Squeezing as an irreducible resource

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(Received 6 March 2005; published 31 May 2005)

Using the Bloch-Messiah reduction we show that squeezing is an "irreducible" resource which remains invariant under transformations by linear optical elements. In particular, this gives a decomposition of any optical circuit with linear input-output relations into a linear multiport interferometer followed by a unique set of single-mode squeezers and then another multiport interferometer. Using this decomposition we derive a no-go theorem for creating superpositions of macroscopically distinct states from single-photon detection. Further, we demonstrate the equivalence between several schemes for randomly creating polarization-entangled states. Finally, we derive minimal quantum optical circuits for ideal quantum nondemolition coupling of quadrature-phase amplitudes.

DOI: 10.1103/PhysRevA.71.055801 PACS number(s): 42.50.Dv, 03.65.Fd

There is still no consensus as to the eventual working material which will be used by large-scale quantum computers to store and process quantum information. By contrast, there seems to be no dispute about using optical or near-infrared photons for quantum communication. The advantages are obvious: high-speed transmission, weak coupling to the environment, negligible thermal noise. Some disadvantages include the difficulties in coupling light to light and in creating suitable input states. Some proposals involve cavity QED [1–3]. However, to date *all* implementations of quantum-communication protocols (over distances larger than micrometers) have used only coherent state inputs and optical components which are no more nonlinear than parametric down-converters or photodetectors. Thus, outside detection, this suggests that near-future quantum-communication experiments will also involve information processing which can be described by at most a, possibly time-dependent, linear mixing of annihilation and creation operators (linear Bogoliubov transformations corresponding to quadratic interactions) for optical modes. In this paper we will demonstrate that for such systems *squeezing* [4] forms an "irreducible" resource which allows us to quantify their power.

It has been known for some time how to analytically and numerically calculate the evolution of systems under the action of linear Bogoliubov transformations [5–7]. In quantum optics, however, and especially in quantum communication, an explicit Fock space description is important. This is equivalent to developing a normal-ordered description of the evolution. The analytic tools for this, however, have been limited to one or two modes at most [8]. Here we develop a tool that can leverage the Campbell-Baker-Hausdorff result to an arbitrary number of modes, in particular, a tool that will allow us to predict the strengths and limitations of devices (and resources). To that end, we would like to formalize our equations in terms of a universal set of irreducible resources and a restricted set of operations.

As a first step, we will see that any optical system that is modeled by linear Bogoliubov transforms can be decomposed into strictly "linear" and strictly "nonlinear" components. For photonic modes, quantum optics provides a well-developed correspondence between laboratory components and theoretical mode couplings. In this correspondence, traditional optics involves only linear elements (beam splitters, mirrors, half-wave plates, etc.). Mathematically, linear optical components have Bogoliubov transformations given by

$$\hat{b}_j = \sum_k U_{jk}\hat{a}_k, \tag{1}$$

where U is an arbitrary unitary matrix and there is no mixing of the mode annihilation and creation operators. Any such unitary U may be explicitly constructed from linear optical primitive components [9].

By contrast, *nonlinear optical* components (in particular squeezers, parametric amplifiers, and down-converters) are used to generate quantum resources (squeezed states, entangled states, etc.). These nonlinear components may produce a *linear* mixing between annihilation and creation operators when some pumping field or fields are strong enough that their quantum fluctuations and pump depletion may be neglected (the so-called parametric approximation). It is this regime of linear transformations on (photonic) modes that is of interest to us. Without attempting to be exhaustive we shall explicitly label three types of nonlinear optical elements which yield linear Bogoliubov transformations.

Squeezers (S). These are single-mode down-converters (also known as parametric amplifiers) and may be described by an interaction Hamiltonian of the form

$$\hat{H}_{\text{int}} = ir(\hat{a}_1^{\dagger 2} - \hat{a}_1^2)/2; \tag{2}$$

here r is the squeezing parameter and we drop extraneous phases from our descriptions without loss of generality.

