Zero-range potentials for Dirac particles: Scattering and related continuum problems

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(Received 18 November 2004; published 18 May 2005)

The zero-range potential model, widely used in nonrelativistic quantum mechanics, is extended to continuum problems involving Dirac particles. A bispinor wave function of a Dirac particle scattered from a system of zero-range potentials is sought in the form of an incident wave superposed with waves emerging from points where targets are located. Interactions between the particle and individual targets are described by imposing certain limiting conditions, relating linearly upper and lower components of the wave function at target locations. This yields an inhomogeneous algebraic system for superposition coefficients appearing in the expression for the wave function. After preliminary considerations, admitting a quite general form of the incident wave, the case of the monochromatic plane-wave scattering is considered in detail. Expressions for 2×2 and 4×4 matrix scattering amplitudes and scattering kernels, as well as for various kinds of differential and total cross sections, are given. An eigenchannel formalism for the model is developed in the manner analogous to that presented in the author's recent work [R. Szmytkowski, Ann. Phys. (N.Y.) **311**, 503 (2004)] on scattering from short-range potentials. Eigenchannel representations of the scattering wave function, of a "final-state" wave function for photodetachment, as well as of outgoing and ingoing matrix Green functions, are derived. Formulas for matrix scattering amplitudes, scattering kernels, and cross sections expressed in terms of eigenphase shifts and eigenchannel spinor harmonics are presented. A possibility to formulate the zero-range potential model for Dirac particles in an alternative way is also discussed.

DOI: 10.1103/PhysRevA.71.052708 PACS number(s): 03.65.Nk, 03.65.Pm, 11.80.−m, 03.65.Ge

I. INTRODUCTION

Among the analytically soluble models used in nonrelativistic quantum mechanics, one of particularly wide applicability is the zero-range potential model (ZRPM). According to Demkov and Ostrovsky [1], this model was first used in 1934 in atomic physics by Fermi [2], in a paper concerning perturbation of spectral lines, and shortly thereafter by the same author [3] in nuclear physics, in a study of neutron scattering from chemically bound protons. At the same time, similar ideas appeared independently in a paper by Thomas [4] on the structure of H^3. Until the 1960s, the model was used primarily in nuclear physics [5]; it found also applications in the theory of multiple scattering of scalar waves [6] and in quantum statistical mechanics [7]. The wide use of the ZRPM in nonrelativistic atomic and molecular physics started in 1964, after publications of Demkov [8], Smirnov and Firsov [9], and Demkov and Drukarev [10]. Comprehensive presentations of the method and its numerous applications in nonrelativistic atomic and molecular theory, with bibliographies covering the period through the mid-1970s, may be found in the monograph by Demkov and Ostrovsky [1] and in the review by Drukarev [11]. Representative works on the subject published since 1980 are listed in Refs. [12,13].

The past three decades have seen a rapid growth of interest in the relativistic effects in atomic and molecular physics [14]. Consequently, it seems quite natural to consider applications of the ZRPM in the relativistic theory of atoms and molecules. It is therefore somewhat surprising that, after an extensive search of the literature, we have found only one relevant publication: in a paper published in 1977, Perel'man [15], extending the nonrelativistic work [10], discussed a Dirac particle bound in a field of a single zero-range potential and perturbed by a homogeneous electric field [16].

In the ZRPM for Schrödinger particles, interactions between a particle and pointlike targets (zero-range potentials) are modeled by imposing suitable limiting conditions on the particle's wave function at points where the targets are located; these conditions relate linearly the wave function and its spatial derivative [cf. Eq. (2.5) in Ref. [13]]. Conditions used by Perel'man [15] for a Dirac particle were of the same kind. Guided by experience gained during our earlier studies on the operator formulation of the *R*-matrix method for the Dirac equation [17], in the present paper we propose the ZRPM for Dirac particles differing from that suggested in Ref. [15]. The difference lies in the fact that we model interactions between a Dirac particle and pointlike targets by using *matrix* limiting conditions, relating linearly *upper and lower components* of the particle's bispinor wave function at the targets' locations. The formalism developed in this work is aimed at dealing with scattering and related continuum problems. Its extension to bound-state problems, as well as its applications to specific atomic and molecular processes, will be presented in later publications.

The structure of the paper is as follows. In Sec. II, we introduce the ZRPM for an unbound Dirac particle, prescribing a form of its wave function and subjecting this function to the aforementioned particular limiting conditions at points where zero-range potentials are located. In Sec. III, we consider the problem of scattering of an initially parallel monoenergetic beam of spin-polarized Dirac particles from a

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system of zero-range potentials. In particular, we derive general expressions for matrix scattering amplitudes and scattering kernels, as well as for various kinds of differential and total cross sections of interest. In Sec. IV, we define an auxiliary generalized matrix eigenproblem and use its solutions to introduce eigenchannels, eigenphase shifts, and eigenchannel bispinor and spinor harmonics in the manner analogous to that presented in Refs. [13,18,19]. In Sec. V A, we return to the general continuum problem discussed earlier in Sec. II. Exploiting the concepts and results of Sec. IV, we split the particle's wave function into two parts, one for which the target is ideally transparent and the remainder, which is a genuine scattering function. Then, in Sec. V B, the results of Sec. V A are applied to the monochromatic plane-wave scattering problem of Sec. III. Explicit expressions for matrix scattering amplitudes and scattering kernels in terms of eigenphase shifts and eigenchannel bispinor or spinor harmonics are obtained in Sec. V C. In Sec. V D, analogous results are derived for three kinds of total cross sections. Keeping in view planned applications of the relativistic ZRPM to atomic and molecular photodetachment problems, in Sec. VI we find a so-called "final-state" wave function, which asymptotically behaves as a plane wave superposed with a radially ingoing wave. Then, in Sec. VII we construct continuum outgoing and ingoing matrix Green functions for a Dirac particle in the presence of a system of zero-range potentials. Two brief examples are presented in Sec. VIII. We conclude in Sec. IX giving there an outlook on some planned extensions of the formalism developed in this work. The paper ends with several Appendixes. In particular, in Appendix A we transform the limiting conditions, built in the definition of our ZRPM for Dirac particles, to a form resembling counterpart conditions underlying the nonrelativistic ZRPM, while in Appendix G we provide a snapshot look at still another ZRPM for Dirac particles.

II. DEFINITION OF THE MODEL AND GENERAL CONSIDERATIONS

We are interested in the situation in which a Dirac particle of rest mass m and fixed real energy E (such that $|E|>mc^2$) scatters elastically from a system of $N\geqslant 1$ pointlike targets (zero-range potentials) located at the points $\{\mathbf{r}_n\}$. In the model proposed in this work, the time-independent four-component wave function $\Psi^{(+)}(E,\mathbf{r})$ describing this process is assumed to have the following form:

$$\Psi^{(+)}(E,\mathbf{r}) = \Phi(E,\mathbf{r}) + \sum_{n=1}^{N}\Psi_n^{(+)}(E,\mathbf{r}) \tag{2.1}$$

and to satisfy the Dirac equation

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathscr{I}]\Psi^{(+)}(E,\mathbf{r}) = 0$$

$$(\mathbf{r}\neq\mathbf{r}_n; n=1,\ldots,N) \tag{2.2}$$

everywhere in $\mathbb{R}^3$ *except at the points where the targets are located*. In Eq. (2.2), $\boldsymbol{\alpha}$ and β are standard 4×4 Dirac matrices [20] and $\mathscr{I}$ denotes the unit 4×4 matrix.

The first component of the right-hand side of Eq. (2.1), i.e., $\Phi(E,\mathbf{r})$, is an incident wave; it satisfies the free-particle Dirac equation everywhere in $\mathbb{R}^3$,

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathscr{I}]\Phi(E,\mathbf{r}) = 0 \quad (\mathbf{r}\in\mathbb{R}^3). \tag{2.3}$$

[Throughout this section, we shall be admitting that $\Phi(E,\mathbf{r})$ is *any* regular solution of Eq. (2.3).] It is evident from Eq. (2.3) that $\Phi(E,\mathbf{r})$ may be expressed as

$$\Phi(E,\mathbf{r}) = \begin{pmatrix} \phi_+(E,\mathbf{r}) \\ \dfrac{-ic\hbar}{E+mc^2}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\,\phi_+(E,\mathbf{r}) \end{pmatrix}, \tag{2.4}$$

where $\boldsymbol{\sigma}$ is the vector composed of the Pauli matrices, while the two-component spinor function $\phi_+(E,\mathbf{r})$ satisfies the free Klein-Gordon equation

$$[-c^2\hbar^2\boldsymbol{\nabla}^2 + (mc^2)^2 - E^2]\phi_+(E,\mathbf{r}) = 0 \quad (\mathbf{r}\in\mathbb{R}^3). \tag{2.5}$$

Each of the functions $\{\Psi_n^{(+)}(E,\mathbf{r})\}$, standing under the sum on the right-hand side of Eq. (2.1), satisfies the free-particle Dirac equation everywhere in $\mathbb{R}^3$ *except* at *one* of the points $\{\mathbf{r}_n\}$,

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathscr{I}]\Psi_n^{(+)}(E,\mathbf{r}) = 0 \quad (\mathbf{r}\neq\mathbf{r}_n) \tag{2.6}$$

and is explicitly given by

$$\Psi_n^{(+)}(E,\mathbf{r}) = \frac{1}{k}\begin{pmatrix} \chi_n^{(+)}(E) \\ -i\varepsilon k^{-1}\boldsymbol{\sigma}\chi_n^{(+)}(E)\cdot\boldsymbol{\nabla} \end{pmatrix}\frac{e^{ik|\mathbf{r}-\mathbf{r}_n|}}{|\mathbf{r}-\mathbf{r}_n|} \tag{2.7a}$$

or, equivalently,

$$\Psi_n^{(+)}(E,\mathbf{r}) = \begin{pmatrix} h_0^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)\chi_n^{(+)}(E) \\ \varepsilon h_1^{(+)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\chi_n^{(+)}(E) \end{pmatrix}, \tag{2.7b}$$

where $\{h_{0\text{ or }1}^{(+)}(z)\}$ [and $\{h_{0\text{ or }1}^{(-)}(z)\}$, to be used later] are defined in terms of the spherical Hankel functions of the first and second kinds [21] as

$$h_0^{(\pm)}(z) = \pm ih_0^{\binom{1}{2}}(z) = \frac{e^{\pm iz}}{z} \tag{2.8a}$$

and

$$h_1^{(\pm)}(z) = -h_1^{\binom{1}{2}}(z) = \frac{e^{\pm iz}}{z} \pm i\frac{e^{\pm iz}}{z^2}. \tag{2.8b}$$

In Eqs. (2.7a) and (2.7b), and also hereafter,

$$k = \operatorname{sgn}(E)\frac{\sqrt{E^2-(mc^2)^2}}{c\hbar} \tag{2.9}$$

is the particle's wave number (observe that it may assume positive as well as negative values),

$$\varepsilon = \sqrt{\frac{E-mc^2}{E+mc^2}}, \tag{2.10}$$

$\boldsymbol{\mu}_n(\mathbf{r})$ is the unit vector along the direction of $(\mathbf{r}-\mathbf{r}_n)$,

$$\boldsymbol{\mu}_n(\mathbf{r}) = \frac{\mathbf{r}-\mathbf{r}_n}{|\mathbf{r}-\mathbf{r}_n|}, \tag{2.11}$$

while $\chi_n^{(+)}(E)$ is some energy-dependent two-component spinor to be determined later.

Since for large $r=|\mathbf{r}|$ it holds that

$$h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|) \overset{r\to\infty}{\sim} \frac{e^{\pm ikr}}{kr} e^{\mp ik\mathbf{n}_r\cdot\mathbf{r}_n}, \tag{2.12a}$$

$$h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|) \overset{r\to\infty}{\sim} \frac{e^{\pm ikr}}{kr} e^{\mp ik\mathbf{n}_r\cdot\mathbf{r}_n}, \tag{2.12b}$$

and

$$\boldsymbol{\mu}_n(\mathbf{r}) \overset{r\to\infty}{\sim} \mathbf{n}_r, \tag{2.13}$$

where $\mathbf{n}_r=\mathbf{r}/r$, from Eq. (2.7b) one deduces that asymptotically the function $\Psi_n^{(+)}(E,\mathbf{r})$ satisfies the condition which may be written as

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_+ - \varepsilon\beta_+]\Psi_n^{(+)}(E,\mathbf{r}) = 0 \tag{2.14a}$$

or *equivalently* (cf. Appendix A)

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_- - \varepsilon^{-1}\beta_-]\Psi_n^{(+)}(E,\mathbf{r}) = 0, \tag{2.14b}$$

with

$$\boldsymbol{\alpha}_\pm = \beta_\pm\boldsymbol{\alpha}, \quad \beta_\pm = \tfrac{1}{2}(\mathscr{I} \pm \beta). \tag{2.15}$$

On the other hand, it follows from Eq. (2.7b) that for large r the radial component of the current density

$$\mathbf{j}_n^{(+)}(E,\mathbf{r}) = c\Psi_n^{(+)\dagger}(E,\mathbf{r})\boldsymbol{\alpha}\Psi_n^{(+)}(E,\mathbf{r}) \tag{2.16}$$

(the dagger denotes the matrix Hermitian conjugation) is

$$\mathbf{n}_r\cdot\mathbf{j}_n^{(+)}(E,\mathbf{r}) \overset{r\to\infty}{\sim} \frac{2c\varepsilon}{k^2r^2}\chi_n^{(+)\dagger}(E)\chi_n^{(+)}(E), \tag{2.17}$$

which is non-negative irrespective of the sign of E. Consequently, the conditions (2.14a) and (2.14b) are *outflow* (or *radiation*) conditions [22] and in the asymptotic zone each of the functions $\{\Psi_n^{(+)}(E,\mathbf{r})\}$, and thus also their superposition standing on the right-hand side of Eq. (2.1), describes a particular mode of *escape* of the Dirac particle.

