Virtual light-by-light scattering and the g factor of a bound electron

R. N. Lee,* A. I. Milstein,† and I. S. Terekhov‡
Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia

S. G. Karshenboim§
D. I. Mendeleev Institute for Metrology (VNIIM), St. Petersburg 198005, Russia and Max-Planck-Institut für Quantenoptik, 85748 Garching, Germany

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The contribution of the light-by-light diagram to the g factor of an electron and muon bound in a Coulomb field is obtained. For an electron in a ground state, our results are in good agreement with the results of other authors obtained numerically for large Z. For relatively small Z our results have essentially higher accuracy as compared to the previous ones. For muonic atoms, the contribution is obtained with a high accuracy in the whole region of Z.

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I. INTRODUCTION

The progress in experimental investigations of the g factor of a bound electron [1] and muon [2,3] in ions stimulated intensive theoretical investigation of various contributions to this quantity. The contributions of self-energy, vacuum polarization, and nuclear effects have been considered [4–10]. An essential part of the theoretical uncertainty has been related to the contribution of the vacuum polarization of an external homogeneous magnetic field in the electric field of an atom (so-called the "magnetic-loop" contribution). The corresponding diagram is shown in Fig. 1. In this diagram, the double line in the fermion loop corresponds to the electron propagator in the Coulomb field. Note that the contribution of the free-electron loop to the vacuum polarization of a homogeneous magnetic field vanishes due to the gauge invariance. The first nonvanishing term of expansion with respect to the Coulomb field shown in Fig. 1 is the contribution of virtual light-by-light scattering with one of the quanta corresponding to the external magnetic field. The results of numerical calculations of the magnetic-loop contribution, which take into account all orders of the parameter $Z\alpha$ (Z is the nuclear charge number, $\alpha=e^2$ is the fine-structure constant, $\hbar=c=1$), are presented in Ref. [6]. At present, the most accurate experimental data are obtained in the region of medium Z. Unfortunately, in this region the uncertainty of the results of Ref. [6] is very big, being, e.g., 100% for $Z=12$. In Ref. [9], the leading in the $Z\alpha$ magnetic-loop contribution to the g factor of an electron in the S state of a hydrogenlike ion has been derived. It reads

$$\frac{\Delta g_0}{g_0}=\frac{\Delta g_0}{2}=\frac{7\alpha(Z\alpha)^5}{432n^3}, \tag{1}$$

where g_0 is the Landé factor equal to 2 for S state. One can compare this correction with the result of [6] for rather large Z where the accuracy of the numerical calculation is reasonable. This comparison shows the noticeable difference which can be attributed to the contribution of the next-to-leading terms in the $Z\alpha$ expansion, starting from $\alpha(Z\alpha)^6$. Since the numerical factor in Eq. (1) is very small ($\sim 1/30$), the next-to-leading terms could give a noticeable contribution to the g factor even at small Z, if the corresponding numerical factor is of order of unity.

In the present paper, we generalize Eq. (1) to the case of an arbitrary bound electron state. We also calculate the next-to-leading contribution of the magnetic loop to the g factor of the electron in an arbitrary state (or the magnetic moment of the electron in this state). It has the form $\Delta g_1=\alpha(Z\alpha)^6[a_1\ln(1/Z\alpha)+a_2]$, where $a_{1,2}$ are some constants and a_1 is not zero only for S states. In order to calculate this contribution, it is sufficient to take into account the diagrams of virtual light-by-light scattering and use the nonrelativistic wave functions of the bound electron. Comparison of the correction $\Delta g_0+\Delta g_1$ for the $1S_{1/2}$ state with the results of [6] shows that the account of Δg_1 does not provide good agreement for relatively small $Z\sim 30$, where the numerical calculations were performed with sufficient accuracy. Thus, for such Z it is necessary to take into account the next terms in $Z\alpha$. These terms have two different origins. First, they come from the relativistic corrections to the wave function of a

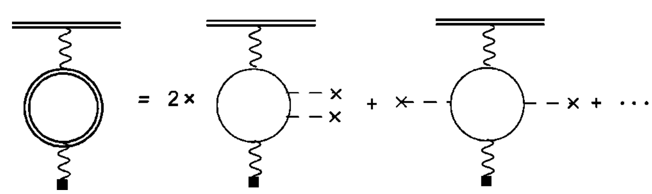

FIG. 1. The diagram corresponding to the magnetic-loop contribution to the g factor of a bound electron and first nonvanishing terms of expansion of this loop with respect to the Coulomb field. The double line denotes the electron propagator and the wave function in a Coulomb field, the dashed line with the cross denotes the Coulomb field, the wavy line with the square denotes the external homogeneous magnetic field, and the internal wavy line corresponds to the photon propagator.

