Coherence protection by the quantum Zeno effect and nonholonomic control in a Rydberg rubidium isotope

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The protection of the coherence of open quantum systems against the influence of their environment is a very topical issue. A scheme is proposed here which protects a general quantum system from the action of a set of arbitrary uncontrolled unitary evolutions. This method draws its inspiration from ideas of standard error correction (ancilla adding, coding and decoding) and the quantum Zeno effect. A demonstration of our method on a simple atomic system—namely, a rubidium isotope—is proposed.

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I. INTRODUCTION

The uncontrollable interaction of an open quantum system with its environment leads to complete loss of the information initially stored in its quantum state. This phenomenon is commonly referred to as "loss of coherence." The question of how it is possible to avoid the negative influence of this process is one of the most interesting issues in modern quantum mechanics and concerns many different fields of physics, in particular the domains of quantum information and computation.

In the context of quantum information, the effects of interactions with the environment, known as "quantum errors," may render information storage and processing unreliable [1,2]. Since Shor's demonstration that error-correcting schemes exist in quantum computation [3], a general framework of error correction has been built upon the formalism of quantum operations. The main contributions concern quantum codes [4] and, particularly, the class of stabilizer codes [5,6]; other strategies developed suggest the use of "noiseless quantum codes" or "decoherence-free subspaces" [7–9]. All these methods usually demand that errors act independently on different qubits (the independent error model) and make use of the symmetry properties associated with these requirements. This implies that the set of errors to be corrected hence is restricted to a special subgroup, called the Clifford group. In this paper, we present a protection method which draws its inspiration from the ideas of standard error correction and the quantum Zeno effect, and requires no specific symmetry of the errors. Moreover, we suggest its physical implementation in an arbitrary quantum system and show how it works for the example of a rubidium isotope.

The phenomenon known as the quantum Zeno effect (QZE) takes place in a system which is subject to frequent measurements projecting it onto its (necessarily known) initial state: if the time interval between two projections is small enough, the evolution of the system is nearly "frozen." This effect and its inverse (the anti-Zeno effect) have been widely investigated theoretically [10–13] as well as experimentally [14,15]. Generalizations have been proposed which employ incomplete measurements [16]: in this setting, the Hilbert space is split into "Zeno subspaces" (degenerate multidimensional eigenspaces of the measured observable), and the state vector of the system is compelled by frequent measurements of the physical observable to remain in its initial Zeno subspace. The dynamics of the system in the Zeno subspaces has also been studied in different specific situations [17].

Employing these ideas, enriched by standard techniques from coding theory [18], we have previously proposed an information protection scheme [19] in Zanardi's spirit [20], except that we do not make any symmetry assumption on the unitary errors we consider. We form a compound system $\mathcal{S}$ which comprises the information system $\mathcal{I}$ to be protected and an auxiliary system $\mathcal{A}$ (called an ancilla). We then apply a controlled unitary operation $\hat{C}$ (the coding matrix) which encodes the information, initially stored in $\mathcal{I}$, in an entangled state of $\mathcal{I}$ and $\mathcal{A}$. After a short time interval, during which infinitesimal errors may have occurred, we apply the unitary transformation $\hat{C}^{-1}$ (the inverse to the previous step), which decodes information. Finally, we measure the ancilla to get rid of the infinitesimal changes that may have been caused by errors. Whereas in classical error-correction theory, the ancilla contains information about the errors allowing them to be corrected, in our QZE-based approach, the quantum state of the ancilla resulting from an elementary (coding-errors-decoding) sequence is close to its initial state, so that the measurement of the ancilla brings it back to its initial state with a probability of nearly 1. The key point of our method is to entangle the initial state of the ancilla with the state of the system in such a way that the detection of the ancilla in its initial state implies that the system is also in its initial state. This is achieved through the coding procedure which has to ensure that after the exposition of the system to the action of errors, the initial state of the ancilla remains entangled only with the initial state of the system, whereas any other entanglement remains small during the time interval between consecutive measurements.

The coding procedure involves a rather complex unitary transformation in the Hilbert space of the compound system: performing such coding operations is a nontrivial quantum control problem in itself. However, one can achieve this objective by adapting the idea of nonholonomic control which we have previously presented [21]. Specific algorithms which allow us to determine and physically implement the coding matrix $\hat{C}$ have been constructed.

In this paper, we provide a comprehensive presentation of our theoretical scheme, including algorithmic aspects which were not dealt with in [19]. Moreover, we propose a "demonstrative" application of our technique to a physical system: more precisely, we show how our method can protect one qubit of information stored in the spin variable of the quantum state of a single rubidium atom, the orbital variable playing the role of the ancilla, against a given set of error inducing Hamiltonians. Here we describe a realistic experimental setting which achieves the different steps of our scheme through the application of a sequence of laser pulses and culminates in a measurement involving spontaneous emission. When dealing with ensembles of atoms, as usually done in current cold atom experiments, experimental drawbacks arise due to dipolar interactions which forbid the actual implementation of our application. Yet even though not completely satisfactory from an experimental point of view, the example we propose here shows the general scope of our method as well as its physical operationality.

The paper is organized as follows. In Sec. II, we present a multidimensional generalization of the QZE and its application to the protection of information contained in compound systems. In Sec. III, we present the algorithms which enable us to calculate the code space and physically implement the coding matrix through the nonholonomic control technique. In Sec. IV, we focus on the application of our method to a rubidium isotope. In the Appendix, we present the explicit derivation of the code subspace.

II. MULTIDIMENSIONAL ZENO EFFECT AND COHERENCE PROTECTION

We start this section by the geometric presentation of a multidimensional QZE which allows us to protect an arbitrary subspace of the Hilbert space against the action of a set of given interaction Hamiltonians. In the second part of this section, we take advantage of this phenomenon to protect an information-carrying subsystem of a compound quantum system from the influence of some uncontrolled error-inducing external fields.

Consider a quantum system $\mathcal{S}$, whose N-dimensional Hilbert space is denoted by $\mathcal{H}$ and whose time-dependent Hamiltonian has the form

$$\hat{H}(\tau) = \sum_{m=1}^{M} f_m(\tau)\hat{E}_m, \tag{1}$$

where $\{\hat{E}_m\}_{m=1,\ldots,M}$ are M given independent Hermitian matrices on $\mathcal{H}$ and $\{f_m(\tau)\}_{m=1,\ldots,M}$ are M unknown functions of time. The Hamiltonian $\hat{H}(\tau)$ accounts for the errors we want to get rid of. Note that the unperturbed part of the Hamiltonian (1) is assumed to be zero (or proportional to the identity so that one can set it to zero). The standard QZE [10–13] implies that we can nearly "freeze" the evolution of the system by measuring it frequently enough in its (known) initial state; in other words, this effect allows us to protect the one-dimensional subspace spanned by the initial state of the system from the influence of the error-inducing Hamiltonian (1). In what follows, we generalize this effect so as to protect an arbitrary multidimensional subspace $\mathcal{C}$ from $\hat{H}(\tau)$.