To make our model complete, we have to build into it interactions between the scattered Dirac particle and individual targets. We do this by imposing the following limiting conditions:

$$\lim_{\mathbf{r}\to\mathbf{r}_n}[i(\mathbf{r}-\mathbf{r}_n)\cdot\boldsymbol{\alpha}_+ + \varepsilon(|\mathbf{r}-\mathbf{r}_n|\mathscr{K}_n + k^{-1}\beta_+)]\Psi^{(+)}(E,\mathbf{r}) = 0$$

$$(n=1,\ldots,N) \tag{2.18}$$

at the points where the targets are located; in Appendix B, we show that these conditions are natural generalizations of the limiting conditions used in the nonrelativistic ZRPM [cf. Eq. (2.5) of Ref. [13]]. The 4×4 matrices $\{\mathscr{K}_n\}$ appearing in Eq. (2.18) possess the property

$$\beta_+\mathscr{K}_n\beta_+ = \mathscr{K}_n \tag{2.19}$$

and are defined in terms of 2×2 matrices $\{K_n\}$ as

$$\mathscr{K}_n = \begin{pmatrix} K_n & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.20}$$

with zeros denoting 2×2 null matrices. In the Pauli basis (consisting of the unit 2×2 matrix I and the Pauli vector $\boldsymbol{\sigma}$), the matrices $\{K_n\}$ have the representations

$$K_n = \varkappa_n I + \boldsymbol{\kappa}_n\cdot\boldsymbol{\sigma}, \tag{2.21}$$

where

$$\varkappa_n = \tfrac{1}{2}\mathrm{Tr}[K_n], \tag{2.22a}$$

$$\boldsymbol{\kappa}_n = \tfrac{1}{2}\mathrm{Tr}[\boldsymbol{\sigma}K_n]. \tag{2.22b}$$

Henceforth, we shall be assuming that the matrices $\{K_n\}$ are Hermitian. It is then evident from Eqs. (2.22a) and (2.22b) that the scalars $\{\varkappa_n\}$ and the vectors $\{\boldsymbol{\kappa}_n\}$ are real. Although, for brevity, hereafter we shall not be marking this explicitly, we shall be admitting that $\{K_n\}$, thus also $\{\varkappa_n\}$ and $\{\boldsymbol{\kappa}_n\}$, may be energy-dependent.

It remains to explain how the spinors $\{\chi_n^{(+)}(E)\}$, in terms of which the outgoing waves $\{\Psi_n^{(+)}(E,\mathbf{r})\}$ have been defined in Eqs. (2.7a) and (2.7b), may be determined. Substituting Eq. (2.1) into Eq. (2.18), after making use of Eqs. (2.4), (2.7b), (2.8a), (2.8b), and (2.20), we find that $\{\chi_n^{(+)}(E)\}$ are solutions of the following linear algebraic system:

$$[K_n + iI]\chi_n^{(+)}(E) + \sum_{\substack{n'=1\\(n'\neq n)}}^{N} h_0^{(+)}(k|\mathbf{r}_n-\mathbf{r}_{n'}|)\chi_{n'}^{(+)}(E)$$

$$= -\phi_+(E,\mathbf{r}_n) \quad (n=1,\ldots,N). \tag{2.23}$$

Arranging $\{\chi_n^{(+)}(E)\}$ and $\{\phi_+(E,\mathbf{r}_n)\}$ into $2N$-component column vectors

$$\mathsf{a}^{(+)}(E) = (\chi_1^{(+)\mathrm{T}}(E)\cdots\chi_N^{(+)\mathrm{T}}(E))^{\mathrm{T}} \tag{2.24}$$

and

$$\mathsf{b}(E) = (\phi_+^{\mathrm{T}}(E,\mathbf{r}_1)\cdots\phi_+^{\mathrm{T}}(E,\mathbf{r}_N))^{\mathrm{T}}, \tag{2.25}$$

respectively, and introducing a $2N\times2N$ complex matrix $\mathsf{L}(E)$ with elements

$$L_{n\nu,n'\nu'}(E) = \{[K_n]_{\nu\nu'} + i\delta_{\nu\nu'}\}\delta_{nn'} + h_0^{(+)}(k|\mathbf{r}_n-\mathbf{r}_{n'}|)$$

$$\times\delta_{\nu\nu'}(1-\delta_{nn'}) \quad (n,n'=1,\ldots,N;\nu,\nu'=1,2), \tag{2.26}$$

we may rewrite the system (2.23) in the compact form

$$\mathsf{L}(E)\mathsf{a}^{(+)}(E) = -\mathsf{b}(E), \tag{2.27}$$

convenient for later purposes.

Apart from the matrix $\mathsf{L}(E)$, in considerations carried out in later sections, use will be made of the matrices

$$\mathsf{L}_H(E) = \tfrac{1}{2}[\mathsf{L}(E) + \mathsf{L}^\dagger(E)] \tag{2.28}$$

and

$$\mathsf{L}_A(E) = \tfrac{1}{2i}[\mathsf{L}(E) - \mathsf{L}^{\dagger}(E)] \tag{2.29}$$

being, respectively, the Hermitian and anti-Hermitian parts of $\mathsf{L}(E)$. Elements of these matrices are given by

$$[\mathsf{L}_H(E)]_{n\nu,n'\nu'} = [K_n]_{\nu\nu'}\delta_{nn'} - y_0(k|\mathbf{r}_n - \mathbf{r}_{n'}|)\delta_{\nu\nu'}(1-\delta_{nn'})$$

$$(n,n' = 1,\ldots,N;\nu,\nu' = 1,2) \tag{2.30}$$

and

$$[\mathsf{L}_A(E)]_{n\nu,n'\nu'} = \delta_{\nu\nu'}\delta_{nn'} + j_0(k|\mathbf{r}_n - \mathbf{r}_{n'}|)\delta_{\nu\nu'}(1-\delta_{nn'})$$

$$(n,n' = 1,\ldots,N;\nu,\nu' = 1,2), \tag{2.31}$$

with

$$j_0(z) = \frac{\sin z}{z}, \quad y_0(z) = -\frac{\cos z}{z} \tag{2.32}$$

being, respectively, the spherical Bessel and Neumann functions of order zero [21].

III. PLANE-WAVE SCATTERING

A. Preliminaries

Let the incoming wave (2.4) be a monochromatic spin-polarized Dirac plane wave propagating in the direction $\mathbf{n}_0$,

$$\Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0), \tag{3.1}$$

with

$$\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \begin{pmatrix} \eta(\boldsymbol{\nu}_0) \\ \varepsilon\mathbf{n}_0\cdot\boldsymbol{\sigma}\eta(\boldsymbol{\nu}_0)\end{pmatrix}. \tag{3.2}$$

In the definition (3.2), $\boldsymbol{\nu}_0$ stands for a unit vector, describing the initial particle's spin polarization in its rest frame, and $\eta(\boldsymbol{\nu}_0)$ is a two-component spinor such that

$$\boldsymbol{\nu}_0\cdot\boldsymbol{\sigma}\eta(\boldsymbol{\nu}_0) = +\eta(\boldsymbol{\nu}_0), \tag{3.3}$$

normalized to unity in the sense of

$$\eta^{\dagger}(\boldsymbol{\nu}_0)\eta(\boldsymbol{\nu}_0) = 1. \tag{3.4}$$

Then, it follows from Eqs. (2.1), (2.7b), (2.12a), (2.12b), and (2.13) that asymptotically the wave function $\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$ excited by the wave (3.1) is of the form

$$\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \overset{r\to\infty}{\sim} \operatorname*{asymp}_{r\to\infty} e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) + \frac{e^{ikr}}{r}\mathcal{F}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0), \tag{3.5}$$

where

$$\mathcal{F}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0) = \begin{pmatrix} f^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0) \\ \varepsilon\mathbf{n}_r\cdot\boldsymbol{\sigma}f^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)\end{pmatrix}, \tag{3.6}$$

with

$$f^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0) = \frac{1}{k}\sum_{n=1}^{N} e^{-ik\mathbf{n}_r\cdot\mathbf{r}_n}\chi_n^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0), \tag{3.7}$$

is the bispinor scattering amplitude. The spinors $\{\chi_n^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\}$ appearing in Eq. (3.7) are solutions to the system (2.23), with $\phi_+(E,\mathbf{r}_n)$ specified to be

$$\phi_+(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}_n) = e^{ik\mathbf{n}_0\cdot\mathbf{r}_n}\eta(\boldsymbol{\nu}_0). \tag{3.8}$$

B. Matrix scattering amplitudes and matrix scattering kernels

We define the 2×2 matrix scattering amplitude $F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)$ through the relationship

$$f^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0) = F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\eta(\boldsymbol{\nu}_0). \tag{3.9}$$

In terms of $F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)$, the 4×4 matrix scattering amplitude is defined as

$$\mathscr{F}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \frac{1}{1+\varepsilon^2} \times\begin{pmatrix} F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) & \varepsilon F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\mathbf{n}_0\cdot\boldsymbol{\sigma} \\ \varepsilon\mathbf{n}_r\cdot\boldsymbol{\sigma}F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) & \varepsilon^2\mathbf{n}_r\cdot\boldsymbol{\sigma}F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\mathbf{n}_0\cdot\boldsymbol{\sigma}\end{pmatrix}. \tag{3.10}$$

It is readily verifiable that it holds that

$$\mathcal{F}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0) = \mathscr{F}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0). \tag{3.11}$$

[Observe that while Eq. (3.11) follows from Eq. (3.10), the converse is *not* true.]

Having defined the matrix scattering amplitudes, we introduce 2×2 and 4×4 matrix scattering kernels through the relationships

$$S^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \delta^{(2)}(\mathbf{n}_r - \mathbf{n}_0)I + \frac{ik}{2\pi}F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) \tag{3.12}$$

and

$$\mathscr{S}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \delta^{(2)}(\mathbf{n}_r - \mathbf{n}_0)\mathscr{I} + \frac{ik}{2\pi}\mathscr{F}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0), \tag{3.13}$$

respectively, where $\delta^{(2)}(\mathbf{n}-\mathbf{n}')$ is the Dirac delta function on the unit sphere. Since

$$\operatorname*{asymp}_{r\to\infty} e^{ik\mathbf{n}_0\cdot\mathbf{r}} = \frac{2\pi i}{k}\left[\frac{e^{-ikr}}{r}\delta^{(2)}(\mathbf{n}_r + \mathbf{n}_0) - \frac{e^{ikr}}{r}\delta^{(2)}(\mathbf{n}_r - \mathbf{n}_0)\right], \tag{3.14}$$

from Eqs. (3.5), (3.1), (3.11), and (3.13) we infer that

$$\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \overset{r\to\infty}{\sim} \frac{2\pi i}{k}\left[\frac{e^{-ikr}}{r}\delta^{(2)}(\mathbf{n}_r+\mathbf{n}_0)\mathscr{I} - \frac{e^{ikr}}{r}\mathscr{S}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\right]\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0). \tag{3.15}$$

Similarly, if we define $\psi_+^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$ to be the upper component of $\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$,

$$\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = \begin{pmatrix} \psi_+^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \\ -i\varepsilon k^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\,\psi_+^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \end{pmatrix}, \tag{3.16}$$

and exploit the asymptotic relation (3.14), from Eqs. (3.5), (3.1), (3.2), (3.6), (3.9), and (3.12) we obtain

$$\psi_+^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \overset{r\to\infty}{\sim} \frac{2\pi i}{k}\left[\frac{e^{-ikr}}{r}\delta^{(2)}(\mathbf{n}_r+\mathbf{n}_0)I - \frac{e^{ikr}}{r}S^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\right]\eta(\boldsymbol{\nu}_0). \tag{3.17}$$

C. Differential cross sections

The angular distribution of scattered particles in the asymptotic zone may be characterized by a differential cross section defined as

$$\frac{d^2Q(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r} = \lim_{r\to\infty} r^2\frac{j_{scat}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})}{j_{inc}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})}, \tag{3.18}$$

where

$$j_{scat}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = c\Psi^{\dagger}_{scat}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})\mathbf{n}_r\cdot\boldsymbol{\alpha}\Psi_{scat}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}), \tag{3.19}$$

with

$$\Psi_{scat}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = \Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) - \Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}), \tag{3.20}$$

is the radial current density in the scattered wave, while

$$j_{inc}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = c\Phi^{\dagger}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})\mathbf{n}_0\cdot\boldsymbol{\alpha}\Phi(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = 2c\varepsilon \tag{3.21}$$

is the current density in the incident plane wave (3.1) [on the right-hand side of Eq. (3.21), use has been made of the normalization condition (3.4)]. On combining Eqs. (3.18)–(3.21), (3.5), and (3.6), one obtains

$$\frac{d^2Q(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r} = f^{(+)\dagger}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)f^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0), \tag{3.22}$$

and further, after exploiting Eq. (3.9),

$$\frac{d^2Q(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r} = \mathrm{Tr}[F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\varrho(\boldsymbol{\nu}_0)F^{(+)\dagger}(E,\mathbf{n}_r,\mathbf{n}_0)], \tag{3.23}$$

where

$$\varrho(\boldsymbol{\nu}_0) = \eta(\boldsymbol{\nu}_0)\eta^{\dagger}(\boldsymbol{\nu}_0) = \tfrac{1}{2}[I+\boldsymbol{\nu}_0\cdot\boldsymbol{\sigma}] \tag{3.24}$$

is the 2×2 spin density matrix for the incident wave. Since

$$\oint_{4\pi} d^2\boldsymbol{\nu}_0\varrho(\boldsymbol{\nu}_0) = 2\pi I, \tag{3.25}$$

a differential cross section averaged over all orientations of $\boldsymbol{\nu}_0$,

$$\frac{d^2Q(E,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r} = \frac{1}{4\pi}\oint_{4\pi} d^2\boldsymbol{\nu}_0\frac{d^2Q(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r}, \tag{3.26}$$

is given by

$$\frac{d^2Q(E,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r} = \frac{1}{2}\mathrm{Tr}[F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)F^{(+)\dagger}(E,\mathbf{n}_r,\mathbf{n}_0)]. \tag{3.27}$$

D. Total cross sections

Three kinds of total cross sections will be considered in this work. The first one is a total cross section for a fixed direction of incidence $\mathbf{n}_0$ and a fixed initial spin orientation $\boldsymbol{\nu}_0$, defined as

$$Q(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \oint_{4\pi} d^2\mathbf{n}_r\frac{d^2Q(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r}. \tag{3.28}$$

The second total cross section of interest is an average of one defined in Eq. (3.28) over all initial spin orientations,

$$Q(E,\mathbf{n}_0) = \frac{1}{4\pi}\oint_{4\pi} d^2\boldsymbol{\nu}_0 Q(E,\boldsymbol{\nu}_0,\mathbf{n}_0). \tag{3.29a}$$

It follows from Eq. (3.26) that it may be also found from the formula

$$Q(E,\mathbf{n}_0) = \oint_{4\pi} d^2\mathbf{n}_r\frac{d^2Q(E,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r}, \tag{3.29b}$$

with the integrand given by Eq. (3.27). Finally, the third kind of the total cross section we shall be interested in is obtained by averaging one defined in Eq. (3.29a) over all directions of incidence,

$$Q(E) = \frac{1}{4\pi}\oint_{4\pi} d^2\mathbf{n}_0 Q(E,\mathbf{n}_0). \tag{3.30}$$

It seems worthwhile to note that the cross section (3.28) may be expressed as

$$Q(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \frac{4\pi}{k^2}\mathsf{a}^{(+)\dagger}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\mathsf{L}_A(E)\mathsf{a}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0). \tag{3.31}$$

Here $\mathsf{a}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)$ is the $2N$-component column vector composed of the spinors $\{\chi_n^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\}$ [cf. Eq. (2.24)] and $\mathsf{L}_A(E)$ has been defined in Eq. (2.29). To prove the validity of

Eq. (3.31), we combine the definition (3.28) with Eqs. (3.22) and (3.7), which yields

$$Q(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \frac{1}{k^2}\sum_{n,n'=1}^{N} \chi_n^{(+)\dagger}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\chi_{n'}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) \times \oint_{4\pi} d^2\mathbf{n}_r e^{ik\mathbf{n}_r\cdot(\mathbf{r}_n-\mathbf{r}_{n'})}. \qquad (3.32)$$

Applying the well-known formula

$$\oint_{4\pi} d^2\mathbf{n} e^{ik\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}')} = 4\pi j_0(k|\mathbf{r}-\mathbf{r}'|) \qquad (3.33)$$

to the integral on the right-hand side of Eq. (3.32) gives

$$\oint_{4\pi} d^2\mathbf{n}_r e^{ik\mathbf{n}_r\cdot(\mathbf{r}_n-\mathbf{r}_{n'})} = 4\pi[\delta_{nn'} + j_0(k|\mathbf{r}_n-\mathbf{r}_{n'}|)(1-\delta_{nn'})]. \qquad (3.34)$$

Hence, after invoking Eq. (2.31), one arrives at Eq. (3.31).