*Electronic address: R.N.Lee@inp.nsk.su
†Electronic address: A.I.Milstein@inp.nsk.su
‡Electronic address: I.S.Terekhov@inp.nsk.su
§Electronic address: sek@mpq.mpg.de

bound electron. Next, they come from the higher-order contributions to the electron loop. Note that the diagram in Fig. 1 can be interpreted as the contribution of the scattering of the magnetic quantum in a Coulomb field (virtual Delbrück scattering) to the g factor. It is known that the Coulomb corrections to the Delbrück amplitude for the momentum of quantum $q \lesssim m$ (m is the electron mass) are numerically small even for large Z [11,12]. In contrast, the account of the corrections to the wave function is very important, starting from relatively small Z. We calculate the correction Δg using the relativistic wave function and the leading approximation for the electron loop. As a result we have obtained good agreement with the numerical data of [6] even for very large Z (difference is 4% for $Z=92$). Using such an approach, we have calculated the corresponding correction of the electron loop to the g factor of a bound muon.

II. GENERAL RELATIONS

Let us consider the amplitude T of interaction of homogeneous magnetic field $\boldsymbol{B}$ with the electron bound in a hydrogenlike ion. In the zero approximation, it reads (see, e.g., [13])

$$T^{(0)} = e\int \frac{d\boldsymbol{k}}{(2\pi)^3}\boldsymbol{A_k}\cdot\boldsymbol{j_k^*} = \frac{e\kappa\boldsymbol{B}\cdot\langle\boldsymbol{J}\rangle}{J(J+1)}\int_0^\infty drr^3 f_1(r)f_2(r), \tag{2}$$

where $\boldsymbol{j_k}$ is the Fourier transform of the electron current $\boldsymbol{j}(\boldsymbol{r})=\bar{\psi}(\boldsymbol{r})\gamma\psi(\boldsymbol{r})$, the wave function ψ has the form

$$\psi(\boldsymbol{r}) = \begin{pmatrix} f_1(r)\Omega \\ if_2(r)\tilde{\Omega}\end{pmatrix}, \tag{3}$$

Ω is the spherical spinor [14] with angular momentum J and orbital momentum L, $\tilde{\Omega}=-(\boldsymbol{\sigma}\cdot\boldsymbol{n})\Omega$, and $\kappa=(J+1/2)\mathrm{sgn}(L-J)$. In Eq. (2) we have used the relation

$$i\boldsymbol{k}\times\boldsymbol{A_k} = (2\pi)^3\delta(\boldsymbol{k})\boldsymbol{B}. \tag{4}$$

Note that the sign of $T^{(0)}$ is opposite to that of the Hamiltonian. Substituting the radial wave functions $f_1(r)$ and $f_2(r)$ for the Coulomb field (see, e.g., [14]), we obtain, for the arbitrary bound state,

$$T^{(0)} = \frac{e\boldsymbol{B}\cdot\langle\boldsymbol{J}\rangle}{2m}g,$$

$$g = \frac{2\kappa}{1-4\kappa^2}\left(1-\frac{2\kappa\varepsilon}{m}\right) = \frac{2\kappa}{1-4\kappa^2}\left(1-\frac{2\kappa}{\sqrt{1+(Z\alpha)^2/(\gamma+n_r)^2}}\right), \tag{5}$$

where n_r is the radial quantum number, ε is the binding energy, and $\gamma=\sqrt{\kappa^2-(Z\alpha)^2}$. The particular cases of this formula obtained earlier are presented in [13]. In the nonrelativistic approximation ($Z\alpha \ll 1$), Eq. (2) turns into

$$T_0^{(0)} = \frac{e\boldsymbol{B}\cdot\langle\boldsymbol{J}\rangle}{2m}g_0, \quad g_0 = \frac{2\kappa}{2\kappa+1}. \tag{6}$$

