched states. For this reason, existing photon number QND measurement techniques [11,12,21] with small θ are insufficient to perform the symmetry analysis step. The technique for nondestructive symmetry analysis that we describe below is one of the principal results of this paper.

In order to describe our scheme for symmetry analysis, we first consider the illustrative example of an analyzer capable of nondestructively distinguishing between the balanced and bunched states, $|1,1\rangle$ and $(|2,0\rangle \pm |0,2\rangle)/\sqrt{2}$. Here,

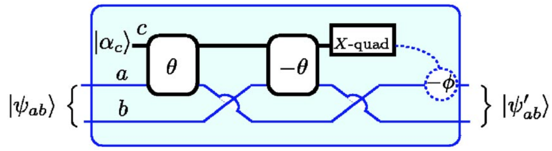

FIG. 1. Schematic circuit showing how two relatively small cross-Kerr nonlinearities can be used to discriminate between $(|2,0\rangle \pm |0,2\rangle)/\sqrt{2}$ and $|1,1\rangle$ nondestructively. The boxes labeled by θ and $-\theta$ represent the cross-Kerr nonlinearities that induce a phase shift in the coherent state $|\alpha_c\rangle$ proportional to the number of photons in the corresponding signal mode. A homodyne measurement of the X quadrature with outcome x will then discriminate between the two input states. Depending on this outcome, a relative phase shift $2\phi(x)$ is needed to restore the $|2,0\rangle \pm |0,2\rangle$ state.

$|j,k\rangle$ represents a two-spatial mode optical state, with j photons in mode a and k photons in mode b, with all photons polarized in the same direction. The setup for discriminating these states is shown in Fig. 1. Mode c is initially prepared in a coherent state $|\alpha_c\rangle = e^{-|\alpha_c|^2/2}\sum_n \alpha_c^n/\sqrt{n!}|n\rangle$, where $|n\rangle$ denotes an n-photon number state [31]. The coherent state can be generated, for example, by a laser pulse. The photons in mode c sequentially interact with those in modes a and b via the two (relatively small) cross-Kerr nonlinear operations, acting with phases θ and $-\theta$, respectively. These operations can be written as $\exp(i\theta\hat{n}_a\hat{n}_c)$ and $\exp(-i\theta\hat{n}_b\hat{n}_c)$, as follows from Eq. (1) [32].

Suppose now that the input state for the two signal modes a and b and the probe mode c is given by

$$|\psi_0\rangle = \left[d_1|1,1\rangle + \frac{d_2}{\sqrt{2}}(|2,0\rangle \pm |0,2\rangle)\right]|\alpha_c\rangle, \qquad (3)$$

where d_1, d_2 are complex coefficients satisfying the usual normalization requirements. The effect of each cross-Kerr operation is to induce a phase shift in the coherent state $|\alpha_c\rangle$, which is proportional to the number of photons in the corresponding signal mode (alternatively, although each Fock state in the coherent state imparts a different Kerr shift to the signal photons, the measurement of the coherent state is designed so that there is only one overall phase shift on the signal photons afterwards). Thus, for the $|1,1\rangle$ component of the state, the total phase shift induced is $\theta+(-\theta)=0$, and for the $|2,0\rangle$ and $|0,2\rangle$ components, the phase shifts are $+2\theta$ and -2θ, respectively. After these cross-Kerr operations, the state of the three modes is thus given by

$$|\psi_1\rangle = d_1|1,1\rangle|\alpha_c\rangle + \frac{d_2}{\sqrt{2}}(|2,0\rangle|\alpha_c e^{2i\theta}\rangle \pm |0,2\rangle|\alpha_c e^{-2i\theta}\rangle). \qquad (4)$$

This state is illustrated in the phase space plot in Fig. 2(a).