Any vector $|\psi\rangle$ of $\mathcal{C}$ evolves according to the operator

$$\hat{U}(t,t_0) = \mathcal{T}\left\{ \exp\left[-i\int_{t_0}^{t} \hat{H}(\tau)d\tau \right] \right\},$$

where $\mathcal{T}$ denotes time ordering and where we set $\hbar=1$. For the QZE to hold, we shall only consider evolution in short time periods, whose duration T is so short that the corresponding action of the M components of the Hamiltonian (1) is small—i.e., $|\hat{E}_m\int_t^{t+T} f_m(\tau)d\tau| \ll 1$. We can thus expand

$$\hat{U}(t+T,t) = \hat{U}_{inf} \simeq \hat{I} - i\sum_{m=1}^{M}\left(\int_t^{t+T} f_m(\tau)d\tau \right)\hat{E}_m. \tag{2}$$

This implies that after a Zeno interval T, the initial state $|\psi\rangle$ is transformed into $|\psi_e\rangle = |\psi\rangle + |\delta\psi_e\rangle$ where $|\delta\psi_e\rangle \simeq -i\Sigma_{m=1}^{M}\varepsilon_m\hat{E}_m|\psi\rangle$ with $\varepsilon_m = (\int f_m(\tau)d\tau)$. Note that, strictly speaking, the operator $\hat{U}_{inf}$ introduced in Eq. (2) is not unitary: nevertheless, the nonunitary part, due to the truncation of the time development of the evolution operator, is of second order in time and is thus negligible in the Zeno limit $(T\to 0)$. Moreover, as we exclusively consider finite-dimensional systems interacting with classical external fields, the approximation, Eq. (2), holds, without raising any mathematical problem. But it is worth emphasizing that this is no longer the case when dealing with systems of infinitely large Hilbert space (for example, see [22]).

Let us assume that we are physically able to perform a measurement-induced projection onto $\mathcal{C}$ in the system $\mathcal{S}$ (see below the discussion of such projections for compound systems comprising an information subsystem and an ancilla). Now if we just follow the standard QZE procedure and merely project the state vector $|\psi_e\rangle$, resulting from the infinitesimal evolution of the initial state $|\psi\rangle$, onto $\mathcal{C}$, we obtain a vector $|\psi_p\rangle$, which (*a priori*) differs from $|\psi\rangle$ [see Fig. 1(a)]. This is due to the fact that usually the operators $\{\hat{E}_m\}_{m=1,\ldots,M}$ do not act orthogonally on $\mathcal{C}$, which means that the vectors $\hat{E}_m|\psi\rangle$ and thus the increment vector $|\delta\psi_e\rangle$ itself are not orthogonal to $\mathcal{C}$. It is a well-known manifestation of the standard Zeno effect: the system is compelled to remain in a "Zeno subspace" (which corresponds here to $\mathcal{C}$) in which it presents a remaining dynamics called "Zeno dynamics" [17]; in our case, this dynamics is unknown and thus threatens the information stored in $\mathcal{C}$. Therefore, we see that the standard Zeno strategy does not suffice to protect a multidimensional subspace: we have to adapt it using some ideas of coding theory.

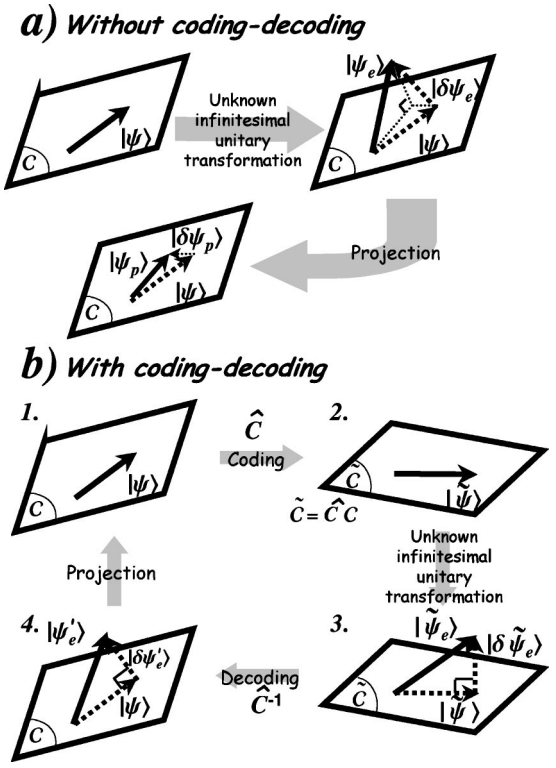

FIG. 1. Multidimensional QZE: (a) a simple projection fails to recover the initial vector, (b) the sequence coding-decoding-projection protects the initial vector.

To this end, we assume a unitary matrix $\hat{C}$ acting on $\mathcal{H}$, which we call the coding matrix, such that the Hermitian operators $\{\hat{E}_m\}_{m=1,\ldots,M}$ act orthogonally on the subspace $\tilde{\mathcal{C}} = \hat{C}\mathcal{C}$, which we call the code space. Let us denote by $I \geqslant 1$ the dimension of $\mathcal{C}$ and by $\{|\gamma_i\rangle\}_{i=1,\ldots,I}$ one of its orthonormal bases; $\{|\tilde{\gamma}_i\rangle = \hat{C}|\gamma_i\rangle\}_{i=1,\ldots,I}$ will denote one of the orthonormal bases of $\tilde{\mathcal{C}}$, the state vectors $|\tilde{\gamma}_i\rangle$ being called the codewords. For any pair $(|\tilde{\gamma}_s\rangle, |\tilde{\gamma}_t\rangle)$ of codewords and any operator $\hat{E}_m \in \{\hat{E}_m\}_{m=1,\ldots,M}$ we have, by the definitions of $\hat{C}$ and $\tilde{\mathcal{C}}$,

$$\langle\tilde{\gamma}_t|\tilde{\gamma}_s\rangle = \delta_{st} \quad \text{(orthonormality condition)}, \tag{3}$$

$$\langle\tilde{\gamma}_t|\hat{E}_m|\tilde{\gamma}_s\rangle = 0 \quad \text{(orthogonality of the errors)}. \tag{4}$$

Equivalently, for any pair $(|\psi\rangle, |\chi\rangle)$ of vectors of $\mathcal{C}$ and for any operator $\hat{E}_m \in \{\hat{E}_m\}_{m=1,\ldots,M}$,

$$\langle\chi|\hat{C}^\dagger\hat{E}_m\hat{C}|\psi\rangle = 0. \tag{5}$$

In particular, for any pair $(|\gamma_s\rangle, |\gamma_t\rangle)$ of basis vectors of $\mathcal{C}$ and for any operator $\hat{E}_m \in \{\hat{E}_m\}_{m=1,\ldots,M}$,