We now pass to the calculation of the amplitude $T^{(1)}$ corresponding to the diagram shown in Fig. 1. It has the form

$$\mathcal{T}^{(1)} = e\int\frac{d\boldsymbol{k}}{(2\pi)^3}\int\frac{d\boldsymbol{q}}{(2\pi)^3}\frac{4\pi}{\boldsymbol{q}^2}A^i_{\boldsymbol{k}}\mathcal{M}^{il}j^{l*}_{\boldsymbol{q}}, \tag{7}$$

where the amplitude $\mathcal{M}^{il}$ of the virtual Delbrück scattering in the case $k \ll m$ has the form following from the gauge invariance:

$$\mathcal{M}^{il} = \frac{\alpha}{m^3}[\delta^{il}(\mathbf{k}\cdot\mathbf{q}) - q^ik^l]\mathcal{F}(q/m,Z\alpha). \tag{8}$$

Note that $\mathcal{F}$ is even function of $Z\alpha$. In the approximation leading in $Z\alpha$ (contribution of light-by-light scattering),

$$\mathcal{F}(q/m,Z\alpha) = (Z\alpha)^2F(q/m), \tag{9}$$

with $F(0)=7/1152$; see Ref. [9]. From Eqs. (4), (7), (8), and (9) we obtain

$$T^{(1)} = e\frac{4\kappa\alpha(Z\alpha)^2\boldsymbol{B}\cdot\langle\boldsymbol{J}\rangle}{\pi m^3 J(J+1)}\int_0^\infty dqF(q/m)\int_0^\infty drrf_1(r)f_2(r) \times\left(\frac{\sin qr}{qr}-\cos qr\right). \tag{10}$$

Using the relation $\mathcal{M}^{ii}=[2\alpha(Z\alpha)^2(\boldsymbol{k}\cdot\boldsymbol{q})/m^3]F(q/m)$, following from Eq. (8), and the gauge invariance of the light-by-light scattering amplitude, we can represent $F(q/m)$ in the form

$$F(q/m) = \frac{m^3}{2\pi}\int\frac{d\boldsymbol{Q}}{\boldsymbol{Q}^2(\boldsymbol{q}-\boldsymbol{Q})^2}\frac{\boldsymbol{q}\cdot(\nabla_{\boldsymbol{k}}M)|_{\boldsymbol{k}=0}}{\boldsymbol{q}^2},$$

$$M = 2i\int\frac{d^4p}{(2\pi)^4}\mathrm{Sp}\{G(p)\gamma^iG(p-k)\gamma^0[G(p+Q-q)\gamma^iG(p+Q)\gamma^0 + G(p+Q-q)\gamma^0G(p-q)\gamma^i + G(p-Q-k)\gamma^0G(p-q)\gamma^i]\}, \tag{11}$$

where $G(p)=[\hat{p}-m]^{-1}$ is a free-electron propagator. Straightforward calculation leads to the representation of the function F in the form of a twofold integral with respect to the Feynman parameters. The resulting formulas, being rather cumbersome, are not presented here explicitly. For $x=q/m \ll 1$, the first two terms of the expansion of the function $F(x)$ have the form

$$F(x) = \frac{7}{1152}\left(1+\frac{8}{35}x\right). \tag{12}$$

The first term in this formula agrees with the result of Ref. [9]. For $x \gg 1$, the asymptotics of the function $F(x)$ reads

$$F(x) = \frac{1}{2x^3}. \tag{13}$$

For arbitrary x, we performed the numerical tabulation of the function $F(x)$. The result is shown in Fig. 2 and Table I.

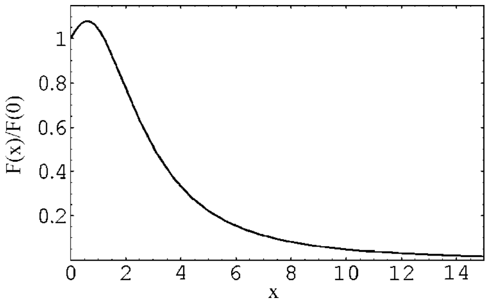

FIG. 2. The ratio $F(x)/F(0)$ as a function of $x=q/m$.