In order to distinguish the balanced and bunched components of $|\psi_1\rangle$, it is sufficient to measure the X quadrature component of the probe mode c. This can be achieved with a standard homodyne measurement. To perform such a measurement, the probe mode is combined at a beam splitter with a local oscillator of the same frequency. The output is then measured with photodetectors [31]. Provided the local

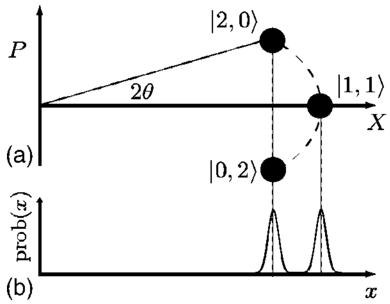

FIG. 2. (a) Schematic phase space illustration of the state $|\psi_1\rangle = d_1|1,1\rangle|\alpha_c\rangle + (d_2/\sqrt{2})(|2,0\rangle|\alpha_c e^{2i\theta}\rangle \pm |0,2\rangle|\alpha_c e^{-2i\theta}\rangle)$. For the $|1,\,1\rangle$ component of the state, the probe mode receives zero total phase shift, whereas for the $|2,\,0\rangle$ and $|0,\,2\rangle$ components, the phase shifts are $+2\theta$ and -2θ, respectively. (b) The corresponding probability distribution for the outcome of the X quadrature measurement of the probe beam. The two peaks in this distribution are associated with the states $|1,\,1\rangle$ and $|2,0\rangle \pm |2,0\rangle$.

oscillator is prepared in a large-amplitude coherent state with the same phase as $|\alpha_c\rangle$, homodyne measurement amounts to a (destructive) measurement of the observable $\hat{X} = \hat{c} + \hat{c}^\dagger$, where $\hat{c}$ is an annihilation operator for photons in the probe mode. Using the result $\langle x|\beta\rangle = (2\pi)^{-1/4}\exp[-\mathrm{Im}(\beta)^2 - (x-2\beta)^2/4]$ [33], where $|x\rangle$ is an eigenstate of X with eigenvalue x, the state of modes a and b after the measurement on c is

$$|\psi_3\rangle = d_1 f(x,\alpha_c)|1,1\rangle + \frac{d_2}{\sqrt{2}} f(x,\alpha_c \cos 2\theta) \times (e^{i\phi(x)}|2,0\rangle \pm e^{-i\phi(x)}|0,2\rangle), \qquad (5)$$

where we have defined

$$f(x,\beta) \equiv (2\pi)^{-1/4} \exp[-(x-2\beta)^2/4],$$

$$\phi(x) \equiv \alpha_c \sin 2\theta (x - 2\alpha_c \cos 2\theta) \bmod 2\pi. \qquad (6)$$

The Gaussian terms $f(x,\alpha_c)$ and $f(x,\alpha_c \cos 2\theta)$ in Eq. (5) correspond to probability amplitudes associated with each of the two states $|1,\,1\rangle$ and $|2,0\rangle \pm |2,0\rangle$, respectively [see Fig. 2(b)]. The phase shift $\phi(x)$ associated with the two-photon components depends on the outcome of the homodyne measurement. This can be corrected by applying the phase shift operation $\exp[-i\phi(x)\hat{n}_a]$, conditional on the obtained value of x. In order to resolve the balanced and the bunched components, we require only a small overlap between their probability distributions. Values of x below the midpoint between the peaks define one measurement outcome, and values of x above it the other outcome. The error probability is thus the sum of the lower x distribution tail above the midpoint and the upper distribution tail below the midpoint, and is given by $P_{\text{error}} = \mathrm{erfc}(\sqrt{2}\alpha_c\theta^2)/2$. This is less than 0.01 provided $\alpha_c\theta^2 > 1.2$. Highly accurate discrimination is therefore possible with weak cross-Kerr nonlinearities ($\theta \ll \pi$) provided α_c can be made sufficiently large. For example, the system described in the NV-diamond example given above generates the error probability $P_{\text{error}} = 0.01$.