Two-mode down-converters (D_2) are described by

$$\hat{H}_{\text{int}} \propto i(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1\hat{a}_2). \tag{3}$$

(Entangling) four-mode down-converters (E_4) are described by

$$\hat{H}_{\text{int}} \propto i(\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_3^\dagger \hat{a}_4^\dagger - \hat{a}_1\hat{a}_2 - \hat{a}_3\hat{a}_4). \tag{4}$$

These latter devices may be thought of as *entangling* down-converters if, for example, the even (odd) numbered modes

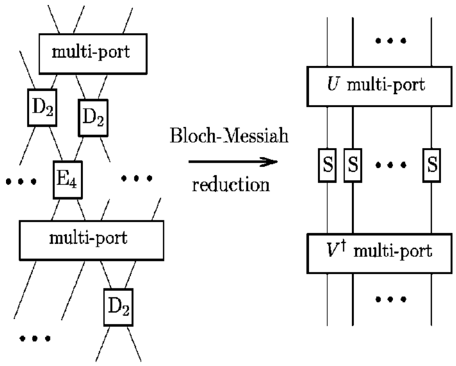

FIG. 1. An arbitrarily complicated combination of linear multiport interferometers, squeezers, down-converters, etc. (S, D_2, E_4, etc.). Each component describable by a quadratic interaction may be decomposed by Bloch-Messiah reduction into a linear multiport described by $V^\dagger$, a parallel set of single-mode squeezers (S), and a second linear multiport U.

represent differing polarization states for a mode heading left (right).

We are now in a position to describe the reduction of linear Bogoliubov transformations. This reduction is given by the so-called Bloch-Messiah theorem for bosons (a formal extension of the original result for fermions in [10]; see the Appendix for a compact proof), which is as follows.

Theorem (Bloch-Messiah reduction). For a general linear unitary Bogoliubov transformation of the form

$$\hat{b}_j = \sum_k (A_{jk}\hat{a}_k + B_{jk}\hat{a}_k^\dagger) + \beta_j, \tag{5}$$

where $\hat{a}_j, \hat{b}_j$ are bosonic annihilation operators, the matrices A and B may be decomposed into a pair of unitary matrices U and V and a pair of non-negative diagonal matrices A_D and B_D satisfying

$$A_D^2 = B_D^2 + \mathbb{1}, \tag{6}$$

with $\mathbb{1}$ the identity matrix, by

$$A = UA_DV^\dagger, \quad B = UB_DV^{\mathrm{T}}. \tag{7}$$

Corollary. For optical modes, Bloch-Messiah reduction says that the general form of multimode evolution with linear Bogoliubov transformations may be decomposed into a multiport linear interferometer, followed by the parallel application of a set of single-mode squeezers followed by yet another multiport linear interferometer [11]. This reduction is illustrated in Fig. 1.

One common application for down-converters is as sources of interesting quantum states. We shall use Bloch-Messiah reduction to tell us something about how versatile such devices may be. The simplest way of operating such a source is unconditionally, for which we state the following result.

Theorem. Given initial vacuum states, an arbitrarily complicated combination of linear multiport interferometers, down-converters, squeezers, etc., will deterministically generate only Gaussian states with normal ordered form

$$|\psi_{\mathrm{out}}\rangle \propto \exp\left(\frac{1}{2}\sum_{jk} B_{jk}\hat{b}_j^\dagger\hat{b}_k^\dagger\right)|0\rangle, \tag{8}$$

where without loss of generality B_{jk} may be chosen to be complex symmetric and $\hat{b}_j^\dagger$ are the outgoing mode creation operators.

Proof. Consider such a combination of components acting on the vacuum. By Bloch-Messiah reduction (see Fig. 1) the initial linear multiport interferometer described by $V^\dagger$ preserves the vacuum state, so only the later components have an effect. Since the individual single-mode squeezers have evolution operators which may be trivially normally ordered we may immediately write out the general form for the outgoing state as shown in Eq. (8). (Note that this result would be nontrivial using traditional techniques.)

There are at least two other modes of state generation which might be considered of interest.

Conditional state generation occurs where the required state leaves some part of the apparatus whenever a suitable sequence of photodetection events is found in another part. For example, a weakly coupled two-mode down-converter (3) can make a single-photon state to a good approximation in either of the two modes *conditioned* on a single-photon count in the other.

Random state generation occurs where the required state is "polluted" by contributions from the vacuum state. In this case, the state may be inferred by destructive photodetection, but then it cannot leave the apparatus. For example, a weakly coupled four-mode down-converter (4) can make polarization-entangled states *randomly* (in the sense given above).

We see from these examples that the "cheap" nonlinearity introduced by particle detection can increase the versatility of linear Bogoliubov transformations. However, there still appear to be limitations.

Theorem (no-go for macrosuperpositions). Detection of a single photon in one mode and no photons in any number of other modes cannot *conditionally* create superpositions of macroscopically distinct states given an initial vacuum state and using an arbitrarily complicated combination of linear multiport interferometers, down-converters, squeezers, etc. (all described by quadratic interactions).