III. CORRECTION TO THE g FACTOR AT SMALL $Z\alpha$

In order to obtain the leading term of expansion in $Z\alpha$ of the amplitude $\mathcal{T}^{(1)}$, it is sufficient to use Eq. (10) with the substitution $F(q/m)\to F(0)$ and the wave functions taken in the nonrelativistic approximation. In this approximation $f_1(r)$ coincides with $R(r)$, the radial part of the nonrelativistic wave function, and

$$f_2(r)=\frac{1}{2m}\left(R'(r)+\frac{1+\kappa}{r}R(r)\right). \tag{14}$$

The correction Δg to the Landé factor is determined by the relation

$$\frac{\Delta g}{g_0}=\frac{\mathcal{T}^{(1)}}{\mathcal{T}_0^{(0)}}. \tag{15}$$

Taking in Eq. (10) the integral over q and then over r, we obtain the leading contribution Δg_0 for the arbitrary state:

$$\frac{\Delta g_0}{g_0}=\frac{7\alpha(Z\alpha)^5}{144n^3(2L+1)\kappa(2\kappa-1)}=\frac{7\alpha(Z\alpha)^5}{288n^3J(J+1)(2J+1)}, \tag{16}$$

where $n=n_r+|\kappa|$ is a principal quantum number. For S states $(L=0,\kappa=-1)$, this result is in agreement with Eq. (1) obtained in Ref. [9].

The relativistic corrections to the wave function as well as the corrections to the magnetic loop have the relative magnitude $(Z\alpha)^2$. Therefore, the term Δg_1 of the order $\alpha(Z\alpha)^6$ can also be obtained with the use of the nonrelativistic wave functions and magnetic loop in the leading approximation (light-by-light scattering diagrams). For $L\neq 0$, it is sufficient to substitute the second term of expansion of $F(x)$ [see Eq. (12)] in Eq. (10). Then we obtain

$$\frac{\Delta g_1}{g_0}=\frac{2\alpha(Z\alpha)^6}{45\pi n^3(2L+1)(2\kappa-1)^2}\left(\frac{3}{L(L+1)}-\frac{1}{n^2}\right). \tag{17}$$

For S states, calculation of Δg_1 is more complicated. For the nS state, $f_1(r)f_2(r)=(\pi/m)\rho_n'(r)$, where $\rho_n(r)$ is the electron density in the nonrelativistic approximation. Substitution of Eq. (12) into Eq. (10) leads to a logarithmic divergence. Therefore, it is convenient to split the region of integration over r in Eq. (10) into two: $[0,r_0]$ and $[r_0,\infty)$ with $1/m \ll r_0 \ll 1/(mZ\alpha)$. In the first region, we can replace $\rho'(r)$ by $\rho'(0)$ and take the integral over r. In the second region, we can use the expansion (12) and take the integral over q. The sum of these two contributions, as it should be, is independent of r_0. The final result reads

TABLE I. Function $F(x)/F(0)$ versus $x=q/m$; $F(0)=7/1152$.