A straightforward generalization of the methodology described above can be used to construct a nondestructive photonic symmetry analyzer on the two-qubit Hilbert space $\mathcal{H}$ of the incoming modes, spanned by the states $|H,H\rangle$, $|H,V\rangle$, $|V,H\rangle$, and $|V,V\rangle$ (see Fig. 3). As noted above, the beam splitter transformation, Eq. (2), transforms the singlet state into a balanced state, whereas the triplet states are bunched. The polarizing beam splitters (PBS) will separate the two polarization modes. A polarization rotation (not shown) applied to the same output of each PBS will then ensure that all the photons have identical polarization, thus satisfying the assumption made about the inputs to the analyzer of Fig. 1. (These rotations are then undone before the outgoing PBS.) By counting the phase shifts (θ and $-\theta$) we can determine the total phase that is acquired by $|\alpha_c\rangle$. As before, the (balanced) singlet state will not induce a phase shift in $|\alpha_c\rangle$. The different components of the (bunched) triplet states will induce phase shifts of $+2\theta$ or -2θ. Therefore, the X-quadrature homodyne measurement of mode c now allows the singlet and triplet states to be distinguished nondestructively. After corrective phase shifts (again conditional on the outcome of the X-quadrature measurement) and recombination on the PBS, the final beam splitter will return the state to the two-qubit Hilbert space $\mathcal{H}$. Note that it is crucial that the measurement does not introduce decoherence between the symmetric amplitudes, as repetition of the symmetry analysis is needed for full Bell-state analysis.

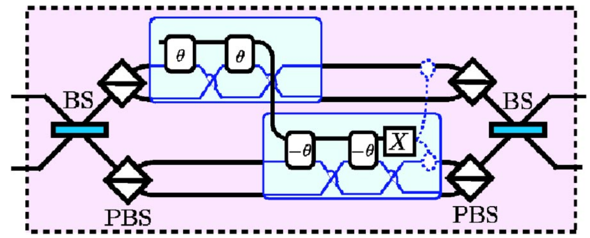

FIG. 3. The symmetry analyzer: photonic two-qubit input states interfere on a 50:50 beam splitter (BS). The polarizing beam splitters (PBS) will transform the beam splitter output into four modes with equal polarization. Using the methodology of Fig. 1, one can distinguish between the resulting bunched and balanced states. This results in a measurement of the symmetry of the photonic two-qubit input state. The PBS and BS after the homodyne measurement return the two modes to the two-qubit space.

Once we have a nondestructive symmetry analyzer (SA), it is straightforward to construct a quantum nondemolition Bell-state detector (depicted in Fig. 4). First, we test whether or not the input state is the singlet by applying the SA. We then apply a bit flip σ_x to move the antisymmetric subspace into the symmetric subspace. We apply the SA again, and if the transformed input state is the singlet, we know that the original state was $|\Phi^-\rangle$. We then apply a relative phase shift

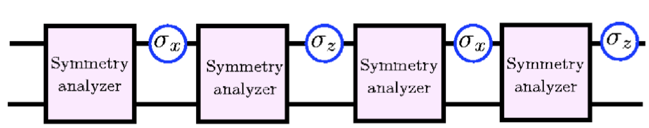

FIG. 4. The Bell-state detector: repeated application of the symmetry analyzer (SA) and local unitary rotations in $\mathcal{H}$ subsequently project onto different Bell states. The local unitaries are simply performed with suitable optical plates for polarization qubits.

σ_z. If the third SA finds the singlet, then the input was $|\Phi^+\rangle$. If no singlet signal has arisen, then the input state must have been $|\Psi^+\rangle$. We can test this by applying another bit flip σ_x and invoke the SA again. The final σ_z ensures that the outgoing state is actually that identified by the analyzer. Clearly if one is prepared to accept not finding the singlet in the first three analyzers as the signature of $|\Psi^+\rangle$, the final analyzer can be omitted. In this case the third and final single-qubit operation is instead σ_y to restore the outgoing state to that identified. Furthermore, if classical switching conditional on the homodyne measurement results is employed, the analysis could be terminated after a singlet signal from any SA, by switching the two-qubit state out and then reconstructing the identified Bell state by a local operation.

To summarize, in this paper we have shown how to construct a Bell-state analyzer from small cross-Kerr nonlinearities—small here means much less than the size of the nonlinearity required to perform a maximally entangling-disentangling gate directly between photons. Our analyzer distinguishes all four polarization Bell states and is near deterministic in operation. We have suggested EIT systems as one potential route for realizing the required cross-Kerr nonlinearities, which could lead to practical QIP in the relatively near future, especially since for our proposed symmetry and Bell-state analyzers there is not a requirement to generate π phase shifts. As we have shown, as long as the probe beam has a sufficient amplitude α_c such that $\alpha_c\theta^2 \gtrsim 1$, we can work with much smaller phase shifts. This makes our analyzers rather easier to implement that those based on standard nonlinear quantum logic.

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