IV. EIGENCHANNELS FOR SYSTEMS OF ZERO-RANGE SCATTERERS

A. An auxiliary matrix eigenproblem

Consider an auxiliary weighted eigenproblem

$$\mathsf{L}_H(E)\mathsf{x}_\gamma(E) = \lambda_\gamma(E)\mathsf{L}_A(E)\mathsf{x}_\gamma(E), \qquad (4.1)$$

with the weight matrix $\mathsf{L}_A(E)$ defined in Eq. (2.29); in Eq. (4.1), $\lambda_\gamma(E)$ is an eigenvalue and $\mathsf{x}_\gamma(E)$ is an associated eigenvector. We already know that both matrices appearing in the problem (4.1) are Hermitian; moreover, it is proved in Appendix C that $\mathsf{L}_A(E)$ is *at least positive semidefinite*. If $\mathsf{L}_A(E)$ is strictly positive definite, then, following a routine procedure, it may be shown that all $2N$ eigenvalues to the problem (4.1) are real and that eigenvectors associated with different eigenvalues are orthogonal in the sense of

$$\mathsf{x}_\gamma^\dagger(E)\mathsf{L}_A(E)\mathsf{x}_{\gamma'}(E) = 0 \quad [\lambda_\gamma(E) \neq \lambda_{\gamma'}(E)]. \qquad (4.2)$$

It is always possible to choose eigenvectors associated with degenerate eigenvalues (if there are any) so that the orthogonality relation (4.2) holds for all eigenvectors. If, in addition, the eigenvectors $\{\mathsf{x}_\gamma(E)\}$ are normalized to unity in the sense of

$$\mathsf{x}_\gamma^\dagger(E)\mathsf{L}_A(E)\mathsf{x}_\gamma(E) = 1, \qquad (4.3)$$

then one has the weighted orthonormality relation

$$\mathsf{x}_\gamma^\dagger(E)\mathsf{L}_A(E)\mathsf{x}_{\gamma'}(E) = \delta_{\gamma\gamma'}, \qquad (4.4)$$

which will play an important role in later considerations. Finally, the Hermiticity of $\mathsf{L}_H(E)$ and $\mathsf{L}_A(E)$ implies that the eigenvectors $\{\mathsf{x}_\gamma(E)\}$ form a complete set; the corresponding closure relation, derived in Appendix D, is

$$\sum_{\gamma=1}^{2N} \mathsf{x}_\gamma(E)\mathsf{x}_\gamma^\dagger(E)\mathsf{L}_A(E) = \mathsf{L}_A(E)\sum_{\gamma=1}^{2N} \mathsf{x}_\gamma(E)\mathsf{x}_\gamma^\dagger(E) = \mathsf{I}, \qquad (4.5)$$

where I is the unit $2N\times 2N$ matrix. The case when the matrix $\mathsf{L}_A(E)$ is positive semidefinite may be treated as a limiting case of $\mathsf{L}_A(E)$ positive definite.

For some systems of zero-range potentials, due to their inherent geometrical or "dynamical" symmetries, the problem of solving Eq. (4.1) may appear to be reducible to that of solving several lower-dimensional eigensystems. Two examples of such situations, being consequences of particular forms of the "interaction" matrices $\{K_n\}$, are considered in Appendix E.

B. Eigenchannels, eigenphase shifts, eigenchannel bispinor and spinor harmonics

Let us express the eigenvectors $\{\mathsf{x}_\gamma(E)\}$ in terms of two-component spinors $\{\xi_{n\gamma}(E)\}$,

$$\mathsf{x}_\gamma(E) = (\xi_{1\gamma}^{\mathrm{T}}(E)\cdots\xi_{N\gamma}^{\mathrm{T}}(E))^{\mathrm{T}}. \qquad (4.6)$$

With these spinors, we define $2N$ *eigenchannels*

$$\mathcal{X}_\gamma(E,\mathbf{r}) = \sqrt{\frac{k(E+mc^2)}{4\pi c^2\hbar^2}}\sum_{n=1}^{N}\left[\begin{pmatrix} y_0(k|\mathbf{r}-\mathbf{r}_n|)\xi_{n\gamma}(E) \\ i\varepsilon y_1(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E)\end{pmatrix} + \lambda_\gamma(E)\begin{pmatrix} j_0(k|\mathbf{r}-\mathbf{r}_n|)\xi_{n\gamma}(E) \\ i\varepsilon j_1(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E)\end{pmatrix}\right], \qquad (4.7)$$

where

$$j_1(z) = -\frac{\cos z}{z} + \frac{\sin z}{z^2}, \quad y_1(z) = -\frac{\sin z}{z} - \frac{\cos z}{z^2} \qquad (4.8)$$

are, respectively, spherical Bessel and Neumann functions of order 1 [21]. [The factor in front of the sum in Eq. (4.7) has been introduced to enforce compatibility of the asymptotic formula (4.13) with Eq. (7.29) of Ref. [18].] Evidently, the eigenchannels satisfy the free-particle Dirac equation everywhere in $\mathbb{R}^3$ except at the points $\{\mathbf{r}_n\}$,

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathcal{I}]\mathcal{X}_\gamma(E,\mathbf{r}) = 0 \quad (\mathbf{r}\neq\mathbf{r}_n; n=1,\ldots,N) \qquad (4.9)$$

[cf. Eq. (2.2)]. With the aid of Eqs. (4.1), (4.6), (2.30), and (2.31), it is also readily verifiable that at the points $\{\mathbf{r}_n\}$ the eigenchannels obey the limiting conditions

$$\lim_{\mathbf{r}\to\mathbf{r}_n}[i(\mathbf{r}-\mathbf{r}_n)\cdot\boldsymbol{\alpha}_+ + \varepsilon(|\mathbf{r}-\mathbf{r}_n|\mathscr{K}_n + k^{-1}\beta_+)]\mathcal{X}_\gamma(E,\mathbf{r}) = 0 \quad (n=1,\ldots,N) \tag{4.10}$$

[cf. Eq. (2.18)].

Introducing real *eigenphase shifts*, related to the real [cf. the discussion preceding Eq. (4.2)] eigenvalues of the system (4.1) through

$$\lambda_\gamma(E) = -\cot\,\delta_\gamma(E), \tag{4.11}$$

we may rewrite Eq. (4.7) in the form

$$\mathcal{X}_\gamma(E,\mathbf{r}) = -\,\mathrm{sgn}(E)\sqrt{\frac{E+mc^2}{4\pi c^2\hbar^2 k}}\frac{1}{\sin\,\delta_\gamma(E)} \times \sum_{n=1}^{N}\begin{pmatrix}\xi_{n\gamma}(E)\\ -i\varepsilon k^{-1}\boldsymbol{\sigma}\xi_{n\gamma}(E)\cdot\boldsymbol{\nabla}\end{pmatrix}\frac{\sin[k|\mathbf{r}-\mathbf{r}_n|+\delta_\gamma(E)]}{|\mathbf{r}-\mathbf{r}_n|}. \tag{4.12}$$

Making in either of Eqs. (4.7) or (4.12) the limiting passage $r\to\infty$, we arrive at

$$\mathcal{X}_\gamma(E,\mathbf{r}) \overset{r\to\infty}{\sim} \mathrm{sgn}(E)\sqrt{\frac{E}{2c^2\hbar^2 k}}\frac{1}{i\,\sin\,\delta_\gamma(E)}\left[\frac{e^{-ikr-i\delta_\gamma(E)}}{r} \times \mathcal{Y}_\gamma(E,-\mathbf{n}_r) - \frac{e^{ikr+i\delta_\gamma(E)}}{r}\mathcal{Y}_\gamma(E,\mathbf{n}_r)\right], \tag{4.13}$$

where

$$\mathcal{Y}_\gamma(E,\mathbf{n}_r) = \frac{1}{\sqrt{4\pi(1+\varepsilon^2)}}\sum_{n=1}^{N} e^{-ik\mathbf{n}_r\cdot\mathbf{r}_n}\begin{pmatrix}\xi_{n\gamma}(E)\\ \varepsilon\mathbf{n}_r\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E)\end{pmatrix} \tag{4.14}$$

are *eigenchannel bispinor harmonics*. Defining two-component *eigenchannel spinor harmonics*

$$\Upsilon_\gamma(E,\mathbf{n}_r) = \frac{1}{\sqrt{4\pi}}\sum_{n=1}^{N} e^{-ik\mathbf{n}_r\cdot\mathbf{r}_n}\xi_{n\gamma}(E), \tag{4.15}$$

we may rewrite the definition (4.14) in the following more compact form:

$$\mathcal{Y}_\gamma(E,\mathbf{n}_r) = \frac{1}{\sqrt{1+\varepsilon^2}}\begin{pmatrix}\Upsilon_\gamma(E,\mathbf{n}_r)\\ \varepsilon\mathbf{n}_r\cdot\boldsymbol{\sigma}\Upsilon_\gamma(E,\mathbf{n}_r)\end{pmatrix}. \tag{4.16}$$

In Appendix F, we show that the spinor harmonics (4.15) form an orthonormal set on the unit sphere, i.e., it holds that

$$\oint_{4\pi} d^2\mathbf{n}_r \Upsilon^\dagger_\gamma(E,\mathbf{n}_r)\Upsilon_{\gamma'}(E,\mathbf{n}_r) = \delta_{\gamma\gamma'}. \tag{4.17}$$

Combining this with Eq. (4.16), we see that the bispinor harmonics are also orthonormal,

$$\oint_{4\pi} d^2\mathbf{n}_r \mathcal{Y}^\dagger_\gamma(E,\mathbf{n}_r)\mathcal{Y}_{\gamma'}(E,\mathbf{n}_r) = \delta_{\gamma\gamma'}. \tag{4.18}$$

V. APPLICATIONS OF EIGENCHANNELS AND RELATED OBJECTS IN SCATTERING PROBLEMS

A. Scattering wave function: General case

Since the eigenchannel vectors form the complete set, we may use them as a basis for expanding the vector $\mathsf{a}^{(+)}(E)$ defined in Eq. (2.24),

$$\mathsf{a}^{(+)}(E) = \sum_{\gamma'=1}^{2N} a^{(+)}_{\gamma'}(E)\mathsf{x}_{\gamma'}(E). \tag{5.1}$$

Substituting this expansion into Eq. (2.27), making use of the fact that

$$\mathsf{L}(E) = \mathsf{L}_H(E) + i\mathsf{L}_A(E), \tag{5.2}$$

and exploiting the eigenequation (4.1), we find

$$\sum_{\gamma'=1}^{2N}[\lambda_{\gamma'}(E)+i]a^{(+)}_{\gamma'}(E)\mathsf{L}_A(E)\mathsf{x}_{\gamma'}(E) = -\,\mathsf{b}(E). \tag{5.3}$$

Premultiplying Eq. (5.3) with $\mathsf{x}^\dagger_\gamma(E)$ and using the weighted orthonormality relation (4.4) yields the expansion coefficient $a^{(+)}_\gamma(E)$ in the form

$$a^{(+)}_\gamma(E) = -\frac{1}{\lambda_\gamma(E)+i}\mathsf{x}^\dagger_\gamma(E)\mathsf{b}(E). \tag{5.4}$$

Substituting this back into Eq. (5.1) gives

$$\mathsf{a}^{(+)}(E) = -\sum_{\gamma=1}^{2N}\frac{1}{\lambda_\gamma(E)+i}[\mathsf{x}^\dagger_\gamma(E)\mathsf{b}(E)]\mathsf{x}_\gamma(E) \tag{5.5a}$$

or, equivalently, in terms of the eigenphase shifts,

$$\mathsf{a}^{(+)}(E) = \sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\,\delta_\gamma(E)[\mathsf{x}^\dagger_\gamma(E)\mathsf{b}(E)]\mathsf{x}_\gamma(E). \tag{5.5b}$$

Hence, for the spinor components of $\mathsf{a}^{(+)}(E)$ we have

$$\chi^{(+)}_n(E) = \sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\,\delta_\gamma(E)\left[\sum_{n'=1}^{N}\xi^\dagger_{n'\gamma}(E)\phi_+(E,\mathbf{r}_{n'})\right]\xi_{n\gamma}(E). \tag{5.6}$$

Combining Eqs. (2.1), (2.7b), and (5.6) results in the following expression for the total wave function induced by the wave (2.4),

$$\Psi^{(+)}(E,\mathbf{r}) = \Phi(E,\mathbf{r}) + \sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\,\delta_\gamma(E) \times\left[\sum_{n'=1}^{N}\xi^\dagger_{n'\gamma}(E)\phi_+(E,\mathbf{r}_{n'})\right]\Theta^{(+)}_\gamma(E,\mathbf{r}), \tag{5.7}$$

where

$$\Theta^{(\pm)}_\gamma(E,\mathbf{r}) = \begin{pmatrix}\theta^{(\pm)}_{0\gamma}(E,\mathbf{r})\\ \pm\varepsilon\theta^{(\pm)}_{1\gamma}(E,\mathbf{r})\end{pmatrix}, \tag{5.8}$$

with

$$\theta_{0\gamma}^{(\pm)}(E,\mathbf{r}) = \sum_{n=1}^{N} h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\xi_{n\gamma}(E) \tag{5.9}$$

and

$$\theta_{1\gamma}^{(\pm)}(E,\mathbf{r}) = \sum_{n=1}^{N} h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E). \tag{5.10}$$

[For the sake of brevity, two kinds of functions have been defined in Eqs. (5.8)–(5.10); those with lower superscripts will find applications in later sections.]