x	$F(x)/F(0)$	x	$F(x)/F(0)$	x	$F(x)/F(0)$	x	$F(x)/F(0)$	x	$F(x)/F(0)$	x	$F(x)/F(0)$
0.0	1.0	0.42	1.07	1.5	0.927	5.0	0.222	26	3.81×10^{-3}	128	3.75×10^{-5}
0.05	1.01	0.44	1.07	1.6	0.897	5.5	0.184	28	3.1×10^{-3}	144	2.66×10^{-5}
0.1	1.02	0.46	1.07	1.7	0.867	6.0	0.154	30	2.55×10^{-3}	160	1.94×10^{-5}
0.15	1.03	0.48	1.08	1.8	0.837	6.5	0.13	32	2.12×10^{-3}	176	1.46×10^{-5}
0.16	1.03	0.5	1.08	1.9	0.806	7.0	0.11	36	1.52×10^{-3}	192	1.13×10^{-5}
0.17	1.04	0.55	1.08	2.0	0.776	7.5	9.45×10^{-2}	40	1.12×10^{-3}	208	8.91×10^{-6}
0.18	1.04	0.6	1.08	2.1	0.746	8.0	8.15×10^{-2}	44	8.55×10^{-4}	224	7.16×10^{-6}
0.19	1.04	0.65	1.08	2.2	0.716	9	6.18×10^{-2}	48	6.66×10^{-4}	240	5.82×10^{-6}
0.2	1.04	0.7	1.08	2.3	0.687	10	4.79×10^{-2}	52	5.28×10^{-4}	256	4.8×10^{-6}
0.22	1.04	0.75	1.08	2.4	0.659	11	3.78×10^{-2}	56	4.26×10^{-4}	288	3.38×10^{-6}
0.24	1.05	0.8	1.07	2.5	0.631	12	3.04×10^{-2}	60	3.49×10^{-4}	320	2.47×10^{-6}
0.26	1.05	0.85	1.07	2.6	0.605	13	2.47×10^{-2}	64	2.89×10^{-4}	352	1.86×10^{-6}
0.28	1.05	0.9	1.06	2.7	0.579	14	2.04×10^{-2}	72	2.05×10^{-4}	384	1.43×10^{-6}
0.3	1.06	0.95	1.05	2.8	0.554	15	1.7×10^{-2}	80	1.5×10^{-4}	416	1.13×10^{-6}
0.32	1.06	1.0	1.05	2.9	0.531	16	1.43×10^{-2}	88	1.14×10^{-4}	448	9.04×10^{-7}
0.34	1.06	1.1	1.03	3.0	0.508	18	1.05×10^{-2}	96	8.8×10^{-5}	480	7.36×10^{-7}
0.36	1.06	1.2	1.01	3.5	0.409	20	7.86×10^{-3}	104	6.95×10^{-5}	512	6.07×10^{-7}
0.38	1.07	1.3	0.981	4.0	0.331	22	6.05×10^{-3}	112	5.58×10^{-5}	1000	8.18×10^{-8}
0.4	1.07	1.4	0.955	4.5	0.27	24	4.76×10^{-3}	120	4.55×10^{-5}	2000	1.03×10^{-8}

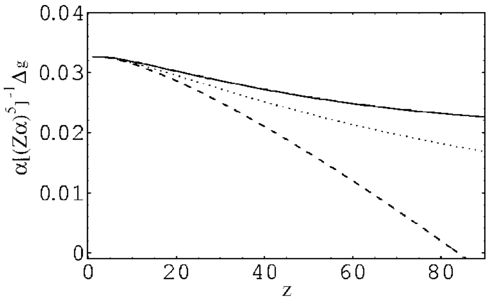

FIG. 3. The correction Δg for $1S_{1/2}$. Solid curve: the exact result. Dashed curve: $\Delta g_0+\Delta g_1$. Dotted curve: Δg_{nr}.

$$\frac{\Delta g_1}{g_0} = \frac{4\alpha(Z\alpha)^6}{135\pi n^3}\left(\ln\frac{1}{Z\alpha} - a - b_n\right),$$

$$a = -\frac{1}{2} + \frac{35}{8}\int_0^\infty dx \ln xF''(x) \approx 2.6,$$

$$b_n = -C + \frac{1}{\rho_n'(0)}\int_0^\infty dr \ln(mZ\alpha r)\rho_n''(r), \qquad (18)$$

where $C=0.577\ldots$ is the Euler constant. For each n, the coefficient b_n can be easily calculated so that $b_1=\ln 2\approx 0.693$, $b_2=5/8=0.625$, $b_3=55/54+\ln(2/3)\approx 0.613$, and $b_\infty=C+\ln 2-2/3\approx 0.604$.

IV. CORRECTION TO THE g FACTOR AT $Z\alpha\sim 1$

As was pointed out in the Introduction, the sum $\Delta g_0+\Delta g_1$ gives a good approximation to Δg only for small Z. For intermediate Z, it is necessary to account for the next terms in $Z\alpha$. The largest corrections are due to the significant difference between the relativistic wave function and the nonrelativistic one already at intermediate Z. At the same time, the difference between the function $\mathcal{F}$ and its leading approximation $(Z\alpha)^2F$ results in corrections which are numerically small even for large Z. Using the numerical results for $F(x)$ and the relativistic wave functions, we have performed the tabulation of Δg for various Z, using $T^{(1)}$ from Eq. (1) as an approximation to $T^{(1)}$. The results of this tabulation for $1S_{1/2}$, $2S_{1/2}$, and $2P_{1/2}$ states are presented in Table II. For the $1S_{1/2}$ state, we also present the contribution of the first two terms of expansion in $Z\alpha$, Eqs. (18) and (16), and the correction Δg_{nr} obtained with the use of nonrelativistic wave functions. The results for $1S_{1/2}$ are also shown in Fig. 3.