$$\langle\gamma_t|\hat{C}^\dagger\hat{E}_m\hat{C}|\gamma_s\rangle = 0. \tag{6}$$

If we apply the coding matrix $\hat{C}$ to the initial state vector $|\psi\rangle$, before exposing it to the action of the Hamiltonian (1), we obtain the new vector $|\tilde{\psi}\rangle = \hat{C}|\psi\rangle \in \tilde{\mathcal{C}}$ [Figs. 1(b1) and 1(b2)] which is transformed after a Zeno interval T into $|\tilde{\psi}_e\rangle = \hat{U}_{inf}|\tilde{\psi}\rangle = |\tilde{\psi}\rangle + |\delta\tilde{\psi}_e\rangle$, where $|\delta\tilde{\psi}_e\rangle \simeq -i\Sigma_{m=1}^M \varepsilon_m \hat{E}_m|\tilde{\psi}\rangle = -i\Sigma_{m=1}^M \varepsilon_m \hat{E}_m\hat{C}|\psi\rangle$ [Fig. 1(b3)]. Decoding $|\tilde{\psi}_e\rangle$ yields the vector $|\psi'_e\rangle = \hat{C}^{-1}|\tilde{\psi}_e\rangle = |\psi\rangle + |\delta\psi'_e\rangle$ where $|\delta\psi'_e\rangle \simeq -i\Sigma_{m=1}^M \times \varepsilon_m \hat{C}^\dagger\hat{E}_m\hat{C}|\psi\rangle$. From Eq. (5) it can be seen that for any vector $|\chi\rangle \in \mathcal{C}$, $\langle\chi|\delta\psi'_e\rangle = -i\Sigma_{m=1}^M \varepsilon_m \langle\chi|\hat{C}^\dagger\hat{E}_m\hat{C}|\psi\rangle = 0$ which means that $|\delta\psi'_e\rangle$ is orthogonal to $\mathcal{C}$ [Fig. 1(b4)]. A measurement-induced projection onto $\mathcal{C}$ finally recovers the initial vector $|\psi\rangle$ with a probability very close to 1 (the error probability is proportional to T^2). If the (coding-decoding-projection) sequence is frequently repeated, any vector $|\psi\rangle$ of the subspace $\mathcal{C}$ can thus be protected from the Hamiltonian (1) for as long as needed. We stress that the role of projective measurements consists both in confining the system in $\mathcal{C}$ (as in the standard quantum Zeno effect) and in clearing out the erroneous component which has been made orthogonal to $\mathcal{C}$ through coding and decoding.

Let us note that a more general version of conditions (4) can be considered. Indeed, if for any pair of codewords $(|\tilde{\gamma}_s\rangle, |\tilde{\gamma}_t\rangle)$ of $\tilde{\mathcal{C}}$ and any error Hamiltonian $\hat{E}_m \in \{\hat{E}_m\}$,

$$\langle\tilde{\gamma}_t|\hat{E}_m|\tilde{\gamma}_s\rangle = \delta_{ts}\xi_m,$$

where δ_{ts} is the Kronecker symbol and ξ_m a real number depending only on the number m of the error Hamiltonian $\hat{E}_m$, the projection onto $\mathcal{C} = \text{Span}\{|\gamma_i\rangle, i = 1, \ldots, I\}$ of the state vector $|\psi'_e\rangle = |\psi\rangle - i\Sigma_{m=1}^M \varepsilon_m \hat{E}_m|\psi\rangle$, obtained after a (coding-decoding) sequence, yields

$$\hat{\Pi}_C|\psi'_e\rangle = |\psi\rangle - i\sum_{m=1}^M \varepsilon_m \hat{\Pi}_C\hat{E}_m|\psi\rangle,$$

where $\hat{\Pi}_C = [\Sigma_{t=1}^I |\gamma_t\rangle\langle\gamma_t|]$; if we denote by $|\psi\rangle = \Sigma_{s=1}^I \alpha_s|\gamma_s\rangle$ the decomposition of the initial information state vector, $\hat{\Pi}_C\hat{E}_m|\psi\rangle$ has the form

$$\begin{aligned}\hat{\Pi}_C\hat{E}_m|\psi\rangle &= \sum_{s,t=1}^I \alpha_s|\gamma_t\rangle\langle\gamma_t|\hat{C}^\dagger\hat{E}_m\hat{C}|\gamma_s\rangle \\ &= \xi_m\sum_{s,t=1}^I \alpha_s\delta_{ts}|\gamma_t\rangle = \xi_m|\psi\rangle,\end{aligned}$$

which finally leads to $\hat{\Pi}_C|\psi'_e\rangle = (1 - i\Sigma_{m=1}^M \varepsilon_m\xi_m)|\psi\rangle$. In other words, the errors $\hat{E}_m$ just introduce a global phase factor in front of the initial information state vector, but leaves its coherence intact. Obviously, the correction conditions (4) are obtained as a particular case of the above conditions, setting $\xi_m = 0$ for all m. Yet, though less general, they will be employed in the rest of the paper for the sake of simplicity.

The multidimensional generalization of the QZE we have just described allows us to protect any subspace $\mathcal{C}$ of a Hilbert space $\mathcal{H}$ against Hamiltonians of the form (1) provided the projection onto $\mathcal{C}$ is physically achievable and the coding matrix $\hat{C}$ exists. This result is very useful in the context of information protection as we will show in the following paragraphs.

Consider an information system $\mathcal{I}$ of Hilbert space $\mathcal{H}_I$ and dimensionality I. This system is subjected to a set of M error-inducing Hamiltonians $\{\hat{E}_m\}_{m=1,\ldots,M}$ which, for instance, represent interactions of the system with M uncontrolled external fields $f_m(t)$: we want to get rid of this external influence which is likely to result in the loss of the information stored in the initial-state vector $|\psi_I\rangle=\Sigma_{i=1}^{I}c_i|\nu_i\rangle$, where $\{|\nu_i\rangle\}_{i=1,\ldots,I}$ denotes an orthonormal basis of $\mathcal{H}_I$. To this end, we will use the multidimensional Zeno effect. As the multidimensional QZE can only protect a subspace of the whole Hilbert space, we first have to add an A-dimensional auxiliary system $\mathcal{A}$ (called ancilla) to our system $\mathcal{I}$, so that the information is transferred from $\mathcal{H}_I$ into an I-dimensional subspace $\mathcal{C}$ of the $(N=I\times A)$-dimensional Hilbert space $\mathcal{H}=\mathcal{H}_I\otimes\mathcal{H}_A$ of the compound system $\mathcal{S}=\mathcal{I}\otimes\mathcal{A}$ (let us note that this ancilla adding procedure is quite standard in quantum error correction [1] and corresponds to the redundancy used in classical coding methods). Furthermore, we shall suppose that all the state vectors of the different Hilbert spaces $\mathcal{H}_I$, $\mathcal{H}_A$, and hence $\mathcal{H}$ are degenerate in energy so that the unperturbed part $\hat{H}_0$ of the Hamiltonian can be set to zero as in the first part of this section: the subspace $\mathcal{C}$ and the information it carries can thus be protected through the multidimensional QZE (in Sec. IV, in the example of Rb, we shall see that the multidimensional QZE may also be used even though $\hat{H}_0$ is not zero, provided $\hat{H}_0$ and the errors have some convenient properties). Note that $\mathcal{A}$ and $\mathcal{I}$ need not be "physically separate" systems, but only have to possess independent Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_I$. For example, in Sec. IV, we shall consider the rubidium atom as the compound of two independent subsystems: namely, its spin (which plays the role of $\mathcal{I}$) and orbital (which plays the role of $\mathcal{A}$) parts. Doing so, we shall use the terms "factorized" and "entangled" in a generalized manner to designate states obtained as a direct product of the spin and orbital parts, and linear combinations of such states, respectively.