Manipulating Eqs. (5.7)–(5.10) with the aid of the relations

$$h_0^{(\pm)}(z) = \pm i j_0(z) - y_0(z), \tag{5.11a}$$

$$h_1^{(\pm)}(z) = - j_1(z) \mp i y_1(z), \tag{5.11b}$$

it is possible to split the function $\Psi^{(+)}(E,\mathbf{r})$ according to

$$\Psi^{(+)}(E,\mathbf{r}) = \Psi_{nint}(E,\mathbf{r}) + \Psi_{int}^{(+)}(E,\mathbf{r}), \tag{5.12}$$

with

$$\Psi_{nint}(E,\mathbf{r}) = \Phi(E,\mathbf{r}) - \sum_{\gamma=1}^{2N}\left[\sum_{n'=1}^{N}\xi_{n'\gamma}^{\dagger}(E)\phi_+(E,\mathbf{r}_{n'})\right] \times \sum_{n=1}^{N}\begin{pmatrix} j_0(k|\mathbf{r}-\mathbf{r}_n|)\xi_{n\gamma}(E) \\ i\varepsilon j_1(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E)\end{pmatrix} \tag{5.13}$$

and

$$\Psi_{int}^{(+)}(E,\mathbf{r}) = -\sqrt{\frac{4\pi c^2\hbar^2}{k(E+mc^2)}}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\delta_\gamma(E) \times\left[\sum_{n'=1}^{N}\xi_{n'\gamma}^{\dagger}(E)\phi_+(E,\mathbf{r}_{n'})\right]\mathcal{X}_\gamma(E,\mathbf{r}). \tag{5.14}$$

Evidently, the function $\Psi_{nint}(E,\mathbf{r})$ satisfies the free-particle Dirac equation *everywhere* in $\mathbb{R}^3$,

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathscr{I}]\Psi_{nint}(E,\mathbf{r}) = 0 \quad (\mathbf{r}\in\mathbb{R}^3). \tag{5.15}$$

This means that it is that part of $\Psi^{(+)}(E,\mathbf{r})$ which does *not* experience scattering, i.e., for which the target is ideally transparent. The remainder $\Psi_{int}^{(+)}(E,\mathbf{r})$ is the superposition of the eigenchannels (4.7) and therefore obeys the free-particle Dirac equation

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathscr{I}]\Psi_{int}^{(+)}(E,\mathbf{r}) = 0 \quad (\mathbf{r}\neq\mathbf{r}_n; n=1,\ldots,N) \tag{5.16}$$

[cf. Eqs. (2.2) and (4.9)] everywhere in $\mathbb{R}^3$ *except* at the target locations, where it is constrained to satisfy the limiting conditions

$$\lim_{\mathbf{r}\to\mathbf{r}_n}[i(\mathbf{r}-\mathbf{r}_n)\cdot\boldsymbol{\alpha}_+ + \varepsilon(|\mathbf{r}-\mathbf{r}_n|\mathscr{K}_n + k^{-1}\beta_+)]\Psi_{int}^{(+)}(E,\mathbf{r}) = 0 \quad (n=1,\ldots,N) \tag{5.17}$$

[cf. Eqs. (2.18) and (4.10)]. Consequently, $\Psi_{int}^{(+)}(E,\mathbf{r})$ is essentially that part of $\Psi^{(+)}(E,\mathbf{r})$ which describes the scattering process.

We conclude this thread observing that two alternative expressions for the noninteracting wave $\Psi_{nint}(E,\mathbf{r})$, derivable from Eq. (5.13), are

$$\Psi_{nint}(E,\mathbf{r}) = \Phi(E,\mathbf{r}) - \frac{1}{k}\sum_{\gamma=1}^{2N}\left[\sum_{n'=1}^{N}\xi_{n'\gamma}^{\dagger}(E)\phi_+(E,\mathbf{r}_{n'})\right] \times\sum_{n=1}^{N}\begin{pmatrix}\xi_{n\gamma}(E) \\ -i\varepsilon k^{-1}\boldsymbol{\sigma}\xi_{n\gamma}(E)\cdot\boldsymbol{\nabla}\end{pmatrix}\frac{\sin k|\mathbf{r}-\mathbf{r}_n|}{|\mathbf{r}-\mathbf{r}_n|} \tag{5.18a}$$

and

$$\Psi_{nint}(E,\mathbf{r}) = \Phi(E,\mathbf{r}) - \sqrt{\frac{1+\varepsilon^2}{4\pi}} \times\sum_{\gamma=1}^{2N}\left[\sum_{n'=1}^{N}\xi_{n'\gamma}^{\dagger}(E)\phi_+(E,\mathbf{r}_{n'})\right] \times\oint_{4\pi}d^2\mathbf{n}\,\mathcal{Y}_\gamma(E,\mathbf{n})e^{ik\mathbf{n}\cdot\mathbf{r}}. \tag{5.18b}$$

B. Scattering wave function: Plane-wave scattering

If the incoming wave is the plane wave (3.1), Eqs. (5.6), (3.8), and (4.15) yield

$$\begin{aligned}\chi_n^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) &= \sqrt{4\pi}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\delta_\gamma(E)\Upsilon_\gamma^{\dagger}(E,\mathbf{n}_0)\boldsymbol{\eta}(\boldsymbol{\nu}_0)\xi_{n\gamma}(E) \\ &= \sqrt{\frac{4\pi}{1+\varepsilon^2}}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\delta_\gamma(E)\mathcal{Y}_\gamma^{\dagger}(E,\mathbf{n}_0) \\ &\quad\times\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\xi_{n\gamma}(E).\end{aligned} \tag{5.19}$$

Further, application of Eqs. (3.2) and (4.14) transforms the total wave function (5.7) into

$$\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) + \sqrt{\frac{4\pi}{1+\varepsilon^2}}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\delta_\gamma(E) \times\mathcal{Y}_\gamma^{\dagger}(E,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\Theta_\gamma^{(+)}(E,\mathbf{r}). \tag{5.20}$$

In the same manner, the genuine scattering wave function (5.14) goes over into

$$\Psi_{int}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = -\sqrt{\frac{8\pi^2c^2\hbar^2}{kE}}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)} \times \sin\delta_\gamma(E)\mathcal{Y}_\gamma^\dagger(E,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\mathcal{X}_\gamma(E,\mathbf{r}), \tag{5.21}$$

while the noninteracting wave (5.13) may be rewritten in either of the following three equivalent forms:

$$\Psi_{nint}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) - \sqrt{\frac{4\pi}{1+\varepsilon^2}}\sum_{\gamma=1}^{2N}\mathcal{Y}_\gamma^\dagger(E,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) \times \sum_{n=1}^{N}\begin{pmatrix} j_0(k|\mathbf{r}-\mathbf{r}_n|)\xi_{n\gamma}(E) \\ i\varepsilon j_1(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E)\end{pmatrix}, \tag{5.22a}$$

$$\Psi_{nint}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) - \frac{1}{k}\sqrt{\frac{4\pi}{1+\varepsilon^2}}\sum_{\gamma=1}^{2N}\mathcal{Y}_\gamma^\dagger(E,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) \times \sum_{n=1}^{N}\begin{pmatrix} \xi_{n\gamma}(E) \\ -i\varepsilon k^{-1}\boldsymbol{\sigma}\xi_{n\gamma}(E)\cdot\boldsymbol{\nabla}\end{pmatrix}\frac{\sin k|\mathbf{r}-\mathbf{r}_n|}{|\mathbf{r}-\mathbf{r}_n|}, \tag{5.22b}$$

$$\Psi_{nint}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = \oint_{4\pi} d^2\mathbf{n}\mathscr{P}(E,\mathbf{n},\mathbf{n}_0)e^{ik\mathbf{n}\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0), \tag{5.22c}$$

with

$$\mathscr{P}(E,\mathbf{n},\mathbf{n}_0) = \delta^{(2)}(\mathbf{n}-\mathbf{n}_0)\mathscr{I} - \sum_{\gamma=1}^{2N}\mathcal{Y}_\gamma(E,\mathbf{n})\mathcal{Y}_\gamma^\dagger(E,\mathbf{n}_0) \tag{5.23}$$

[we note parenthetically that the kernel $\mathscr{P}(E,\mathbf{n},\mathbf{n}_0)$ is Hermitian and idempotent, so it is a projecting kernel]. The representation in Eq. (5.22b) is suited for making comparison with its nonrelativistic analogue [cf. Eq. (4.16) of Ref. [13]], while that in Eq. (5.22c) is particularly convenient for investigating the asymptotics of $\Psi_{nint}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$. Indeed, on using in Eq. (5.22c) the asymptotic formula (3.14), we obtain

$$\Psi_{nint}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \overset{r\to\infty}{\sim} \frac{2\pi i}{k}\left[\frac{e^{-ikr}}{r}\mathscr{P}(E,-\mathbf{n}_r,\mathbf{n}_0) - \frac{e^{ikr}}{r}\mathscr{P}(E,\mathbf{n}_r,\mathbf{n}_0)\right]\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0). \tag{5.24}$$

Similarly, for the asymptotic form of the interacting wave (5.21) we find

$$\Psi_{int}^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \overset{r\to\infty}{\sim} \frac{2\pi i}{k}\left[\frac{e^{-ikr}}{r}[\delta^{(2)}(\mathbf{n}_r+\mathbf{n}_0)\mathscr{I} - \mathscr{P}(E,-\mathbf{n}_r,\mathbf{n}_0)] - \frac{e^{ikr}}{r}\mathscr{S}_{red}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\right]\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0), \tag{5.25}$$

where

$$\mathscr{S}_{red}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \sum_{\gamma=1}^{2N} e^{2i\delta_\gamma(E)}\mathcal{Y}_\gamma(E,\mathbf{n}_r)\mathcal{Y}_\gamma^\dagger(E,\mathbf{n}_0) \tag{5.26}$$

is the 4×4 *reduced* scattering kernel.

C. Matrix scattering amplitudes and matrix scattering kernels

Combining Eqs. (3.7) and (5.19) gives

$$f^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0) = \frac{4\pi}{k}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\delta_\gamma(E) \times \Upsilon_\gamma(E,\mathbf{n}_r)\Upsilon_\gamma^\dagger(E,\mathbf{n}_0)\boldsymbol{\eta}(\boldsymbol{\nu}_0). \tag{5.27}$$

Hence, upon invoking Eq. (3.9), we deduce the expansion

$$F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \frac{4\pi}{k}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\delta_\gamma(E)\Upsilon_\gamma(E,\mathbf{n}_r)\Upsilon_\gamma^\dagger(E,\mathbf{n}_0). \tag{5.28}$$

Substituting this result into Eq. (3.10) and making use of Eq. (4.16) yields the analogous expansion

$$\mathscr{F}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \frac{4\pi}{k}\sum_{\gamma=1}^{2N} e^{i\delta_\gamma(E)}\sin\delta_\gamma(E)\mathcal{Y}_\gamma(E,\mathbf{n}_r)\mathcal{Y}_\gamma^\dagger(E,\mathbf{n}_0). \tag{5.29}$$

Further, exploiting the expansions (5.28) and (5.29) in the definitions (3.12) and (3.13) leads to the following representations of the matrix scattering kernels:

$$S^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \delta^{(2)}(\mathbf{n}_r-\mathbf{n}_0)I + \sum_{\gamma=1}^{2N}[e^{2i\delta_\gamma(E)}-1] \times\Upsilon_\gamma(E,\mathbf{n}_r)\Upsilon_\gamma^\dagger(E,\mathbf{n}_0), \tag{5.30}$$

$$\mathscr{S}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \delta^{(2)}(\mathbf{n}_r-\mathbf{n}_0)\mathscr{I} + \sum_{\gamma=1}^{2N}[e^{2i\delta_\gamma(E)}-1] \times\mathcal{Y}_\gamma(E,\mathbf{n}_r)\mathcal{Y}_\gamma^\dagger(E,\mathbf{n}_0). \tag{5.31}$$

Finally, from Eqs. (5.31), (5.26), and (5.23) we obtain the relationship

$$\mathscr{S}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0) = \mathscr{P}(E,\mathbf{n}_r,\mathbf{n}_0) + \mathscr{S}_{red}^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0). \tag{5.32}$$

The representations (5.29) and (5.31) may be used for various purposes. For instance, with the aid of the latter one immediately proves that the scattering kernel $\mathscr{S}^{(+)}(E,\mathbf{n},\mathbf{n}')$ is unitary, because one has

$$\oint_{4\pi} d^2\mathbf{n}'' \mathscr{S}^{(+)}(E,\mathbf{n},\mathbf{n}'')\mathscr{S}^{(+)\dagger}(E,\mathbf{n}',\mathbf{n}'') = \delta^{(2)}(\mathbf{n}-\mathbf{n}')\mathscr{I} \tag{5.33a}$$

and

$$\oint_{4\pi} d^2\mathbf{n}'' \mathscr{S}^{(+)\dagger}(E,\mathbf{n}'',\mathbf{n})\mathscr{S}^{(+)}(E,\mathbf{n}'',\mathbf{n}') = \delta^{(2)}(\mathbf{n}-\mathbf{n}')\mathscr{I}. \tag{5.33b}$$

[The reader may wish to show that the same is *not* true for the reduced scattering kernel $\mathscr{S}^{(+)}_{red}(E,\mathbf{n},\mathbf{n}')$, which appears to be unitary only in the subspace spanned by the bispinor harmonics (4.14).] Similarly, exploiting the expansion (5.29), one arrives at the generalized optical relations

$$\oint_{4\pi} d^2\mathbf{n}'' \mathscr{F}^{(+)}(E,\mathbf{n},\mathbf{n}'')\mathscr{F}^{(+)\dagger}(E,\mathbf{n}',\mathbf{n}'') = \frac{2\pi}{ik}[\mathscr{F}^{(+)}(E,\mathbf{n},\mathbf{n}') - \mathscr{F}^{(+)\dagger}(E,\mathbf{n}',\mathbf{n})] \tag{5.34a}$$

and

$$\oint_{4\pi} d^2\mathbf{n}'' \mathscr{F}^{(+)\dagger}(E,\mathbf{n}'',\mathbf{n})\mathscr{F}^{(+)}(E,\mathbf{n}'',\mathbf{n}') = \frac{2\pi}{ik}[\mathscr{F}^{(+)}(E,\mathbf{n},\mathbf{n}') - \mathscr{F}^{(+)\dagger}(E,\mathbf{n}',\mathbf{n})]. \tag{5.34b}$$

Unitarity relations, analogous to those in Eqs. (5.33a) and (5.33b), may be derived for the 2×2 scattering kernel $S^{(+)}(E,\mathbf{n},\mathbf{n}')$, while optical relations, counterpart to those in Eqs. (5.34a) and (5.34b), may be obtained for the 2×2 scattering amplitude $F^{(+)}(E,\mathbf{n},\mathbf{n}')$.