For $Z<10$, both $\Delta g_0+\Delta g_1$ and Δg_{nr} coincide with Δg with an accuracy better than 1%. The difference grows with Z, reaching 10% at $Z\sim 30$ for $\Delta g_0+\Delta g_1$ and at $Z\sim 50$ for Δg_{nr}.

In Table II, we also show the results of numerical tabulation from Ref. [6] for the $1S_{1/2}$ state. For $30<Z<70$, our result for Δg agrees with that obtained in Ref. [6] within 1%–2% percent. The difference between these two results for $Z<30$ is due to the poor accuracy of the numerical results of Ref. [6]. For $Z>70$ the difference increases and becomes 8% for $Z=92$. This difference corresponds to the contribution of next-to-leading terms in the magnetic loop, which was taken into account in Ref. [6] and omitted in our paper. Thus, the effect of these terms is small in a wide region of Z, while the relativistic effects in the wave function become important already at relatively small Z.

V. CORRECTION Δg FOR MUONIC ATOMS

The correction Δg to the g factor of a bound muon due to the electron magnetic loop can be obtained from Eq. (10) with $f_1(r)$ and $f_2(r)$ being the wave functions of the muon. The asymptotics of Δg for $\mu Z\alpha/(mn^2)\approx 1.5Z/n^2\gg 1$ (μ is the muon mass) can be calculated as follows. We split the region of integration over q in Eq. (10) into two: $[0,q_0]$ and $[q_0,\infty)$ with $m\ll q_0\ll \mu Z\alpha/n^2$. In the first region we can replace $[(qr)^{-1}\sin qr-\cos qr]$ by $(qr)^2/3$ and take the integral over r. In the second region we can use the asymptotics, Eq. (13), and take the integral over q. Summing these two contributions, we obtain

$$\Delta g_{as} = g\frac{2\alpha(Z\alpha)^2}{3\pi}[\ln(\mu Z\alpha/m) - A - B],$$

$$A = 2\int_0^\infty dy \ln y\partial_y(y^3F(y)) \approx 2.24,$$

$$B = C - \frac{4}{3} - \frac{4}{Z\alpha(1-2\kappa\varepsilon/\mu)}\int_0^\infty dx x^3\tilde{f}_1(x)\tilde{f}_2(x)\ln x,$$

$$\tilde{f}_1(x) = (\mu Z\alpha)^{-3/2}f_1(x/\mu Z\alpha), \quad \tilde{f}_2(x) = (\mu Z\alpha)^{-3/2}f_2(x/\mu Z\alpha), \qquad (19)$$

where g is defined in Eq. (5). For the $1S_{1/2}$ state we obtain

$$g = \frac{2}{3}(1+2\gamma), \quad B = C - \frac{4}{3} + \psi(2\gamma+2) - \ln 2,$$
$$\gamma = \sqrt{1-(Z\alpha)^2}. \qquad (20)$$

For $n=n_r+|\kappa|\gg 1$, we have

$$g = g_0 = \frac{2\kappa}{2\kappa+1}, \quad B = C + \ln(n^2/2). \qquad (21)$$

Formula (19) can be interpreted as follows. In Ref. [15] the logarithmic contribution of the electron vacuum polarization to the magnetic moment of a heavy nucleus was calculated. The result obtained has the form

$$\frac{\Delta g}{g} = \frac{2\alpha(Z\alpha)^2H(Z\alpha)}{3\pi}\ln(1/mR_{nucl}), \qquad (22)$$

where R_{nucl} is the nuclear radius, $R_{nucl}\ll 1/m$. The coefficient $(Z\alpha)^2H(Z\alpha)$ was calculated exactly in $Z\alpha$—i.e., with an account of all Coulomb corrections to the electron loop. The function $H(Z\alpha)$ tends to unity when $Z\alpha\to 0$ and significantly differs from unity only for very large Z. The large logarithm $\ln(1/mR_{nucl})$ in Eq. (22) appears as a result of integration

TABLE II. The quantity $n^3\Delta g$ in units 10^{-6}, calculated in various approximations for $1S_{1/2}$, $2S_{1/2}$, and $2P_{1/2}$ states. Our results are obtained with the account for the magnetic loop in the leading approximation (contribution of light-by-light scattering). The quantity Δg_{nr} denotes the correction obtained with the use of Eq. (10) with the functions $f_1(r)$ and $f_2(r)$ taken in the nonrelativistic approximation; see Eq. (14).