Let us now return to our problem and first consider the simple case in which the ancilla is initially in the pure state $|\alpha\rangle$. The information previously carried by $|\psi_I\rangle\in\mathcal{H}_I$ is then transferred into the factorized state $|\psi\rangle=|\psi_I\rangle\otimes|\alpha\rangle=\Sigma_{i=1}^{I}c_i|\nu_i\rangle\otimes|\alpha\rangle=\Sigma_{i=1}^{I}c_i|\gamma_i\rangle$ which belongs to the tensor product subspace $\mathcal{C}=\mathcal{H}_I\otimes\mathrm{Span}[|\alpha\rangle]=\mathrm{Span}[\{|\gamma_i\rangle=|\nu_i\rangle\otimes|\alpha\rangle\}_{i=1,\ldots,I}]$. In other words, the initial density matrix of the compound system $\mathcal{S}$ is $\hat{\rho}=(|\psi_I\rangle\langle\psi_I|)\otimes(|\alpha\rangle\langle\alpha|)$. After coding (through the matrix $\hat{C}$) it reads $\hat{\tilde{\rho}}=\hat{C}^\dagger\hat{\rho}\hat{C}$; at the end of the action of the errors it is transformed into $\hat{\tilde{\rho}}_e=\hat{U}_{inf}^\dagger\hat{C}^\dagger\hat{\rho}\hat{C}\hat{U}_{inf}$; finally it takes the form $\hat{\rho}_e=\hat{C}\hat{U}_{inf}^\dagger\hat{C}^\dagger\hat{\rho}\hat{C}\hat{U}_{inf}\hat{C}^\dagger$ after decoding. In this setting, the projection onto $\mathcal{C}$ can be simply achieved by measuring the ancilla in its initial state $|\alpha\rangle$. As T is very short, the state of the ancilla evolves just a little within a Zeno interval, such that the probability of detecting it in its initial state $|\alpha\rangle$ and thus of projecting the state of the compound system onto $\mathcal{C}$ is very close to 1. After projection, we trace out the ancilla to obtain the final reduced density matrix $\hat{\rho}_I'=\langle\alpha|\hat{C}\hat{U}_{inf}^\dagger\hat{C}^\dagger\hat{\rho}\hat{C}\hat{U}_{inf}\hat{C}^\dagger|\alpha\rangle$ for the information system $\mathcal{I}$; in the same way, one can calculate the initial reduced density matrix is $\hat{\rho}_I=|\psi_I\rangle\langle\psi_I|$. The variation $\delta\hat{\rho}_I=\hat{\rho}_I'-\hat{\rho}_I$ of the information-space density matrix during the whole process can be expressed as the commutator

$$\delta\hat{\rho}_I=-i\left[\sum_{m=1}^{M}\int f_m(\tau)d\tau\langle\alpha|\hat{C}^\dagger\hat{E}_m\hat{C}|\alpha\rangle,\hat{\rho}_I\right],$$

from which we infer that $\hat{\rho}_I$ satisfies the equation

$$i\frac{d\hat{\rho}_I}{dt}=[\hat{h}_e,\hat{\rho}_I],$$

$$\hat{h}_e=\sum_{m=1}^{M}f_m\langle\alpha|\hat{C}^\dagger\hat{E}_m\hat{C}|\alpha\rangle,$$

where $\hat{h}_e$ is an effective Hamiltonian which is determined by the error-inducing Hamiltonians transformed by the coding and decoding and projected onto the initial state of the ancilla. From Eq. (5) one can infer that $\hat{h}_e=0$ and hence $\hat{\rho}_I$ remains constant in time: as long as we repeat the coding-decoding-ancilla resetting sequence, the information initially stored in $\mathcal{I}$ is protected.

It is not always feasible to directly measure the ancilla independently from the information system; in other words, it is sometimes impossible to perform a projection onto disentangled subspaces of $\mathcal{H}$ of the form $\mathcal{H}_I\otimes\mathrm{Span}[|\alpha\rangle]$: in some cases, as for the Rb atom (Sec. IV), one can only project onto entangled subspaces of the total Hilbert space $\mathcal{H}$. In such a case the information initially stored in the vector $|\psi_I\rangle=\Sigma_{i=1}^{I}c_i|\nu_i\rangle\in\mathcal{H}_I$ is transferred into an entangled state of $\mathcal{I}$ and $\mathcal{A}$ of the form $|\psi\rangle=\Sigma_{i=1}^{I}c_i|\gamma_i\rangle$ where the I vectors $|\gamma_i\rangle$ $(i=1,\ldots,I)$, which form an orthonormal basis of the information-carrying subspace $\mathcal{C}$, are not factorized as earlier but are in general entangled states. Nevertheless, the same method as before can be used in that case to protect information, albeit in a different subspace $\mathcal{C}$.

To conclude this section, let us make a few remarks about our method. We first emphasize that our technique, though inspired by quantum error-correcting codes [1], is very different from them: indeed, in those schemes, the information is encoded in such a way that it can be corrected from the action of a set of errors through a syndrome measurement, followed by a (conditioned) recovery operation, depending on the result of the measurement; on the other hand, in our technique, information is continuously protected by the frequent repetition of a three-step cycle (coding-decoding-projective measurement), in which the projective measurement does not give any indication about which error occurred, but simply clears out the erroneous component of the state vector, which has been made orthogonal to the initial information-carrying subspace through coding and decoding. Let us now return to conditions (3) and (4) imposed on the codewords $\{|\tilde{\gamma}_i\rangle, i=1,\ldots,I\}$ and make two points about them.

(A) We can establish a useful relation between the dimension of the ancilla and the number of correctable error Hamiltonians. The set of the I codewords can be seen as a collection of $2I\times N=2I^2A$ real numbers on which $2I^2+2MI^2=2I^2(1+M)$ constraints, directly derived from Eqs.

(3) and (4), are imposed. The number of free parameters must be larger than the number of constraints; hence, we necessarily have $2I^2A \geqslant 2I^2(1+M)$, which satisfies

$$A - 1 \geqslant M. \tag{7}$$

This condition gives an upper bound on the number of independent error-inducing Hamiltonians that our method can correct simultaneously and is called the "Hamming bound."