D. Total cross sections

It is possible to derive remarkably simple expressions, in terms of the eigenphase shifts and the spinor harmonics, for the three kinds of total cross sections introduced in Sec. III D. From the definition (3.28), the relation (3.23), and the expansion (5.28), after making use of the orthonormality property (4.17), one obtains

$$Q(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \frac{16\pi^2}{k^2}\sum_{\gamma=1}^{2N} \sin^2 \delta_\gamma(E) \Upsilon^\dagger_\gamma(E,\mathbf{n}_0)\varrho(\boldsymbol{\nu}_0)\Upsilon_\gamma(E,\mathbf{n}_0). \tag{5.35}$$

Averaging this result over $\boldsymbol{\nu}_0$ with the aid of Eq. (3.25) gives

$$Q(E,\mathbf{n}_0) = \frac{8\pi^2}{k^2}\sum_{\gamma=1}^{2N} \sin^2 \delta_\gamma(E) \Upsilon^\dagger_\gamma(E,\mathbf{n}_0)\Upsilon_\gamma(E,\mathbf{n}_0). \tag{5.36}$$

Hence, after further averaging over all possible directions of incidence $\mathbf{n}_0$, again exploiting Eq. (4.17), one arrives at

$$Q(E) = \frac{2\pi}{k^2}\sum_{\gamma=1}^{2N} \sin^2 \delta_\gamma(E). \tag{5.37}$$

VI. "FINAL-STATE" WAVE FUNCTION FOR PHOTODETACHMENT

Thus far, we have been concerned with scattering. A related process, belonging to the category of half-collisions, is photodetachment. Attempting to describe this process theoretically, one encounters an auxiliary mathematical object, a so-called "final-state" wave function $\Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$. This function is a counterpart of the scattering wave function $\Psi^{(+)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$ considered before, but differs from the latter in that asymptotically it has the form of a plane wave superposed with a radially *ingoing*, rather than outgoing, wave.

Proceeding in the same spirit as in Sec. II in the presence of N zero-range scatterers, we shall seek $\Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$ in the form

$$\Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) + \sum_{n=1}^{N}\Psi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}). \tag{6.1}$$

Here $\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)$ is the plane-wave amplitude (3.2), while the N functions $\{\Psi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})\}$, obeying the asymptotic *inflow* conditions

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_+ + \varepsilon\beta_+]\Psi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = 0 \tag{6.2a}$$

or *equivalently* (cf. Appendix A)

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_- + \varepsilon^{-1}\beta_-]\Psi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = 0 \tag{6.2b}$$

[cf. Eqs. (2.14a) and (2.14b) and notice differences in signs], are explicitly given by

$$\Psi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = \begin{pmatrix} h^{(-)}_0(k|\mathbf{r}-\mathbf{r}_n|)\chi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0) \\ -\varepsilon h^{(-)}_1(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\chi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\end{pmatrix} \tag{6.3}$$

[cf. Eq. (2.7b) and again notice the difference in signs], with the functions $\{h^{(-)}_{0\text{ or }1}(z)\}$ defined in Eqs. (2.8a) and (2.8b). An algebraic system for the spinor coefficients $\{\chi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\}$ results after inserting Eqs. (6.1) and (6.3) into the limiting conditions

$$\lim_{\mathbf{r}\to\mathbf{r}_n}[i(\mathbf{r}-\mathbf{r}_n)\cdot\boldsymbol{\alpha}_+ + \varepsilon(|\mathbf{r}-\mathbf{r}_n|\mathscr{K}_n + k^{-1}\beta_+)] \times \Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = 0 \quad (n=1,\ldots,N) \tag{6.4}$$

[cf. Eq. (2.18)]; one obtains

$$[K_n - iI]\chi^{(-)}_n(E,\boldsymbol{\nu}_0,\mathbf{n}_0) + \sum_{\substack{n'=1\\(n'\neq n)}}^{N} h^{(-)}_0(k|\mathbf{r}_n-\mathbf{r}_{n'}|)\chi^{(-)}_{n'}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = -e^{ik\mathbf{n}_0\cdot\mathbf{r}_n}\eta(\boldsymbol{\nu}_0) \quad (n=1,\ldots,N) \tag{6.5}$$

[cf. Eq. (2.23)]. The system (6.5) may be solved in the manner completely analogous to that in which in Sec. V we have solved the system (2.23). Therefore, we shall skip details and go directly to the final result,

$$\chi_n^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \sqrt{4\pi}\sum_{\gamma=1}^{2N} e^{-i\delta_\gamma(E)}\sin\delta_\gamma(E) \times \Upsilon_\gamma^{\dagger}(E,\mathbf{n}_0)\,\boldsymbol{\eta}(\boldsymbol{\nu}_0)\,\xi_{n\gamma}(E) = \sqrt{\frac{4\pi}{1+\varepsilon^2}}\sum_{\gamma=1}^{2N} e^{-i\delta_\gamma(E)}\sin\delta_\gamma(E) \times \mathcal{Y}_\gamma^{\dagger}(E,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\xi_{n\gamma}(E) \qquad (6.6)$$

[cf. Eq. (5.19)]. Hence, after combining Eqs. (6.1), (6.3), and (6.6), one obtains

$$\Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) + \sqrt{\frac{4\pi}{1+\varepsilon^2}}\sum_{\gamma=1}^{2N} e^{-i\delta_\gamma(E)}\sin\delta_\gamma(E) \times \mathcal{Y}_\gamma^{\dagger}(E,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\Theta_\gamma^{(-)}(E,\mathbf{r}), \qquad (6.7)$$

with $\Theta_\gamma^{(-)}(E,\mathbf{r})$ defined in Eqs. (5.8)–(5.10). Asymptotically, Eq. (6.7) becomes

$$\Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \overset{r\to\infty}{\sim} \left[\operatorname*{asymp}_{r\to\infty} e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{I} + \frac{e^{-ikr}}{r}\mathcal{F}^{(-)}(E,\mathbf{n}_r,\mathbf{n}_0)\right] \times \mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0), \qquad (6.8)$$

with

$$\mathcal{F}^{(-)}(E,\mathbf{n}_r,\mathbf{n}_0) = \frac{4\pi}{k}\sum_{\gamma=1}^{2N} e^{-i\delta_\gamma(E)}\sin\delta_\gamma(E)\mathcal{Y}_\gamma(E,-\mathbf{n}_r)\mathcal{Y}_\gamma^{\dagger}(E,\mathbf{n}_0), \qquad (6.9)$$

i.e., $\Psi^{(-)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r})$ does indeed show the required asymptotic behavior.

VII. GREEN FUNCTIONS

Finally, we turn to the problem of determining the outgoing (the upper superscript) and ingoing (the lower superscript) matrix Green functions $\mathcal{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}')$ in the presence of N zero-range potentials. As before, we restrict our considerations to energies real and such that $|E|>mc^2$. Everywhere in $\mathbb{R}^3$ except at the points where the potentials are located, these functions obey the inhomogeneous equation

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathcal{I}]\mathcal{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\mathcal{I} \quad (\mathbf{r}\neq\mathbf{r}_n; n=1,\ldots,N). \qquad (7.1)$$

Asymptotically, they satisfy the analogues of the Sommerfeld conditions,

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_+ \mp \varepsilon\beta_+]\mathcal{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = 0 \qquad (7.2a)$$

or *equivalently* (cf. Appendix A)

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_- \mp \varepsilon^{-1}\beta_-]\mathcal{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = 0, \qquad (7.2b)$$

while at the locations of individual potentials they are constrained to obey the limiting conditions

$$\lim_{\mathbf{r}\to\mathbf{r}_n}[i(\mathbf{r}-\mathbf{r}_n)\cdot\boldsymbol{\alpha}_+ + \varepsilon(|\mathbf{r}-\mathbf{r}_n|\mathcal{K}_n + k^{-1}\beta_+)]\mathcal{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = 0 \quad (n=1,\ldots,N). \qquad (7.3)$$

We seek $\mathcal{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}')$ in the forms

$$\mathcal{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \mathcal{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') + \sum_{n=1}^{N}\mathcal{G}_n^{(\pm)}(E,\mathbf{r},\mathbf{r}'), \qquad (7.4)$$

where

$$\mathcal{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \frac{1}{4\pi c^2\hbar^2}[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta + E\mathcal{I}]\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \qquad (7.5)$$

are respective free-particle Dirac Green functions, while

$$\mathcal{G}_n^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \frac{k}{4\pi c^2\hbar^2}\begin{pmatrix} (E+mc^2)h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\Gamma_{n0}^{(\pm)}(E,\mathbf{r}') & (E+mc^2)h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\Gamma_{n1}^{(\pm)}(E,\mathbf{r}') \\ \pm c\hbar k h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\Gamma_{n0}^{(\pm)}(E,\mathbf{r}') & \pm c\hbar k h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\Gamma_{n1}^{(\pm)}(E,\mathbf{r}') \end{pmatrix} \quad (n=1,\ldots,N) \qquad (7.6)$$

are contributions due to the nth zero-range potential. The 2×2 matrices $\{\Gamma_{n0}^{(\pm)}(E,\mathbf{r}')\}$ and $\{\Gamma_{n1}^{(\pm)}(E,\mathbf{r}')\}$ are to be determined.

To find $\{\Gamma_{n0}^{(\pm)}(E,\mathbf{r}')\}$ and $\{\Gamma_{n1}^{(\pm)}(E,\mathbf{r}')\}$, we exploit the fact that Eq. (7.5) may be rewritten as

$$\mathcal{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \frac{k}{4\pi c^2\hbar^2}\begin{pmatrix} (E+mc^2)h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}'|)I & \pm c\hbar k h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}'|)\boldsymbol{\mu}(\mathbf{r},\mathbf{r}')\cdot\boldsymbol{\sigma} \\ \pm c\hbar k h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}'|)\boldsymbol{\mu}(\mathbf{r},\mathbf{r}')\cdot\boldsymbol{\sigma} & (E-mc^2)h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}'|)I \end{pmatrix}, \qquad (7.7)$$

with

$$\boldsymbol{\mu}(\mathbf{r},\mathbf{r}') = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \qquad (7.8)$$

combine Eqs. (7.4), (7.6), and (7.7), and substitute the result into the limiting conditions (7.3). This yields the following systems of algebraic equations:

$$[K_n \pm iI]\Gamma_{n0}^{(\pm)}(E,\mathbf{r}') + \sum_{\substack{n'=1\\(n'\neq n)}}^{N} h_0^{(\pm)}(k|\mathbf{r}_n - \mathbf{r}_{n'}|)\Gamma_{n'0}^{(\pm)}(E,\mathbf{r}') = -h_0^{(\pm)}(k|\mathbf{r}'-\mathbf{r}_n|)I \quad (n=1,\ldots,N), \qquad (7.9)$$

$$[K_n \pm iI]\Gamma_{n1}^{(\pm)}(E,\mathbf{r}') + \sum_{\substack{n'=1\\(n'\neq n)}}^{N} h_0^{(\pm)}(k|\mathbf{r}_n - \mathbf{r}_{n'}|)\Gamma_{n'1}^{(\pm)}(E,\mathbf{r}') = \pm \varepsilon h_1^{(\pm)}(k|\mathbf{r}'-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r}')\cdot\boldsymbol{\sigma} \quad (n=1,\ldots,N). \qquad (7.10)$$

The systems (7.9) and (7.10) are structurally similar either to the system (2.23) (for upper superscripts) or to the system (6.5) (for lower superscripts); the only difference is that now inhomogeneities and unknowns are 2×2 matrices rather than two-component vectors. Solving the systems (7.9) and (7.10) by the expansion method presented in Sec. V A gives

$$\Gamma_{n0}^{(\pm)}(E,\mathbf{r}') = \sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)} \sin\delta_\gamma(E)\xi_{n\gamma}(E)\theta_{0\gamma}^{(\mp)\dagger}(E,\mathbf{r}') \qquad (7.11)$$

and

$$\Gamma_{n1}^{(\pm)}(E,\mathbf{r}') = \mp\varepsilon\sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)} \sin\delta_\gamma(E)\xi_{n\gamma}(E)\theta_{1\gamma}^{(\mp)\dagger}(E,\mathbf{r}'), \qquad (7.12)$$

with $\{\theta_{0\gamma}^{(\pm)}(E,\mathbf{r})\}$ and $\{\theta_{1\gamma}^{(\pm)}(E,\mathbf{r})\}$ defined in Eqs. (5.9) and (5.10), respectively. Hence, upon inserting Eqs. (7.11) and (7.12) into Eq. (7.6), and the result into Eq. (7.4), one arrives at the following explicit expressions for the sought Green functions:

$$\mathscr{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \mathscr{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') + \frac{k}{4\pi c^2\hbar^2}\sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)}\sin\delta_\gamma(E)\begin{pmatrix} (E+mc^2)\theta_{0\gamma}^{(\pm)}(E,\mathbf{r})\theta_{0\gamma}^{(\mp)\dagger}(E,\mathbf{r}') & \mp c\hbar k\theta_{0\gamma}^{(\pm)}(E,\mathbf{r})\theta_{1\gamma}^{(\mp)\dagger}(E,\mathbf{r}') \\ \pm c\hbar k\theta_{1\gamma}^{(\pm)}(E,\mathbf{r})\theta_{0\gamma}^{(\mp)\dagger}(E,\mathbf{r}') & -(E-mc^2)\theta_{1\gamma}^{(\pm)}(E,\mathbf{r})\theta_{1\gamma}^{(\mp)\dagger}(E,\mathbf{r}') \end{pmatrix}, \qquad (7.13)$$

or equivalently, but more compactly,

$$\mathscr{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \mathscr{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') + \frac{k(E+mc^2)}{4\pi c^2\hbar^2}\sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)}\sin\delta_\gamma(E)\Theta_\gamma^{(\pm)}(E,\mathbf{r})\Theta_\gamma^{(\mp)\dagger}(E,\mathbf{r}'). \qquad (7.14)$$