Proof. Consider such a combination of components acting on the vacuum prior to detection. The above theorem gives the form Eq. (8) of the outgoing state $|\psi_{\mathrm{out}}\rangle$. Suppose now a single photon is detected in some mode $\hat{b}_\ell$ and vacuum in several others, the conditioned state is

$$|\psi_{\mathrm{cond}}\rangle \propto {}_{\mathrm{det}}\langle 0|\hat{b}_\ell|\psi_{\mathrm{out}}\rangle \propto \sum_m{}' B_{\ell m}\hat{b}_m^\dagger\,{}_{\mathrm{det}}\langle 0|\psi_{\mathrm{out}}, \tag{9}$$

where $|0\rangle_{\mathrm{det}}$ is the vacuum state for the subset of detected modes and the sum runs only over nondetected modes. It is easy to see that ${}_{\mathrm{det}}\langle 0|\psi_{\mathrm{out}}\rangle$ is a Gaussian state on the remaining modes, so the conditionally created state from *single*-photon detection is seen to be a sum of branches which differ by the placement of only a single photon in one mode or another. ■

Remark. Large-amplitude coherent states are "macroscopic superpositions" only in the sense that they are super-

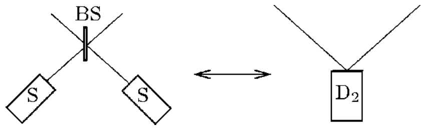

FIG. 2. Bloch-Messiah equivalence: Here we illustrate the equivalence between a pair of squeezers (S) combined at a 50:50 beam splitter (BS) and a single two-mode down-converter (D_2).

positions of macroscopic states (although these states are not macroscopically distinguishable). Thus, we have given a no-go theorem against creating so-called Schrödinger cat states for any such scheme without regard to the specific details of any particular implementation. A consequence of this result is that *entanglement* may not be "amplified" by, say, injecting microscopic superpositions into strongly pumped down-converters as has been suggested by (the title of) Ref. [12] (though obviously superposition states may be sent through an amplifier [13]). A detailed analysis of Ref. [12] supports our more general result in Eq. (9) [14].

The Bloch-Messiah reduction theorem teaches us some important lessons about the interconvertibility of different kinds of sources. For example, we find that a single squeezed state is an irreducible resource which cannot be made from any number of lesser squeezed states and linear optics. Similarly, if some device requires some given number of squeezers in Bloch-Messiah reduced form then fewer squeezers plus linear optics will never suffice for the device's construction. Let us use these observations to relate the three types of down-converters S, D_2, and E_4.

A nonentangling two-mode down-converter (D_2) with coupling (3) requires two squeezers in reduced form as is illustrated in Fig. 2. For weak coupling this device is a source of random photon pairs generated into distinct modes. The Bloch-Messiah reduction into two squeezers and a 50:50 beam-splitter gives us a more sophisticated understanding of the Hong-Ou-Mandel interferometer [15]. Away from the weak-coupling limit we retrieve the twin-beam scheme for making two-mode squeezed states from a pair of independently squeezed states [16]. Bloch-Messiah reduction neatly formalizes these multiphoton interference phenomena.

Similarly, Bloch-Messiah reduction applied to the entangling four-mode down-converter (E_4) of Eq. (4) shows that four squeezers are required in reduced form. Thus, a random polarization-entangled state cannot be formed from a *single* pass through a single nonentangling down-converter [D_2, Eq. (3)]. Nonetheless, it may be made easily enough with two such devices [17]. In Fig. 3 we give just such an equivalence. Curiously, this construction produces entanglement without erasing the which-way information about the photons. It should be noted that this scheme is very different (in terms of the irreducible resources used) than the entanglement swapping scheme of Zukowski *et al.* [18] which starts with a pair of entangling down-converters.

As a final application for the Bloch-Messiah reduction theorem we consider constructing optimal optical circuits using as little squeezing as possible. Consider the ideal quantum nondemolition (QND) coupling between a pair of optical quadrature-phase amplitudes

$$\hat{b}_1 = \hat{a}_1 - \frac{1}{2}\hat{a}_2 + \frac{1}{2}\hat{a}_2^\dagger,$$

$$\hat{b}_2 = \frac{1}{2}\hat{a}_1 + \hat{a}_2 + \frac{1}{2}\hat{a}_1^\dagger. \tag{10}$$

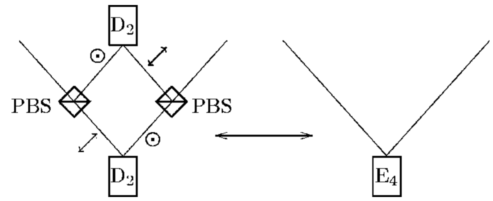

FIG. 3. Polarization entanglement without loss of which-way information. Here we illustrate the equivalence between an entangling four-mode down-converter (E_4) with a pair of nonentangling two-mode down-converters (D_2) which are *randomly* creating photon pairs with opposite polarizations ⊙,⤢. The polarization-dependent beam splitters (PBS) direct all photons to the upper paths. Bloch-Messiah reduction shows that it is impossible to (randomly) create such entangled states with only a *single* pass through a single nonentangling two-mode down-converter.