(B) We may compare our correctability conditions (4) with the more general conditions (see [1], p. 436) of standard quantum error correction:

$$\forall(|\tilde{\gamma}_s\rangle,|\tilde{\gamma}_t\rangle) \in \tilde{\mathcal{C}}^2, \quad \forall\,(\hat{\mathbf{E}}_k,\hat{\mathbf{E}}_l) \in \{\hat{\mathbf{E}}_j(\{\hat{E}_m\})\},$$

$$\langle\tilde{\gamma}_t|\hat{\mathbf{E}}_k^\dagger\hat{\mathbf{E}}_l|\tilde{\gamma}_s\rangle = \alpha_{kl}\langle\tilde{\gamma}_t|\tilde{\gamma}_s\rangle, \tag{8}$$

which ensure the existence of a code space that is completely protected against the error-inducing Hamiltonians $\hat{E}_m$. Here α_{kl} are complex numbers, and the set $\{\hat{E}_m\}$ of Hermitian operators $\hat{E}_m$ generates a group $\mathcal{G}(\{\hat{E}_m\})$ of all possible error-induced evolutions (2). By $\{\hat{\mathbf{E}}_j(\{\hat{E}_m\})\}$ we denote a complete basis set of operators which spans the space of evolution operators $\hat{U}$ and allows one to represent any $\hat{U}$ as a linear combination of the basis operators $\hat{\mathbf{E}}_j$. The variety of all linear combinations of $\hat{\mathbf{E}}_j$ includes not only all $\hat{E}_m$ but also many other operators given by commutators of all orders in $\hat{E}_m$ entering the expansion of $\hat{U}$ for long times. The condition (8) is therefore much more restrictive than Eq. (4). Moreover, even for just two generic matrices $\hat{E}_m$, the basis $\{\hat{\mathbf{E}}_j\}$ spans the entire Hilbert space $\mathcal{H}$, yielding $\tilde{\mathcal{C}}=\varnothing$. Only if the set $\{\hat{E}_m\}$ belongs to an extraspecial algebra restricting the error evolution operators $\hat{U}$ to a subgroup $\mathcal{G}(\{\hat{E}_m\}) \subset \mathcal{G}_U(\mathcal{H})$ of the full unitary group in $\mathcal{H}$ may a nontrivial code space $\tilde{\mathcal{C}}$ exist. The Zeno effect is the only way to suppress loss of coherence if it is not the case.

III. CODE SPACE AND THE CODING MATRIX

It is sometimes possible to build the code space $\tilde{\mathcal{C}}$ explicitly from physical considerations: The Appendix gives an example of a situation in which the code basis can be found directly. In general, however, we need an algorithm to calculate the code basis $\{|\tilde{\gamma}_i\rangle\}_{i=1,\ldots,I}$ or, equivalently, the coding matrix $\hat{C}$. We start this section by describing this algorithm. Then, in a second part, we show that the nonholonomic control technique [21] can be employed to implement the coding matrix physically. We also provide an algorithm which achieves the appropriate control.

Let us first make a remark which will be useful in what follows. Consider a vector $|\mathsf{C}\rangle$ of some Hilbert space and a matrix $\hat{\mathsf{E}}$ on this space. From the vector $|\mathsf{C}\rangle$ we want to calculate a vector $|\tilde{\mathsf{C}}\rangle$ such that $\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle=0$. If $\langle\mathsf{C}|\hat{\mathsf{E}}|\mathsf{C}\rangle=0$, then $|\mathsf{C}\rangle=|\tilde{\mathsf{C}}\rangle$ and the function

$$f_{\tilde{\mathsf{C}}}(\lambda) = \|\|\tilde{\mathsf{C}}\rangle + \lambda\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle\|^2,$$

depending on the c number λ, is minimal for $\lambda=0$: indeed,

$$\|\|\tilde{\mathsf{C}}\rangle + \lambda\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle\|^2 = \langle\tilde{\mathsf{C}}|\tilde{\mathsf{C}}\rangle + \lambda\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle + \lambda^*\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}^\dagger|\tilde{\mathsf{C}}\rangle + |\lambda|^2\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}^\dagger\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle = 1 + |\lambda|^2\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}^\dagger\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle,$$

and as $\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}^\dagger\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle \geqslant 0$, $f_{\tilde{\mathsf{C}}}(\lambda)$ is minimal for $|\lambda|=0$—that is, $\lambda=0$. But if $\langle\mathsf{C}|\hat{\mathsf{E}}|\mathsf{C}\rangle \neq 0$, we can apply the following iterative method: we minimize $f_{\mathsf{C}}(\lambda)$ with respect to λ, and then we set $|\mathsf{C}'\rangle=|\mathsf{C}\rangle+(\lambda/2)\hat{\mathsf{E}}|\mathsf{C}\rangle$ and take $|\mathsf{C}'\rangle/\sqrt{\langle\mathsf{C}'|\mathsf{C}'\rangle}$ as our new $|\mathsf{C}\rangle$; we repeat this sequence as long as needed: $|\mathsf{C}\rangle$ finally tends to $|\tilde{\mathsf{C}}\rangle$, such that $\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}|\tilde{\mathsf{C}}\rangle=0$.

Let us now return to our problem and show how the previous remark can help us. What we want is to find I vectors $|\tilde{\gamma}_i\rangle$ which meet the conditions (3) and (4); equivalently, we can say that we look for an orthonormal basis in which all the matrices $\hat{\mathsf{E}}_k$ have their $I\times I$ upper left blocks equal to zero. To solve this problem, one can first be tempted to use standard techniques of linear algebra, in particular matrix diagonalization: however, it appears that these methods do not work, except in the trivial case when all the matrices $\hat{\mathsf{E}}_k$ have a common kernel, which is much more than what conditions (3) and (4) require. So we propose to transform our initial problem in such a way that it can be dealt with by the iterative algorithm presented in the previous paragraph. Let us combine the I vectors $|\tilde{\gamma}_i\rangle$ into a $(N\times I)$ "supervector"

$$|\tilde{\mathsf{C}}\rangle = \begin{pmatrix} |\tilde{\gamma}_1\rangle \\ \vdots \\ |\tilde{\gamma}_I\rangle \end{pmatrix}.$$

Then let us build

$$E = \left(\frac{I(I-1)}{2} + M\frac{I(I+1)}{2}\right)$$

different $[(N\times I)\times(N\times I)]$-dimensional supermatrices $\hat{\mathsf{E}}_k$ in the following way: we consider them as made of I^2 blocks of dimension $N\times N$ and we successively fill each of these blocks with the different Hamiltonians $\hat{E}_m$ or the identity matrix $\hat{I}$ or 0. To be more explicit, the first $I(I-1)/2$ matrices are built by simply placing the $N\times N$ identity matrix in each of the $I(I-1)/2$ blocks situated above the diagonal. In the last $MI(I+1)/2$ ones, the M operators $\hat{E}_m$ are successively placed in each of the $I(I+1)/2$ blocks on and above the diagonal. One can thus reformulate the conditions (3) as follows: for $1 \leqslant k \leqslant I(I-1)/2$,

$$\langle\tilde{\mathsf{C}}|\hat{\mathsf{E}}_k|\tilde{\mathsf{C}}\rangle = 0.$$

Note that this form does not take the normalization condition into account, which will be imposed differently. Similarly, the conditions (4) are translated into the following form: for $I(I-1)/2+1 \leqslant k \leqslant I(I-1)/2+MI(I+1)/2$,

$$\langle \tilde{\mathsf{C}} | \hat{\mathsf{E}}_k | \tilde{\mathsf{C}} \rangle = 0.$$

Our initial multivectorial problem given by Eqs. (3) and (4) has thus been transformed into a simpler one which can be handled by the same kind of iterative algorithm as in our preliminary remark: we just have to find a $(N \times I)$-dimensional supervector $|\tilde{\mathsf{C}}\rangle$ such that for $1 \leqslant k \leqslant I(I-1)/2 + MI(I+1)/2$,

$$\langle \tilde{\mathsf{C}} | \hat{\mathsf{E}}_k | \tilde{\mathsf{C}} \rangle = 0.$$

Let us now review our iterative algorithm in more detail. First we randomly pick a supervector $|\mathsf{C}_0\rangle$ which will be the starting point of the first step: we normalize this vector by imposing to each of its I components to have norm $= 1/I$. If one of the components of $|\mathsf{C}_0\rangle$ is non-normalizable—that is, equals zero—we pick up a new random supervector $|\mathsf{C}_0\rangle$ as a starting point.