	$1S_{1/2}$				$2S_{1/2}$	$2P_{1/2}$
Z	$\Delta g_0+\Delta g_1$	Δg_{nr}	Δg	Δg (Ref. [6])	$8\Delta g$	$8\Delta g$
1	4.935×10^{-9}	4.934×10^{-9}	4.934×10^{-9}		4.936×10^{-9}	1.638×10^{-9}
2	1.58×10^{-7}	1.58×10^{-7}	1.58×10^{-7}		1.58×10^{-7}	5.26×10^{-8}
3	1.2×10^{-6}	1.2×10^{-6}	1.2×10^{-6}		1.2×10^{-6}	4.01×10^{-7}
4	5.04×10^{-6}	5.04×10^{-6}	5.04×10^{-6}		5.05×10^{-6}	1.69×10^{-6}
5	1.53×10^{-5}	1.53×10^{-5}	1.54×10^{-5}		1.54×10^{-5}	5.18×10^{-6}
6	3.79×10^{-5}	3.8×10^{-5}	3.81×10^{-5}		3.82×10^{-5}	1.29×10^{-5}
7	8.16×10^{-5}	8.17×10^{-5}	8.2×10^{-5}		8.23×10^{-5}	2.81×10^{-5}
8	1.58×10^{-4}	1.58×10^{-4}	1.59×10^{-4}		1.6×10^{-4}	5.49×10^{-5}
9	2.83×10^{-4}	2.84×10^{-4}	2.86×10^{-4}		2.87×10^{-4}	9.94×10^{-5}
10	4.76×10^{-4}	4.78×10^{-4}	4.82×10^{-4}		4.84×10^{-4}	1.69×10^{-4}
11	7.61×10^{-4}	7.66×10^{-4}	7.72×10^{-4}	$3(3)\times10^{-4}$	7.76×10^{-4}	2.73×10^{-4}
12	1.17×10^{-3}	1.18×10^{-3}	1.19×10^{-3}	$4(5)\times10^{-4}$	1.19×10^{-3}	4.24×10^{-4}
13	1.72×10^{-3}	1.74×10^{-3}	1.76×10^{-3}	$8(5)\times10^{-4}$	1.77×10^{-3}	6.35×10^{-4}
14	2.48×10^{-3}	2.51×10^{-3}	2.54×10^{-3}	$1.4(1.0)\times10^{-3}$	2.56×10^{-3}	9.25×10^{-4}
15	3.46×10^{-3}	3.52×10^{-3}	3.57×10^{-3}	$2(1)\times10^{-3}$	3.6×10^{-3}	1.31×10^{-3}
16	4.74×10^{-3}	4.82×10^{-3}	4.9×10^{-3}	$3(1)\times10^{-3}$	4.95×10^{-3}	1.82×10^{-3}
17	6.35×10^{-3}	6.48×10^{-3}	6.6×10^{-3}	$5(2)\times10^{-3}$	6.67×10^{-3}	2.48×10^{-3}
18	8.36×10^{-3}	8.56×10^{-3}	8.73×10^{-3}	$6(2)\times10^{-3}$	8.83×10^{-3}	3.31×10^{-3}
20	1.39×10^{-2}	1.43×10^{-2}	1.46×10^{-2}	$1.0(3)\times10^{-2}$	1.48×10^{-2}	5.66×10^{-3}
24	3.28×10^{-2}	3.45×10^{-2}	3.56×10^{-2}	$3.3(3)\times10^{-2}$	3.63×10^{-2}	1.44×10^{-2}
28	6.72×10^{-2}	7.22×10^{-2}	7.53×10^{-2}	$6.9(3)\times10^{-2}$	7.7×10^{-2}	3.19×10^{-2}
32	0.123	0.136	0.144	0.138	0.148	6.37×10^{-2}
36	0.207	0.238	0.254	0.249	0.262	0.118
40	0.325	0.389	0.421	0.410	0.437	0.206
44	0.481	0.607	0.665	0.658	0.695	0.341
48	0.676	0.907	1.01	1.01	1.06	0.545
52	0.904	1.31	1.48	1.48	1.56	0.841
56	1.15	1.84	2.1	2.12	2.24	1.26
60	1.41	2.51	2.92	2.95	3.13	1.85
64	1.63	3.35	3.97	4.03	4.29	2.66
68	1.77	4.4	5.3	5.39	5.77	3.76
72	1.78	5.67	6.96	7.11	7.62	5.23
76	1.55	7.2	9.0	9.24	9.93	7.18
80	0.983	9.02	11.5	11.9	12.8	9.75
83	0.252	10.6	13.7	14.2	15.3	12.2
88	−1.77	13.7	18.1	18.9	20.5	17.5
92	−4.34	16.5	22.5	23.5	25.5	23.1