From Eqs. (7.7), (7.8), and (7.14), one easily verifies that the functions $\mathscr{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}')$ obey the symmetry relations

$$\mathscr{G}^{(\pm)\dagger}(E,\mathbf{r},\mathbf{r}') = \mathscr{G}^{(\mp)}(E,\mathbf{r}',\mathbf{r}). \qquad (7.15)$$

As the source recedes to infinity, i.e., $r'\to\infty$, the free-particle Green functions (7.5) behave as

$$\mathscr{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') \overset{r'\to\infty}{\sim} \frac{E+mc^2}{4\pi c^2\hbar^2}\frac{e^{\pm ikr'}}{r'}\begin{pmatrix} I & \mp\varepsilon\mathbf{n}_r'\cdot\boldsymbol{\sigma} \\ \mp\varepsilon\mathbf{n}_r'\cdot\boldsymbol{\sigma} & \varepsilon^2 I\end{pmatrix} e^{\mp ik\mathbf{n}_r'\cdot\mathbf{r}}. \qquad (7.16)$$

Moreover, since

$$\theta_{0\gamma}^{(\pm)}(E,\mathbf{r}') \overset{r'\to\infty}{\sim} \frac{\sqrt{4\pi}}{k}\frac{e^{\pm ikr'}}{r'}\Upsilon_\gamma(E,\pm\mathbf{n}_r') \qquad (7.17)$$

and

$$\theta_{1\gamma}^{(\pm)}(E,\mathbf{r}') \overset{r'\to\infty}{\sim} \frac{\sqrt{4\pi}}{k}\frac{e^{\pm ikr'}}{r'}\mathbf{n}_r'\cdot\boldsymbol{\sigma}\Upsilon_\gamma(E,\pm\mathbf{n}_r'), \qquad (7.18)$$

one has

$$\Theta_\gamma^{(\pm)}(E,\mathbf{r}') \overset{r'\to\infty}{\sim} \frac{\sqrt{4\pi(1+\varepsilon^2)}}{k}\frac{e^{\pm ikr'}}{r'}\mathcal{Y}_\gamma(E,\pm\mathbf{n}_r'). \qquad (7.19)$$

Hence, after employing Eqs. (3.2), (5.20), and (6.7), it follows that

$$\mathscr{G}^{(\pm)}(E,\mathbf{r},\mathbf{r}')\mathcal{U}(E,\boldsymbol{\nu}_0,\mp\mathbf{n}_r') \overset{r'\to\infty}{\sim} \frac{E}{2\pi c^2\hbar^2}\frac{e^{\pm ikr'}}{r'}\Psi^{(\pm)}(E,\boldsymbol{\nu}_0,\mp\mathbf{n}_r',\mathbf{r}). \qquad (7.20)$$

VIII. TWO EXAMPLES

A. Cross sections for scattering from a single zero-range potential

As the first example, consider a Dirac particle scattered from a single zero-range potential located at the origin of a coordinate system and characterized by the "interaction" matrix,

$$K = \varkappa I + \boldsymbol{\kappa} \cdot \boldsymbol{\sigma}. \tag{8.1}$$

Then, the matrices in the spectral problem (4.1) are

$$\mathsf{L}_H(E) = K, \quad \mathsf{L}_A(E) = I, \tag{8.2}$$

and the problem has two eigenvalues,

$$\lambda_\pm(E) \equiv -\cot \delta_\pm(E) = \varkappa \pm \kappa, \tag{8.3}$$

with associated orthonormal (in the standard sense, since in this particular case the weight matrix is the unit matrix) eigenvectors

$$\mathsf{x}_\pm(E) \equiv \xi_\pm(E) = \eta(\pm \mathbf{n}_\kappa). \tag{8.4}$$

In Eqs. (8.3) and (8.4), $\kappa = |\boldsymbol{\kappa}|$, $\mathbf{n}_\kappa = \boldsymbol{\kappa}/\kappa$, while the spinors $\eta(\pm \mathbf{n}_\kappa)$ are defined, up to a phase factor, by equations analogous to Eqs. (3.3) and (3.4). The resulting eigenchannel spinor harmonics are

$$\Upsilon_\pm(E, \mathbf{n}_r) = \frac{1}{\sqrt{4\pi}} \eta(\pm \mathbf{n}_\kappa). \tag{8.5}$$

It appears that neither the 2×2 scattering amplitude

$$F^{(+)}(E, \mathbf{n}_r, \mathbf{n}_0) = -\frac{1}{k} \frac{\varkappa + i}{(\varkappa^2 - \kappa^2 - 1) + 2i\varkappa} I + \frac{1}{k} \frac{1}{(\varkappa^2 - \kappa^2 - 1) + 2i\varkappa} \boldsymbol{\kappa} \cdot \boldsymbol{\sigma} \tag{8.6}$$

nor the differential cross sections

$$\frac{d^2 Q(E, \boldsymbol{\nu}_0, \mathbf{n}_r, \mathbf{n}_0)}{d^2 \mathbf{n}_r} = \frac{1}{k^2} \frac{(\varkappa^2 + \kappa^2 + 1) - 2\varkappa \boldsymbol{\kappa} \cdot \boldsymbol{\nu}_0}{(\varkappa^2 + \kappa^2 + 1)^2 - 4\varkappa^2 \kappa^2} \tag{8.7}$$

and

$$\frac{d^2 Q(E, \mathbf{n}_r, \mathbf{n}_0)}{d^2 \mathbf{n}_r} = \frac{1}{k^2} \frac{\varkappa^2 + \kappa^2 + 1}{(\varkappa^2 + \kappa^2 + 1)^2 - 4\varkappa^2 \kappa^2} \tag{8.8}$$

depend on the direction of incidence $\mathbf{n}_0$ and the observation direction $\mathbf{n}_r$. The total cross section (3.28) is

$$Q(E, \boldsymbol{\nu}_0, \mathbf{n}_0) = \frac{4\pi}{k^2} \frac{(\varkappa^2 + \kappa^2 + 1) - 2\varkappa \boldsymbol{\kappa} \cdot \boldsymbol{\nu}_0}{(\varkappa^2 + \kappa^2 + 1)^2 - 4\varkappa^2 \kappa^2} \tag{8.9}$$

[notice that if $\varkappa = 0$ or $\kappa = 0$, the right-hand side of Eq. (8.9) becomes independent of $\boldsymbol{\nu}_0$], while the total cross sections (3.29a) and (3.30) are found to be identical and given by

$$Q(E, \mathbf{n}_0) = Q(E) = \frac{4\pi}{k^2} \frac{\varkappa^2 + \kappa^2 + 1}{(\varkappa^2 + \kappa^2 + 1)^2 - 4\varkappa^2 \kappa^2}. \tag{8.10}$$

B. Eigenphase shifts and eigenchannel spinor harmonics for two identical zero-range potentials

As the second example, we shall find eigenphase shifts and eigenchannel spinor harmonics for a Dirac particle in a field of two identical zero-range potentials located at the points

$$\mathbf{r}_1 = \tfrac{1}{2}\mathbf{R}, \quad \mathbf{r}_2 = -\tfrac{1}{2}\mathbf{R}, \tag{8.11}$$

respectively. Each potential is characterized by the interaction matrix (8.1). A brief analysis shows that in this case

$$\mathsf{L}_H(E) = \mathsf{L}'_H(E) \otimes I + I \otimes \boldsymbol{\kappa} \cdot \boldsymbol{\sigma} \tag{8.12}$$

and

$$\mathsf{L}_A(E) = \mathsf{L}'_A(E) \otimes I, \tag{8.13}$$

where $\mathsf{L}'_H(E)$ and $\mathsf{L}'_A(E)$ are 2×2 matrices with elements

$$[\mathsf{L}'_H(E)]_{nn'} = \varkappa \delta_{nn'} - y_0(kR)(1 - \delta_{nn'}) \quad (n, n' = 1, 2) \tag{8.14}$$

and

$$[\mathsf{L}'_A(E)]_{nn'} = \delta_{nn'} + j_0(kR)(1 - \delta_{nn'}) \quad (n, n' = 1, 2), \tag{8.15}$$

respectively; here $R = |\mathbf{R}|$. After employing the results of Appendix E 2, with no difficulty we find that the eigenvalues of the spectral problem (4.1) are

$$\lambda_1(E) \equiv -\cot \delta_1(E) = \frac{\varkappa + \kappa - y_0(kR)}{1 + j_0(kR)}, \tag{8.16a}$$

$$\lambda_2(E) \equiv -\cot \delta_2(E) = \frac{\varkappa - \kappa - y_0(kR)}{1 + j_0(kR)}, \tag{8.16b}$$

$$\lambda_3(E) \equiv -\cot \delta_3(E) = \frac{\varkappa + \kappa + y_0(kR)}{1 - j_0(kR)}, \tag{8.16c}$$

$$\lambda_4(E) \equiv -\cot \delta_4(E) = \frac{\varkappa - \kappa + y_0(kR)}{1 - j_0(kR)}, \tag{8.16d}$$

while the suitably chosen associated orthonormalized [in the sense of Eq. (4.4)] eigenvectors are

$$\mathsf{x}_1(E) = \frac{1}{\sqrt{2[1 + j_0(kR)]}} \begin{pmatrix} \eta(\mathbf{n}_\kappa) \\ \eta(\mathbf{n}_\kappa) \end{pmatrix}, \tag{8.17a}$$

$$\mathsf{x}_2(E) = \frac{1}{\sqrt{2[1 + j_0(kR)]}} \begin{pmatrix} \eta(-\mathbf{n}_\kappa) \\ \eta(-\mathbf{n}_\kappa) \end{pmatrix}, \tag{8.17b}$$

$$\mathsf{x}_3(E) = \frac{i}{\sqrt{2[1 - j_0(kR)]}} \begin{pmatrix} \eta(\mathbf{n}_\kappa) \\ -\eta(\mathbf{n}_\kappa) \end{pmatrix}, \tag{8.17c}$$

$$\mathsf{x}_4(E) = \frac{i}{\sqrt{2[1 - j_0(kR)]}} \begin{pmatrix} \eta(-\mathbf{n}_\kappa) \\ -\eta(-\mathbf{n}_\kappa) \end{pmatrix}. \tag{8.17d}$$

Hence, the relevant eigenchannel spinor harmonics are found to be

$$\Upsilon_1(E, \mathbf{n}_r) = \frac{\cos(k\mathbf{n}_r \cdot \mathbf{R}/2)}{\sqrt{2\pi[1 + j_0(kR)]}} \eta(\mathbf{n}_\kappa), \tag{8.18a}$$

$$\Upsilon_2(E, \mathbf{n}_r) = \frac{\cos(k\mathbf{n}_r \cdot \mathbf{R}/2)}{\sqrt{2\pi[1 + j_0(kR)]}} \eta(-\mathbf{n}_\kappa), \tag{8.18b}$$

$$\Upsilon_3(E,\mathbf{n}_r) = \frac{\sin(k\mathbf{n}_r \cdot \mathbf{R}/2)}{\sqrt{2\pi[1 - j_0(kR)]}} \eta(\mathbf{n}_\kappa), \tag{8.18c}$$

$$\Upsilon_4(E,\mathbf{n}_r) = \frac{\sin(k\mathbf{n}_r \cdot \mathbf{R}/2)}{\sqrt{2\pi[1 - j_0(kR)]}} \eta(-\mathbf{n}_\kappa). \tag{8.18d}$$

Once the eigenvalues (8.16a), (8.16b), (8.16c), and (8.16d) and the eigenchannel spinor harmonics (8.18a), (8.18b), (8.18c), and (8.18d) have been determined, with no difficulty, from Eqs. (5.28), (3.23), (3.27), and (5.35)–(5.37), one may derive explicit expressions for the 2×2 scattering amplitude as well as for these particular differential and total cross sections which are of interest. We do not present these formulas here for they are quite long.

IX. CONCLUSIONS

In this paper, we have presented the zero-range potential model for scattering of Dirac particles. There are two directions in which we plan to continue this work in the near future. First, we are engaged in extending the formalism developed in this paper to bound-state problems. Such an extension will be interesting not only for its own sake, but also because, in conjunction with the results of Sec. VI of the present paper, it will offer the immediate possibility to carry out perturbative calculations on photodetachment induced by a *weak* time-harmonic electromagnetic field. Significant progress towards achieving this goal has been already made, and results, with applications, will be presented in a forthcoming publication. Second, we plan to extend the ZRPM for Dirac particles to become applicable to time-dependent processes in external electromagnetic fields of arbitrary strengths. In view of the fact that the nonrelativistic ZRPM is frequently used for investigating processes in strong laser fields, the need for such an extension is evident.

ACKNOWLEDGMENTS

Discussions with R. Augusiak are acknowledged. I thank Dr. AM. Alhasan for commenting on the manuscript.

APPENDIX A: EQUIVALENCE OF TWO FORMS OF OUTFLOW (OR INFLOW) ASYMPTOTIC CONDITIONS

Let $\Xi^{(\pm)}(E,\mathbf{r})$ obey the asymptotic conditions

$$\lim_{r\to\infty} r[\mathbf{n}_r \cdot \boldsymbol{\alpha}_+ \mp \varepsilon\beta_+]\Xi^{(\pm)}(E,\mathbf{r}) = 0 \tag{A1}$$

[cf. Eqs. (2.14a), (6.2a), and (7.2a)]. If the upper (lower) sign in the square brackets is chosen, Eq. (A1) represents the outflow (inflow) condition.