Then, as in our preliminary remark, we minimize the function

$$F_{\mathsf{C}_0}(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_E^{(0)}) = \sum_{k=1}^{E} \| |\mathsf{C}_0\rangle + \lambda_k^{(0)} \hat{\mathsf{E}}_k |\mathsf{C}_0\rangle \|^2,$$

with respect to the E c numbers $\lambda_k^{(0)}$: actually, we separate the real and imaginary parts of $\lambda_k^{(0)} = \alpha_k^{(0)} + i\beta_k^{(0)}$ and calculate the appropriate $\alpha_k^{(0)}$'s and $\beta_k^{(0)}$'s by solving the set of $2E$ equations

$$\frac{\partial F_{\mathsf{C}_0}}{\partial \alpha_k^{(0)}} = 0,$$

$$\frac{\partial F_{\mathsf{C}_0}}{\partial \beta_k^{(0)}} = 0,$$

which can be translated into the linear system

$$\hat{K}(|\mathsf{C}_0\rangle) \cdot \vec{\Lambda}^{(0)} = \vec{D}(|\mathsf{C}_0\rangle),$$

where $\hat{K}(|\mathsf{C}_0\rangle)$ is a $(2E \times 2E)$-dimensional real matrix defined by

$$\hat{K}_{ij}(|\mathsf{C}_0\rangle) = \begin{cases} \mathrm{Re}(\langle \mathsf{C}_0 | \hat{\mathsf{E}}_i^\dagger \hat{\mathsf{E}}_j | \mathsf{C}_0 \rangle) & \text{for } 1 \leqslant i \leqslant E \text{ and } 1 \leqslant j \leqslant E, \\ -\mathrm{Im}(\langle \mathsf{C}_0 | \hat{\mathsf{E}}_i^\dagger \hat{\mathsf{E}}_{j-E} | \mathsf{C}_0 \rangle) & \text{for } 1 \leqslant i \leqslant E \text{ and } 1 + E \leqslant j \leqslant 2E, \\ \mathrm{Im}(\langle \mathsf{C}_0 | \hat{\mathsf{E}}_{i-E}^\dagger \hat{\mathsf{E}}_j | \mathsf{C}_0 \rangle) & \text{for } 1 + E \leqslant i \leqslant 2E \text{ and } 1 \leqslant j \leqslant E, \\ \mathrm{Re}(\langle \mathsf{C}_0 | \hat{\mathsf{E}}_{i-E}^\dagger \hat{\mathsf{E}}_{j-E} | \mathsf{C}_0 \rangle) & \text{for } 1 + E \leqslant i \leqslant 2E \text{ and } 1 + E \leqslant j \leqslant 2E, \end{cases}$$

$\vec{D}(|\mathsf{C}_0\rangle)$ is a $2E$-dimensional real vector defined by

$$\vec{D}(|\mathsf{C}_0\rangle) = \begin{cases} -\mathrm{Re}(\langle \mathsf{C}_0 | \hat{\mathsf{E}}_i | \mathsf{C}_0 \rangle) & \text{for } 1 \leqslant i \leqslant E, \\ \mathrm{Im}(\langle \mathsf{C}_0 | \hat{\mathsf{E}}_{i-E} | \mathsf{C}_0 \rangle) & \text{for } E + 1 \leqslant i \leqslant 2E, \end{cases}$$

and $\vec{\Lambda}^{(0)}$ is a $2E$-dimensional real vector containing the parameters $\alpha_k^{(0)}$'s and $\beta_k^{(0)}$'s:

$$\vec{\Lambda}^{(0)} = \begin{pmatrix} \alpha_1^{(0)} \\ \vdots \\ \alpha_E^{(0)} \\ \beta_1^{(0)} \\ \vdots \\ \beta_E^{(0)} \end{pmatrix}.$$

Once the c numbers $\lambda_k^{(0)} = (\alpha_k^{(0)} + i\beta_k^{(0)})$'s have been found, we calculate $|\Delta \mathsf{C}_0\rangle = \Sigma_k \lambda_k^{(0)} \hat{\mathsf{E}}_k |\mathsf{C}_0\rangle$ and $|\mathsf{C}_0'\rangle = |\mathsf{C}_0\rangle + \frac{1}{2}|\Delta \mathsf{C}_0\rangle$. We normalize $|\mathsf{C}_0'\rangle$ by requiring each of its I components to have the norm $= 1/I$ and take the result of this operation as our new starting point $|\mathsf{C}_1\rangle$. If one of the components of $|\mathsf{C}_0'\rangle$ is non-normalizable—that is, equals zero—we pick up a new random supervector $|\mathsf{C}_0\rangle$ as a starting point.

We repeat this sequence of operations as long as needed. Thus, at the mth step, we minimize the function

$$F_{\mathsf{C}_{m-1}}(\lambda_1^{(m-1)}, \lambda_2^{(m-1)}, \ldots, \lambda_E^{(m-1)}) = \sum_{k=1}^{E} \| |\mathsf{C}_{m-1}\rangle + \lambda_k^{(m-1)} \hat{\mathsf{E}}_k |\mathsf{C}_{m-1}\rangle \|^2$$

by solving the real linear system

$$\hat{K}(|\mathsf{C}_{m-1}\rangle) \,.\, \vec{\Lambda}^{(m-1)} = \vec{D}(|\mathsf{C}_{m-1}\rangle).$$

This yields the $\lambda_k^{(m-1)}$'s and $|\Delta \mathsf{C}_{m-1}\rangle$ from which we calculate $|\mathsf{C}_{m-1}'\rangle = |\mathsf{C}_{m-1}\rangle + \frac{1}{2}|\Delta \mathsf{C}_{m-1}\rangle$. If possible, we normalize $|\mathsf{C}_{m-1}'\rangle$ and take the resulting vector as the starting point $|\mathsf{C}_m\rangle$ of the $(m+1)$th step; otherwise, we pick up a new vector $|\mathsf{C}_0\rangle$ as a starting point. Finally $|\mathsf{C}_m\rangle$ tends to $|\tilde{\mathsf{C}}\rangle$ such that $\forall k \in [1, I(I-1)/2], \langle \tilde{\mathsf{C}} | \hat{\mathsf{E}}_k | \tilde{\mathsf{C}} \rangle = 0$.

This algorithm was numerically implemented and allowed us to exhibit new codes: we protected 2 qubits among 7 against the action of 31 errors (21 individual errors + 10 collective errors) and 4 qubits among 9 against the action of 27 individual errors [19].