over distance r in the region $R_{nucl} \ll r \ll 1/m$. We can consider the muonic atom as some nucleus with the effective radius $R_{nucl} \sim n^2/\mu Z\alpha$. In the case $\mu Z\alpha/(mn^2) \gg 1$, we have $R_{nucl} \ll 1/m$. Substituting this radius into Eq. (22) and replacing $H(Z\alpha) \to 1$ (which corresponds to the contribution of light-by-light scattering), we obtain the logarithmically amplified term in Eq. (19). Note that the coefficient n^2 in R_{nucl} corresponds to the asymptotics of B in Eq. (19) at $n \gg 1$. Strictly speaking, the charge of such effective nucleus is $Z-1$, but not Z. However, under the condition $\mu Z\alpha/(mn^2) \approx 1.5Z/n^2 \gg 1$, this difference is not important.

In Table III, we present Δg for the $1S_{1/2}$ state of a muonic atom calculated for arbitrary Z. For comparison, we present also the asymptotics, Eq. (19). As it should be, the accuracy of the asymptotics (19) increases with Z being 4% for $Z=40$ and 1% for $Z=92$.

TABLE III. Δg in units 10^{-6} for $1S_{1/2}$ state of muonic atom. Δg_{as} is the asymptotics (19).

Z	Δg	Δg_{as}	Z	Δg	Δg_{as}
1	1.043×10^{-2}	−0.2701	24	159.5	145.9
2	0.1274	−0.6226	28	233.7	217.9
3	0.484	−0.7983	32	324.0	306.1
4	1.186	−0.6592	36	430.8	411.0
5	2.317	−0.109	40	554.4	532.8
6	3.944	0.9268	44	694.9	671.5
7	6.124	2.508	48	852.3	827.4
8	8.904	4.687	52	1026.0	1000.0
9	12.33	7.506	56	1217.0	1189.0
10	16.43	11.0	60	1424.0	1395.0
11	21.24	15.22	64	1646.0	1616.0
12	26.8	20.17	68	1883.0	1852.0
13	33.13	25.9	72	2134.0	2103.0
14	40.26	32.42	76	2398.0	2366.0
15	48.2	39.77	80	2673.0	2641.0
16	56.98	47.96	83	2886.0	2854.0
17	66.62	57.01	88	3251.0	3219.0
18	77.14	66.94	90	3400.0	3368.0
20	100.9	89.52	92	3550.0	3519.0

In summary, we found higher-order magnetic-loop corrections to the bound g factor in order $\alpha(Z\alpha)^6$ for an arbitrary state. Despite a small coefficient in the leading term of order $\alpha(Z\alpha)^5$ and the logarithmic enhancement of the higher-order contribution, the leading term still dominates for $Z=6$ and $Z=8$, important for experiment. Previously used numerical results show a certain underestimation of the magnetic-loop contribution for $Z<20$. The theoretical description of this contribution presented in this paper is more reliable. The difference of less than a few percent between our analytic results and the numerical calculations of Ref. [6] at high Z (80–90) shows that the contribution of the higher-order terms in the magnetic loop may be safely neglected for $Z\lesssim 50$. We also calculated the correction Δg for the bound muon, and its behavior is very peculiar. All known contributions to the bound g factor scale as n^{-2} or n^{-3}. The correction found in this paper does not contain such a strong suppression factor. This correction is a dominant bound-state QED correction for a bound muon, which even for the $1S_{1/2}$ state supersedes the free-vacuum polarization term [7]. The results obtained significantly diminish the uncertainty of the theoretical predictions for the g factor of a bound particle.

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