Operating on Eq. (A1) from the left with $\mp\varepsilon^{-1}\mathbf{n}_r\cdot\boldsymbol{\alpha}_-$ yields, after rearrangement,

$$\lim_{r\to\infty} r[(\mathbf{n}_r \cdot \boldsymbol{\alpha}_-)\beta_+ \mp \varepsilon^{-1}(\mathbf{n}_r \cdot \boldsymbol{\alpha}_-)(\mathbf{n}_r \cdot \boldsymbol{\alpha}_+)]\Xi^{(\pm)}(E,\mathbf{r}) = 0. \tag{A2}$$

From the definitions (2.15) and from the well-known [20] properties of the Dirac matrices $\boldsymbol{\alpha}$ and β, it follows that

$$(\mathbf{n}_r \cdot \boldsymbol{\alpha}_\pm)\beta_\mp = \mathbf{n}_r \cdot \boldsymbol{\alpha}_\pm, \tag{A3a}$$

$$(\mathbf{n}_r \cdot \boldsymbol{\alpha}_\pm)(\mathbf{n}_r \cdot \boldsymbol{\alpha}_\mp) = \beta_\pm. \tag{A3b}$$

Hence, we deduce that Eq. (A2) may be simplified to

$$\lim_{r\to\infty} r[\mathbf{n}_r \cdot \boldsymbol{\alpha}_- \mp \varepsilon^{-1}\beta_-]\Xi^{(\pm)}(E,\mathbf{r}) = 0 \tag{A4}$$

[cf. Eqs. (2.14b), (6.2b), and (7.2b)].

To show that Eq. (A4) implies Eq. (A1), one should premultiply the former with $\mp\varepsilon\mathbf{n}_r\cdot\boldsymbol{\alpha}_+$ and simplify the result, again making use of Eqs. (A3a) and (A3b).

APPENDIX B: THE LIMITING CONDITIONS (2.18) IN A TWO-COMPONENT FORM

We shall show that it is possible to transform the limiting conditions (2.18) to a form very similar to one used in the nonrelativistic theory [1,13]. Let $\psi_+^{(+)}(E,\mathbf{r})$ and $\psi_-^{(+)}(E,\mathbf{r})$ denote the upper and the lower components of the function (2.1), respectively. Expressed in terms of $\psi_\pm^{(+)}(E,\mathbf{r})$, the limiting conditions (2.18) are

$$\lim_{\mathbf{r}\to\mathbf{r}_n} [(I + k|\mathbf{r} - \mathbf{r}_n|K_n)\psi_+^{(+)}(E,\mathbf{r}) + i\varepsilon^{-1}k(\mathbf{r} - \mathbf{r}_n) \cdot \boldsymbol{\sigma}\psi_-^{(+)}(E,\mathbf{r})] = 0 \qquad (n = 1, \ldots, N). \tag{B1}$$

On the other hand, it follows from the Dirac equation (2.2) that

$$\psi_-^{(+)}(E,\mathbf{r}) = -i\varepsilon k^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\,\psi_+^{(+)}(E,\mathbf{r}) \qquad (\mathbf{r} \neq \mathbf{r}_n; n = 1, \ldots, N). \tag{B2}$$

Inserting this relationship into Eq. (B1) gives

$$\lim_{\mathbf{r}\to\mathbf{r}_n} \{I + k|\mathbf{r} - \mathbf{r}_n|K_n + [(\mathbf{r} - \mathbf{r}_n) \cdot \boldsymbol{\sigma}](\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\}\psi_+^{(+)}(E,\mathbf{r}) = 0 \qquad (n = 1, \ldots, N), \tag{B3}$$

which, after exploiting properties of the Pauli matrices, may be rewritten as

$$\lim_{\mathbf{r}\to\mathbf{r}_n} \{I + k|\mathbf{r} - \mathbf{r}_n|K_n + (\mathbf{r} - \mathbf{r}_n) \cdot I\boldsymbol{\nabla} + i\boldsymbol{\sigma}\cdot[(\mathbf{r} - \mathbf{r}_n)\times\boldsymbol{\nabla}]\} \times\psi_+^{(+)}(E,\mathbf{r}) = 0 \qquad (n = 1, \ldots, N). \tag{B4}$$

In the next step, we make use of the explicit form of $\psi_+^{(+)}(E,\mathbf{r})$ resulting from Eqs. (2.1), (2.4), and (2.7b). Because

$$(\mathbf{r} - \mathbf{r}_n)\times\boldsymbol{\nabla}\, h_0^{(+)}(k|\mathbf{r} - \mathbf{r}_n|) = 0, \tag{B5}$$

we have

$$\lim_{\mathbf{r}\to\mathbf{r}_n} (\mathbf{r} - \mathbf{r}_n)\times\boldsymbol{\nabla}\,\psi_+^{(+)}(E,\mathbf{r}) = 0; \tag{B6}$$

hence, it follows that Eq. (B4) may be replaced by

$$\lim_{\mathbf{r}\to\mathbf{r}_n} [I + k|\mathbf{r} - \mathbf{r}_n|K_n + (\mathbf{r} - \mathbf{r}_n) \cdot I\boldsymbol{\nabla}]\psi_+^{(+)}(E,\mathbf{r}) = 0 \qquad (n = 1, \ldots, N). \tag{B7}$$

If the matrices $\{K_n\}$ are simple multiplies of the unit 2×2

matrix, in Eq. (B7) there is no mixing between components of $\psi_{+}^{(+)}(E,\mathbf{r})$ and both these components satisfy limiting conditions which, apart from unimportant notational differences, are identical with those used in the nonrelativistic theory [cf. Eq. (2.5) in Ref. [13]].

APPENDIX C: POSITIVE SEMIDEFINITENESS OF THE MATRIX $\mathsf{L}_A(E)$

Let z be an arbitrary $2N$-component column vector, composed of N two-component spinors $\{\zeta_n\}$,

$$\mathsf{z} = (\zeta_1^{\mathrm{T}} \cdots \zeta_N^{\mathrm{T}})^{\mathrm{T}}. \tag{C1}$$

Since, as it follows from Eqs. (2.31) and (3.33), elements of the matrix $\mathsf{L}_A(E)$ may be rewritten in the form

$$[\mathsf{L}_A(E)]_{n\nu,n'\nu'}(E) = \frac{1}{4\pi}\delta_{\nu\nu'}\oint_{4\pi} d^2\mathbf{n}_r e^{ik\mathbf{n}_r\cdot(\mathbf{r}_n-\mathbf{r}_{n'})}, \tag{C2}$$

employing Eq. (C1) one has

$$\mathsf{z}^{\dagger}\mathsf{L}_A(E)\mathsf{z} = \frac{1}{4\pi}\sum_{n,n'=1}^{N}\zeta_n^{\dagger}\zeta_{n'}\oint_{4\pi} d^2\mathbf{n}_r e^{ik\mathbf{n}_r\cdot(\mathbf{r}_n-\mathbf{r}_{n'})}. \tag{C3}$$

If we define

$$\tau(E,\mathbf{n}_r) = \frac{1}{\sqrt{4\pi}}\sum_{n=1}^{N} e^{-ik\mathbf{n}_r\cdot\mathbf{r}_n}\zeta_n, \tag{C4}$$

Eq. (C3) may be cast into the form

$$\mathsf{z}^{\dagger}\mathsf{L}_A(E)\mathsf{z} = \oint_{4\pi} d^2\mathbf{n}_r \tau^{\dagger}(E,\mathbf{n}_r)\tau(E,\mathbf{n}_r) \geqslant 0. \tag{C5}$$

Equation (C5) implies that the matrix $\mathsf{L}_A(E)$ is at least positive semidefinite.

APPENDIX D: VALIDATION OF THE WEIGHTED CLOSURE RELATIONS (4.5)

Assume that the Hermitian matrix $\mathsf{L}_A(E)$, defined in Eq. (2.29), is positive definite. Then, it may be written as

$$\mathsf{L}_A(E) = \mathsf{L}_A^{1/2}(E)\mathsf{L}_A^{1/2}(E) \tag{D1}$$

with $\mathsf{L}_A^{1/2}(E)$ being also Hermitian and positive definite. Operating on Eq. (4.1) from the left with the matrix $\mathsf{L}_A^{-1/2}(E)$, transforms this equation into

$$\tilde{\mathsf{L}}_H(E)\tilde{\mathsf{x}}_\gamma(E) = \lambda_\gamma(E)\tilde{\mathsf{x}}_\gamma(E), \tag{D2}$$

with the Hermitian matrix

$$\tilde{\mathsf{L}}_H(E) = \mathsf{L}_A^{-1/2}(E)\mathsf{L}_H(E)\mathsf{L}_A^{-1/2}(E) \tag{D3}$$

and with

$$\tilde{\mathsf{x}}_\gamma(E) = \mathsf{L}_A^{1/2}(E)\mathsf{x}_\gamma(E). \tag{D4}$$

Equation (D2) constitutes the standard Hermitian matrix eigenvalue problem; it results from Eqs. (4.4) and (D4) that all its all eigenvectors $\{\tilde{\mathsf{x}}_\gamma(E)\}$ are orthonormal in the standard sense,

$$\tilde{\mathsf{x}}_\gamma^{\dagger}(E)\tilde{\mathsf{x}}_{\gamma'}(E) = \delta_{\gamma\gamma'}. \tag{D5}$$

For finite-dimensional Hermitian matrices we are working here with, Eq. (D5) implies the closure relation

$$\sum_{\gamma=1}^{2N}\tilde{\mathsf{x}}_\gamma(E)\tilde{\mathsf{x}}_\gamma^{\dagger}(E) = \mathsf{I} \tag{D6}$$

[just observe that Eqs. (D5) and (D6) express, in alternative ways, the fact that the modal matrix for the eigensystem (D2) is unitary]. Referring to the definition (D4), we may rewrite Eq. (D6) in the form

$$\mathsf{L}_A^{1/2}(E)\sum_{\gamma=1}^{2N}\mathsf{x}_\gamma(E)\mathsf{x}_\gamma^{\dagger}(E)\mathsf{L}_A^{1/2}(E) = \mathsf{I}. \tag{D7}$$

Operating on Eq. (D7) from the left with $\mathsf{L}_A^{1/2}(E)$ and from the right with $\mathsf{L}_A^{-1/2}(E)$ gives

$$\mathsf{L}_A(E)\sum_{\gamma=1}^{2N}\mathsf{x}_\gamma(E)\mathsf{x}_\gamma^{\dagger}(E) = \mathsf{I}. \tag{D8}$$

Similarly, acting on Eq. (D7) from the left with $\mathsf{L}_A^{-1/2}(E)$ and from the right with $\mathsf{L}_A^{1/2}(E)$, yields

$$\sum_{\gamma=1}^{2N}\mathsf{x}_\gamma(E)\mathsf{x}_\gamma^{\dagger}(E)\mathsf{L}_A(E) = \mathsf{I}. \tag{D9}$$

Combining Eqs. (D8) and (D9) results in Eq. (4.5).

APPENDIX E: SOME PARTICULAR CASES WHEN THE EIGENSYSTEM (4.1) MAY BE SIMPLIFIED

In this appendix, we shall discuss briefly two particular situations when it is possible to simplify the process of solving the eigensystem (4.1). In both cases, we shall be exploiting the fact that, as it follows directly from Eq. (2.31), the weight matrix $\mathsf{L}_A(E)$ may be written as the Kronecker product

$$\mathsf{L}_A(E) = \mathsf{L}_A'(E) \otimes I, \tag{E1}$$

where the $N\times N$ matrix $\mathsf{L}_A'(E)$ has elements

$$[\mathsf{L}_A'(E)]_{nn'} = \delta_{nn'} + j_0(k|\mathbf{r}_n-\mathbf{r}_{n'}|)(1-\delta_{nn'})$$

$$(n,n' = 1,\ldots,N). \tag{E2}$$

1. "Scalar" zero-range potentials

If all zero-range potentials are "scalar," i.e., such that the matrices $\{K_n\}$ are simple multiples of the unit 2×2 matrix,

$$K_n = \varkappa_n I \tag{E3}$$

[cf. Eq. (2.21)], from Eq. (2.30) one has

$$\mathsf{L}_H(E) = \mathsf{L}_H'(E) \otimes I, \tag{E4}$$

where the $N\times N$ matrix $\mathsf{L}_H'(E)$ has elements

$$[\mathsf{L}'_H(E)]_{nn'} = \varkappa_n \delta_{nn'} - y_0(k|\mathbf{r}_n - \mathbf{r}_{n'}|)(1 - \delta_{nn'}) \quad (n,n' = 1, \ldots ,N). \tag{E5}$$

Consider the $N \times N$ eigensystem

$$\mathsf{L}'_H(E)\mathsf{x}'_g(E) = \lambda'_g(E)\mathsf{L}'_A(E)\mathsf{x}'_g(E), \tag{E6}$$

with its eigenvectors orthonormalized according to

$$\mathsf{x}'^{\dagger}_g(E)\mathsf{L}'_A(E)\mathsf{x}'_{g'}(E) = \delta_{gg'}. \tag{E7}$$

It is evident that if $\lambda'_g(E)$ is some particular eigenvalue to the system (E6) and if $\mathsf{x}'_g(E)$ is its associated eigenvector, then $\lambda_g(E)=\lambda'_g(E)$ is a doubly degenerate (at this moment, we disregard all possible additional degeneracies caused by other factors) eigenvalue to the system (4.1) and its associated eigenvectors, orthonormal in the sense of Eq. (4.4), may be chosen to be

$$\mathsf{x}_{g,1}(E) = \mathsf{x}'_g(E) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathsf{x}_{g,2}(E) = \mathsf{x}'_g(E) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{E8}$$

2. Zero-range potentials with identical vector parts

The second situation we consider is that when all zero-range potentials have identical vector parts, i.e.,

$$K_n = \varkappa_n I + \boldsymbol{\kappa} \cdot \boldsymbol{\sigma} \tag{E9}$$

(one may think about the target composed of such potentials as being, in some sense, "completely polarized"). Then the matrix $\mathsf{L}_H(E)$ is

$$\mathsf{L}_H(E) = \mathsf{L}'_H(E) \otimes I + \mathsf{I}' \otimes \boldsymbol{\kappa} \cdot \boldsymbol{\sigma}, \tag{E10}$$

where $\mathsf{L}'_H(E)$ is defined by Eq. (E5) and I' is the unit $N \times N$ matrix. Since

$$\boldsymbol{\kappa} \cdot \boldsymbol{\sigma} \eta(\pm \mathbf{n}_\kappa) = \pm \kappa \eta(\pm \mathbf{n}_\kappa), \tag{E11}$$

a spectrum of the $2N \times 2N$ eigensystem (4.1) is the union of spectra of the two $N \times N$ eigensystems

$$[\mathsf{L}'_H(E) \pm \kappa \mathsf{I}']\mathsf{x}''_{g,\pm}(E) = \lambda''_{g,\pm}(E)\mathsf{L}'_A(E)\mathsf{x}''_{g,\pm}(E). \tag{E12}$$

Provided the eigenvectors in Eq. (E12) have been orthonormalized so that

$$\mathsf{x}''^{\dagger}_{g,\pm}(E)\mathsf{L}'_A(E)\mathsf{x}''_{g',\pm}(E) = \delta_{gg'}, \tag{E13}$$

eigenvectors to the system (4.1), orthonormal in accordance with Eq. (4.4), are

$$\mathsf{x}_{g,\pm}(E) = \mathsf{x}''_{g,\pm}(E) \otimes \eta(\pm \mathbf{n}_\kappa). \tag{E14}$$

APPENDIX F: ORTHONORMALITY OF THE EIGENCHANNEL SPINOR HARMONICS

Consider the integral

$$I_{\gamma\gamma'}(E) = \oint_{4\pi} d^2\mathbf{n}_r \Upsilon^{\dagger}_{\gamma}(E,\mathbf{n}_r)\Upsilon_{\gamma'}(E,\mathbf{n}_r), \tag{F1}$$

which is the scalar product of two eigenchannel spinor harmonics over the unit sphere. Making explicit use of the definition (4.15), we transform Eq. (F1) into

$$I_{\gamma\gamma'}(E) = \frac{1}{4\pi} \sum_{n,n'=1}^{N} \xi^{\dagger}_{n\gamma}(E)\xi_{n'\gamma'}(E) \oint_{4\pi} d^2\mathbf{n}_r e^{ik\mathbf{n}_r \cdot (\mathbf{r}_n - \mathbf{r}_{n'})}, \tag{F2}$$

and further, after exploiting Eqs. (3.34) and (2.31), into

$$I_{\gamma\gamma'}(E) = \mathsf{x}^{\dagger}_{\gamma}(E)\mathsf{L}_A(E)\mathsf{x}_{\gamma'}(E). \tag{F3}$$

The right-hand side of Eq. (F3) may be simplified with the aid of Eq. (4.4); one obtains

$$I_{\gamma\gamma'}(E) = \delta_{\gamma\gamma'}, \tag{F4}$$

which means that the eigenchannel spinor harmonics (4.15) form an orthonormal set on the unit sphere.