The relevant decomposition is given by

$$A = \begin{pmatrix} \sin\theta & -i\cos\theta \\ \cos\theta & i\sin\theta \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \cos\theta & -i\sin\theta \\ \sin\theta & i\cos\theta \end{pmatrix}^\dagger,$$

$$B = \begin{pmatrix} \sin\theta & -i\cos\theta \\ \cos\theta & i\sin\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \cos\theta & -i\sin\theta \\ \sin\theta & i\cos\theta \end{pmatrix}^{\mathrm{T}}, \tag{11}$$

where $\theta = \frac{1}{2}\sin^{-1}(2/\sqrt{5}) \simeq 31.72°$. The circuit consists of a pair of squeezers with equal squeezing parameters of $r = \ln[(1+\sqrt{5})/2]$ (corresponding to roughly 4.18 dB) and a pair of unequal unbalanced beam splitters with energy transmission coefficients of 27.64% and 72.36%.

In fact, this circuit is equivalent to one derived by Yurke [19]; however, Bloch-Messiah reduction guarantees its optimality. We can improve on it further by noting that the singular-value eigenvalues in Eq. (11) are degenerate and so the decomposition is not unique; a construction with much simpler 50:50 beam splitters is given by

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} ie^{i\theta} & ie^{-i\theta} \\ -e^{i\theta} & e^{-i\theta} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U^*, \tag{12}$$

with θ as above. We note that the QND coupling (10) has recently been used in error-correction codes for quantum-optical fields [20,21].

In conclusion, we have illustrated the utility of the Bloch-Messiah reduction theorem for linear bosonic Bogoliubov transformations in the context of quantum optics. We have shown the equivalence between a number of elementary sources of weak random states, including a simple scheme to randomly generate polarization entanglement without a loss of which-way information. When supplemented by detection of a single photon we have shown that superpositions of macroscopically distinct states cannot be created out of vacuum using linear optics and down-converters, squeezers, etc. (all corresponding to linear Bogoliubov transformations). Finally, we used Bloch-Messiah reduction to study the construction of minimal optical circuits. Although we have concentrated on applications for photonic modes in quantum optics the Bloch-Messiah reduction theorem holds for all bosonic modes.

APPENDIX: PROOF OF BLOCH-MESSIAH REDUCTION

Without loss of generality, we may set the displacements in Eq. (5) to zero, i.e., $\beta_j=0$. The canonical commutation relations for $\hat{b}_j$ in Eq. (5) impose the conditions [7]

$$AB^{\mathrm{T}} = (AB^{\mathrm{T}})^{\mathrm{T}}, \tag{A1}$$

$$AA^{\dagger} = BB^{\dagger} + \mathbb{1}. \tag{A2}$$

Since $AA^{\dagger}$ and $BB^{\dagger}$ are Hermitian and according to Eq. (A2) must commute, they also may be diagonalized in the same basis by some unitary matrix U. However, using the singular-value decomposition theorem [22] we can always diagonalize $A=UA_DV^{\dagger}$ and $B=UB_DW^{\dagger}$ into non-negative matrices A_D and B_D satisfying Eq. (6) where V and W are a pair of unitary matrices. Unitarity of Eq. (5) guarantees a unique inverse which with the aid of Eqs. (A1) and (A2) may be easily computed to be [7]

$$\hat{a}_j = \sum_k (A^*_{kj}\hat{b}_k - B_{kj}\hat{b}^{\dagger}_k). \tag{A3}$$

Imposing the canonical commutation relations again here yields the conditions

$$A^{\dagger}B = (A^{\dagger}B)^{\mathrm{T}}, \tag{A4}$$

$$A^{\dagger}A = (B^{\dagger}B)^{\mathrm{T}} + \mathbb{1}. \tag{A5}$$

Thus we see that $A^{\dagger}A$ and $(B^{\dagger}B)^{\mathrm{T}}$ may be diagonalized in the same basis by a unitary matrix $V=W^*$ which yields Eq. (7) as required. Finally, we note that this form for A and B automatically satisfies the subsidiary conditions of Eqs. (A1) and (A4). ■

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