The coding matrix $\hat{C}$ which allows us to transfer the information from the space $\mathcal{C}$ to the code space $\tilde{\mathcal{C}}$ is a rather complex unitary operator on the Hilbert space of the compound system $\mathcal{S}=\mathcal{I}\otimes\mathcal{A}$. We have just shown how to calculate the codewords, which actually form the first I columns of $\hat{C}$, but one can wonder how to implement it physically. The question of the physical feasibility of the coding matrix $\hat{C}$ can be solved by the nonholonomic control technique.

The nonholonomic control technique has been suggested by our team as a means of controlling the evolution of quantum systems [21]. Basically, it consists in alternately applying two "well-chosen" perturbations $\hat{V}_a$ and $\hat{V}_b$ to the system $\mathcal{S}$ we want to control during pulses with timings t_i. The total Hamiltonian $\hat{H}=\hat{H}_0+\hat{V}$ thus has a pulsed shape and alternately takes the two values $\hat{H}_a\equiv\hat{H}_0+\hat{V}_a$ (during odd-numbered pulses) and $\hat{H}_b\equiv\hat{H}_0+\hat{V}_b$ (even-numbered pulses). The timings t_i play the role of free parameters one has to adjust in order to perform the desired control operation. To be more explicit, the perturbations $\hat{V}_a$ and $\hat{V}_b$ must be chosen so that the commutators of all orders of $\hat{H}_a\equiv\hat{H}_0+\hat{V}_a$ and $\hat{H}_b\equiv\hat{H}_0+\hat{V}_b$ span the whole space of Hermitian matrices acting on the system we want to control: this is called the *bracket generation condition* (BGC). From the Campbell-Baker-Hausdorf formula, it follows that this is a necessary condition of controllability. It also proves to be sufficient in all the practical cases we dealt with. For that reason, we consider that we have "good controllability conditions" as soon as the BGC is checked. The number n_C of control timings depends on the problem to be solved. For instance, if we want to impose the arbitrary evolution $\hat{U}_{arb}$ on an N-dimensional system, we need at least $n_C=N^2$ timings t_i, since N^2 is the total number of free real parameters characterizing a $N\times N$ unitary matrix. We dealt with this problem of complete control in previous papers [21] and developed a general algorithm to find the appropriate timings t_i which realize

$$\hat{U}(t_1,t_2,\ldots,t_{N^2})=\exp(-i\hat{H}_a t_{N^2})\times\exp(-i\hat{H}_b t_{N^2-1})\cdots\exp(-i\hat{H}_b t_1)=\hat{U}_{arb}.$$

We can directly apply this result to our coding problem in the following way: first, we find the codewords $\{|\tilde{\gamma}_i\rangle, i=1,\ldots,I\}$ by the iterative algorithm we have presented in the first part of this section; then, we complete the set of I vectors $\{|\tilde{\gamma}_i\rangle, i=1,\ldots,I\}$ with $(N-I)$ vectors $\{|\tilde{\gamma}_j\rangle, j=I+1,\ldots,N\}$ to form an orthonormal basis of $\mathcal{H}$. We build the coding matrix by taking the vectors $\{|\tilde{\gamma}_i\rangle, i=1,\ldots,N\}$ as columns of $\hat{C}$, and finally we calculate the $n_C=N^2$ appropriate timings $\{t_i\}$ such that

$$\hat{U}(t_1,t_2,\ldots,t_{N^2})=\exp(-i\hat{H}_a t_{N^2})\times\exp(-i\hat{H}_b t_{N^2-1})\cdots\exp(-i\hat{H}_b t_1)=\hat{C}$$

through the complete control algorithm presented in [21]. Note that we assume $\hat{H}_0=0$ (Sec. II); hence, $\hat{H}_a=\hat{V}_a$ and $\hat{H}_b=\hat{V}_b$

Actually, this procedure provides a lot of useless work: indeed, most of the information contained in the coding matrix is irrelevant and the N^2 real parameters of $\hat{C}$ do not all have to be controlled exactly: the number n_C of necessary control parameters $\{t_i\}$ is much less than N^2. Let us examine this point in more detail.

The coding matrix is characterized by the relations (6). The problem of control thus reduces to finding n_C timings t_i, which we will formally gather in a time vector

$$\vec{t}=\begin{pmatrix} t_1 \\ \vdots \\ t_{n_C} \end{pmatrix},$$

such that the nonholonomic evolution matrix

$$\hat{U}(\vec{t})=\exp(-i\hat{H}_a t_{n_C})\exp(-i\hat{H}_b t_{n_C-1})\cdots\exp(-i\hat{H}_a t_1)$$

meets conditions (6): for any pair $(|\gamma_s\rangle,|\gamma_t\rangle)_{1\leqslant s,t\leqslant I}$ of basis vectors of $\mathcal{C}$ and any operator $\hat{E}_m\in\{\hat{E}_m\}_{m=1,\ldots,M}$

$$\langle\gamma_t|\hat{U}^\dagger(\vec{t})\hat{E}_m\hat{U}(\vec{t})|\gamma_s\rangle=0. \tag{9}$$

The number n_C of control parameters must exceed the number of independent constraints which is clearly $\sim MI^2$—that is, $n_C\gtrsim MI^2$. The number of really necessary control parameters appears to be much smaller than N^2. We have to design a new algorithm which achieves a partial and less expensive control of the evolution operator of the system.

The algorithm we shall use to calculate the appropriate control timings t_i mixes the iterative algorithm presented at the beginning of this section and the nonholonomic control technique. If we introduce the $[(N\times I)\times(N\times I)]$-dimensional block-diagonal matrix

$$\hat{\mathsf{U}}(\vec{t})=\begin{pmatrix} \hat{U}(\vec{t}) & 0 & \cdots & 0 \\ 0 & \hat{U}(\vec{t}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \hat{U}(\vec{t}) \end{pmatrix}$$

and the $(N\times I)$-dimensional supervector

$$|\mathsf{C}\rangle=\begin{pmatrix} |\gamma_1\rangle \\ \vdots \\ |\gamma_I\rangle \end{pmatrix}$$

composed of the coordinates of the I basis vectors of $\mathcal{C}$, we can set the problem of control, Eq. (9), in the following equivalent form: we look for a time vector $\vec{t}$ such that

$$\forall k,\quad \langle\mathsf{C}|\hat{\mathsf{U}}^\dagger(\vec{t})\hat{\mathsf{E}}_k\hat{\mathsf{U}}(\vec{t})|\mathsf{C}\rangle=0, \tag{10}$$

where the matrices $\{\hat{\mathsf{E}}_k\}_{k=1,\ldots,E}$ denote E different matrices of dimension $(N\times I)\times(N\times I)$ which have been introduced in the beginning of this section. In other words, we look for the time vector $\vec{t}$ which sets to zero the test function $G(\vec{t})$

$\equiv \Sigma_{k=1}^{E} |\langle \mathsf{C}|\hat{\mathsf{U}}^{\dagger}(\vec{t})\hat{\mathbf{E}}_k\hat{\mathsf{U}}(\vec{t})|\mathsf{C}\rangle|^2$. The idea of our algorithm is to take the supervector $|\mathsf{C}_0\rangle = \hat{\mathsf{U}}(\vec{t}_0)|\mathsf{C}\rangle$, where $\vec{t}_0$ is a random time vector, as the starting point for an elementary step of the iterative algorithm and look for the small time increment $\vec{dt}_0$ such that $\hat{\mathsf{U}}(\vec{t}_0+\vec{dt}_0)|\mathsf{C}\rangle$ follows the direction provided by the result $|\mathsf{C}_0\rangle + |\Delta\mathsf{C}_0\rangle$ of the iterative algorithm. The repetition of this sequence finally yields $\vec{t}=\vec{t}_0+\vec{dt}_0+\vec{dt}_1+\cdots$ which meets Eq. (10).