APPENDIX G: AN ALTERNATIVE MODEL OF ZERO-RANGE POTENTIALS FOR DIRAC PARTICLES

Assume that interactions between a Dirac particle and N point targets are described by the following limiting conditions:

$$\lim_{\mathbf{r}\to\mathbf{r}_n} [i(\mathbf{r} - \mathbf{r}_n) \cdot \boldsymbol{\alpha}_- + \varepsilon^{-1}(|\mathbf{r} - \mathbf{r}_n|\overline{\mathscr{K}}_n + k^{-1}\beta_-)]\overline{\Psi}(E,\mathbf{r}) = 0 \quad (n = 1, \ldots ,N) \tag{G1}$$

imposed on the particle's wave function $\overline{\Psi}(E,\mathbf{r})$ at target locations; here

$$\overline{\mathscr{K}}_n = \begin{pmatrix} 0 & 0 \\ 0 & K_n \end{pmatrix}. \tag{G2}$$

(Throughout this appendix, all objects *without* overlines are defined as in the main text.) Let

$$\overline{\Phi}(E,\mathbf{r}) = \begin{pmatrix} -i\varepsilon^{-1}k^{-1}\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\, \overline{\phi}_-(E,\mathbf{r}) \\ \overline{\phi}_-(E,\mathbf{r}) \end{pmatrix} \tag{G3}$$

be some particular solution to the free-particle Dirac equation in $\mathbb{R}^3$. If we look for these wave functions excited by the wave (G3) which are of the form

$$\overline{\Psi}^{(\pm)}(E,\mathbf{r}) = \overline{\Phi}(E,\mathbf{r}) + \sum_{n=1}^{N} \overline{\Psi}^{(\pm)}_n(E,\mathbf{r}), \tag{G4}$$

with

$$\overline{\Psi}^{(\pm)}_n(E,\mathbf{r}) = \begin{pmatrix} \pm\varepsilon^{-1}h_1^{(\pm)}(k|\mathbf{r} - \mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r}) \cdot \boldsymbol{\sigma}\overline{\chi}^{(\pm)}_n(E) \\ h_0^{(\pm)}(k|\mathbf{r} - \mathbf{r}_n|)\overline{\chi}^{(\pm)}_n(E) \end{pmatrix} \tag{G5}$$

[cf. Eqs. (2.7b) and (6.3)], then, proceeding as in the main text, we find that the spinor coefficients $\{\overline{\chi}_n^{(\pm)}(E)\}$ obey

$$[K_n \pm iI]\overline{\chi}_n^{(\pm)}(E) + \sum_{\substack{n'=1 \\ (n'\neq n)}}^{N} h_0^{(\pm)}(k|\mathbf{r}_n - \mathbf{r}_{n'}|)\overline{\chi}_{n'}^{(\pm)}(E) = -\overline{\phi}_-(E,\mathbf{r}_n) \quad (n=1,\ldots,N). \tag{G6}$$

Solving this system by the method presented in Sec. V A, one eventually arrives at

$$\overline{\Psi}^{(\pm)}(E,\mathbf{r}) = \overline{\Phi}(E,\mathbf{r}) + \sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)} \sin\delta_\gamma(E) \times \left[\sum_{n'=1}^{N} \xi_{n'\gamma}^{\dagger}(E)\overline{\phi}_-(E,\mathbf{r}_{n'})\right]\overline{\Theta}_\gamma^{(\pm)}(E,\mathbf{r}), \tag{G7}$$

with

$$\overline{\Theta}_\gamma^{(\pm)}(E,\mathbf{r}) = \begin{pmatrix} \pm\varepsilon^{-1}\theta_{1\gamma}^{(\pm)}(E,\mathbf{r}) \\ \theta_{0\gamma}^{(\pm)}(E,\mathbf{r}) \end{pmatrix}. \tag{G8}$$

Relevant eigenchannels are

$$\overline{\mathcal{X}}_\gamma(E,\mathbf{r}) = \sqrt{\frac{k(E-mc^2)}{4\pi c^2\hbar^2}}\sum_{n=1}^{N}\left[\begin{pmatrix} i\varepsilon^{-1}y_1(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E) \\ y_0(k|\mathbf{r}-\mathbf{r}_n|)\xi_{n\gamma}(E)\end{pmatrix} - \cot\delta_\gamma(E)\begin{pmatrix} i\varepsilon^{-1}j_1(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\xi_{n\gamma}(E) \\ j_0(k|\mathbf{r}-\mathbf{r}_n|)\xi_{n\gamma}(E)\end{pmatrix}\right]. \tag{G9}$$

Asymptotically, they behave as

$$\overline{\mathcal{X}}_\gamma(E,\mathbf{r}) \overset{r\to\infty}{\sim} \operatorname{sgn}(E)\sqrt{\frac{E}{2c^2\hbar^2 k}}\frac{1}{i\sin\delta_\gamma(E)}\left[\frac{e^{-ikr-i\delta_\gamma(E)}}{r} \times\overline{\mathcal{Y}}_\gamma(E,-\mathbf{n}_r) - \frac{e^{ikr+i\delta_\gamma(E)}}{r}\overline{\mathcal{Y}}_\gamma(E,\mathbf{n}_r)\right], \tag{G10}$$

with

$$\overline{\mathcal{Y}}_\gamma(E,\mathbf{n}_r) = \frac{1}{\sqrt{1+\varepsilon^{-2}}}\begin{pmatrix}\varepsilon^{-1}\mathbf{n}_r\cdot\boldsymbol{\sigma}\Upsilon_\gamma(E,\mathbf{n}_r) \\ \Upsilon_\gamma(E,\mathbf{n}_r)\end{pmatrix} \tag{G11}$$

being orthonormal bispinor eigenchannel harmonics for the problem at hand.

If the exciting wave (G3) is the spin-polarized monochromatic plane wave (3.1), the wave functions (G7) become

$$\overline{\Psi}^{(\pm)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) = e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) + \sqrt{\frac{4\pi}{1+\varepsilon^{-2}}}\sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)}\sin\delta_\gamma(E) \times\overline{\mathcal{Y}}_\gamma^{\dagger}(E,\mathbf{n}_0)\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0)\overline{\Theta}_\gamma^{(\pm)}(E,\mathbf{r}). \tag{G12}$$

Asymptotically, one has

$$\overline{\Psi}^{(\pm)}(E,\boldsymbol{\nu}_0,\mathbf{n}_0,\mathbf{r}) \overset{r\to\infty}{\sim}\left[\operatorname*{asymp}_{r\to\infty} e^{ik\mathbf{n}_0\cdot\mathbf{r}}\mathscr{I} + \frac{e^{\pm ikr}}{r} \times\overline{\mathscr{F}}^{(\pm)}(E,\mathbf{n}_r,\mathbf{n}_0)\right]\mathcal{U}(E,\boldsymbol{\nu}_0,\mathbf{n}_0), \tag{G13}$$

with the far-field matrix amplitudes

$$\overline{\mathscr{F}}^{(\pm)}(E,\mathbf{n}_r,\mathbf{n}_0) = \frac{4\pi}{k}\sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)}\sin\delta_\gamma(E)\overline{\mathcal{Y}}_\gamma(E,\pm\mathbf{n}_r)\overline{\mathcal{Y}}_\gamma^{\dagger}(E,\mathbf{n}_0). \tag{G14}$$

The differential cross section for scattering of a spin-polarized plane wave (3.1) is

$$\frac{d^2\overline{Q}(E,\boldsymbol{\nu}_0,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r} = \operatorname{Tr}[F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)\overline{\varrho}(\boldsymbol{\nu}_0,\mathbf{n}_0)F^{(+)\dagger}(E,\mathbf{n}_r,\mathbf{n}_0)], \tag{G15}$$

where

$$\overline{\varrho}(\boldsymbol{\nu}_0,\mathbf{n}_0) = \mathbf{n}_0\cdot\boldsymbol{\sigma}\varrho(\boldsymbol{\nu}_0)\mathbf{n}_0\cdot\boldsymbol{\sigma}. \tag{G16}$$

Its average over all orientations of $\boldsymbol{\nu}_0$ is

$$\frac{d^2\overline{Q}(E,\mathbf{n}_r,\mathbf{n}_0)}{d^2\mathbf{n}_r} = \frac{1}{2}\operatorname{Tr}[F^{(+)}(E,\mathbf{n}_r,\mathbf{n}_0)F^{(+)\dagger}(E,\mathbf{n}_r,\mathbf{n}_0)] \tag{G17}$$

and is identical with the averaged differential cross section (3.27). The total cross section, the total cross section averaged over $\boldsymbol{\nu}_0$, and the total cross section averaged both over $\boldsymbol{\nu}_0$ and over $\mathbf{n}_0$ are given by

$$\overline{Q}(E,\boldsymbol{\nu}_0,\mathbf{n}_0) = \frac{16\pi^2}{k^2}\sum_{\gamma=1}^{2N}\sin^2\delta_\gamma(E) \times\Upsilon_\gamma^{\dagger}(E,\mathbf{n}_0)\overline{\varrho}(\boldsymbol{\nu}_0,\mathbf{n}_0)\Upsilon_\gamma(E,\mathbf{n}_0), \tag{G18}$$

$$\overline{Q}(E,\mathbf{n}_0) = \frac{8\pi^2}{k^2}\sum_{\gamma=1}^{2N} \sin^2 \delta_\gamma(E) \Upsilon^\dagger_\gamma(E,\mathbf{n}_0)\Upsilon_\gamma(E,\mathbf{n}_0), \tag{G19}$$

and

$$\overline{Q}(E) = \frac{2\pi}{k^2}\sum_{\gamma=1}^{2N} \sin^2 \delta_\gamma(E), \tag{G20}$$

respectively. Notice that the averaged total cross sections (G19) and (G20) are identical with these in Eqs. (5.36) and (5.37), respectively, but, in general, the analogous statement is *not* true for the cross sections (G18) and (5.35).

Finally, the Green functions, satisfying

$$[-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta - E\mathcal{I}]\overline{\mathcal{G}}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\mathcal{I} \quad (\mathbf{r}\neq\mathbf{r}_n; n=1,\ldots,N) \tag{G21}$$

subject to the asymptotic conditions

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_+ \mp \varepsilon\beta_+]\overline{\mathcal{G}}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = 0 \tag{G22a}$$

or *equivalently*

$$\lim_{r\to\infty} r[\mathbf{n}_r\cdot\boldsymbol{\alpha}_- \mp \varepsilon^{-1}\beta_-]\overline{\mathcal{G}}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = 0, \tag{G22b}$$

and to the "interaction" limiting conditions

$$\lim_{\mathbf{r}\to\mathbf{r}_n}[i(\mathbf{r}-\mathbf{r}_n)\cdot\boldsymbol{\alpha}_- + \varepsilon^{-1}(|\mathbf{r}-\mathbf{r}_n|\overline{\mathcal{K}}_n + k^{-1}\beta_-)]\overline{\mathcal{G}}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = 0 \quad (n=1,\ldots,N), \tag{G23}$$

sought in the forms

$$\overline{\mathcal{G}}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \mathcal{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') + \sum_{n=1}^{N}\overline{\mathcal{G}}_n^{(\pm)}(E,\mathbf{r},\mathbf{r}'), \tag{G24}$$

with

$$\overline{\mathcal{G}}_n^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \frac{k}{4\pi c^2\hbar^2}\begin{pmatrix} \pm c\hbar k h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\overline{\Gamma}_{n1}^{(\pm)}(E,\mathbf{r}') & \pm c\hbar k h_1^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\boldsymbol{\mu}_n(\mathbf{r})\cdot\boldsymbol{\sigma}\overline{\Gamma}_{n0}^{(\pm)}(E,\mathbf{r}') \\ (E-mc^2)h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\overline{\Gamma}_{n1}^{(\pm)}(E,\mathbf{r}') & (E-mc^2)h_0^{(\pm)}(k|\mathbf{r}-\mathbf{r}_n|)\overline{\Gamma}_{n0}^{(\pm)}(E,\mathbf{r}') \end{pmatrix} \quad (n=1,\ldots,N), \tag{G25}$$

are found to be

$$\overline{\mathcal{G}}^{(\pm)}(E,\mathbf{r},\mathbf{r}') = \mathcal{G}_0^{(\pm)}(E,\mathbf{r},\mathbf{r}') + \frac{k(E-mc^2)}{4\pi c^2\hbar^2}\sum_{\gamma=1}^{2N} e^{\pm i\delta_\gamma(E)}\sin\delta_\gamma(E)\overline{\Theta}_\gamma^{(\pm)}(E,\mathbf{r})\overline{\Theta}_\gamma^{(\mp)\dagger}(E,\mathbf{r}'). \tag{G26}$$

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