Let us now describe the algorithm in more detail. First, we randomly pick a set of timings $t_{0,i}$ in a "realistic range," dictated by the system under consideration: in particular, control-pulse timings have to be much shorter than the typical lifetime of the system and be much longer than the typical response delay required by the experiment. Then we minimize the function

$$F_{\mathsf{C}_0}(\lambda_1^{(0)},\lambda_2^{(0)},\ldots,\lambda_E^{(0)}) = \sum_{k=1}^{E} \| |\mathsf{C}_0\rangle + \lambda_k^{(0)}\hat{\mathsf{E}}_k|\mathsf{C}_0\rangle\|^2$$

as we did in the algorithm presented at the beginning of this section: we obtain the $\lambda_k^{(0)}$'s and $|\Delta\mathsf{C}_0\rangle = \Sigma_k \lambda_k \hat{\mathbf{E}}_k|\mathsf{C}_0\rangle$. At that point, we look for the small increment $\vec{dt}_0$ of the time vector $\vec{t}_0$ such that

$$\forall k, \quad \left\langle \mathsf{C} \left| \left(\frac{\partial \hat{\mathsf{U}}^{\dagger}}{\partial \vec{t}}(\vec{t}_0)\cdot \vec{dt}_0 \right)\hat{\mathsf{E}}_k\hat{\mathsf{U}}(\vec{t}_0) + \hat{\mathsf{U}}^{\dagger}(\vec{t}_0)\hat{\mathsf{E}}_k\left(\frac{\partial \hat{\mathsf{U}}}{\partial \vec{t}}(\vec{t}_0)\cdot \vec{dt}_0 \right) \right| \mathsf{C} \right\rangle = \frac{\left\langle \mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0|\hat{\mathsf{E}}_k|\mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0 \right\rangle - \langle \mathsf{C}_0|\hat{\mathsf{E}}_k|\mathsf{C}_0\rangle}{\left\langle \mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0|\mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0 \right\rangle}. \quad (11)$$

It should be noticed that we do not consider the error supermatrices $\hat{\mathbf{E}}_k$ corresponding to orthonormality conditions: in other words, we just take matrices $\{\hat{\mathbf{E}}_k\}_{k\in[I(I-1)/2+1,I(I-1)/2+MI(I+1)/2]}$ into account. Thus we deal with $MI(I+1)/2$ complex equations. This set of equations can be reduced to the real linear system

$$\hat{S}(\vec{t}_0)\cdot\vec{dt}_0 = \vec{W}(|\Delta\mathsf{C}_0\rangle), \quad (12)$$

where $\hat{S}(\vec{t}_0)$ and $\vec{W}(|\Delta\mathsf{C}_0\rangle)$ are, respectively, an $MI^2\times n_C$ real matrix and a MI^2-dimensional real vector. We obtained Eq. (12) by splitting the set of $MI(I+1)/2$ complex equations (11) into two sets of $MI(I+1)/2$ real equations and rejecting those which are trivial $(0=0)$ or redundant. Even though this procedure is straightforward, the explicit expressions of the different elements of $\hat{S}$ and $\vec{W}$ involve many indices and are so unpleasant that we prefer not to reproduce them here.

The linear system we have just found is a priori rectangular $(MI^2\times n_C)$, but actually we have not fixed the number n_C yet. Previously, we stated that $n_C \geqslant MI^2$: we could be tempted to set $n_C = MI^2$ so as to obtain a square system, easily solvable by standard techniques of linear algebra. Yet we will proceed in a slightly different way. We set $n_C > MI^2$—say, $n_C = MI^2+\delta n$ where δn is an integer of order 1. Then we randomly pick MI^2 timings t_i among the n_C which will be considered as free parameters, whereas the other δn ones will be regarded as frozen. In other words, we randomly choose a permutation $\sigma_0 \in \mathcal{S}_{n_C}$ (symmetric group of order n_C) and take the timings $\{t_i' = t_{\sigma_0(i)}\}_{i=1,\ldots,MI^2}$ as free parameters whereas the timings $\{t_i' = t_{\sigma_0(i)}\}_{i=1+MI^2,n_C}$ are frozen. This leads to new versions of Eqs. (11) and (12):

$$\forall k, \quad \left\langle \mathsf{C} \left| \left(\frac{\partial \hat{\mathsf{U}}^{\dagger}}{\partial \vec{t}'}(\vec{t}_0)\cdot \vec{dt}'_0 \right)\hat{\mathsf{E}}_k\hat{\mathsf{U}}(\vec{t}_0) + \hat{\mathsf{U}}^{\dagger}(\vec{t}_0)\hat{\mathsf{E}}_k\left(\frac{\partial \hat{\mathsf{U}}}{\partial \vec{t}'}(\vec{t}_0)\cdot \vec{dt}'_0 \right) \right| C \right\rangle = \frac{\left\langle \mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0|\hat{\mathsf{E}}_k|\mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0 \right\rangle - \langle \mathsf{C}_0|\hat{\mathsf{E}}_k|\mathsf{C}_0\rangle}{\left\langle \mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0|\mathsf{C}_0 + \frac{1}{2}\Delta\mathsf{C}_0 \right\rangle}, \quad (13)$$

$$\hat{S}(\vec{t}_0)\cdot\vec{dt}'_0 = \vec{W}(|\Delta\mathsf{C}_0\rangle). \quad (14)$$

Equation (14) is now clearly a square system. Solving Eq. (14) yields the MI^2-dimensional increment $\vec{dt}'_0$ which we complete with δn zeros into a n_C-dimensional vector; by reordering timings, we obtain the total time-vector increment $\vec{dt}_0$. Thus we have for $i\in[1,MI^2]$, $dt_{0,\sigma_0(i)} \neq 0$ (free parameters), whereas for $i\in[1+MI^2,n_C]$, $dt_{0,\sigma_0(i)}=0$ (frozen timings). Then we set $\vec{t}_1^{\alpha} = \vec{t}_0 + \alpha\vec{dt}_0$ where α is a convergence coefficient and calculate the test function $G(\vec{t}) = \Sigma_k |\langle \mathsf{C}|\hat{\mathsf{U}}^{\dagger}(\vec{t})\hat{\mathsf{E}}_k\hat{\mathsf{U}}(\vec{t})|\mathsf{C}\rangle|^2$ in $\vec{t}=\vec{t}_1^{\alpha}$ for different values of $\alpha \in [0,1]$. If we find an α_1 such that $G(\vec{t}_1^{\alpha}) < G(\vec{t}_0)$, we take $\vec{t}_1 \equiv \vec{t}_1^{\alpha_1}$ as our new time vector and keep the same free-varying timings: in other words, the permutation σ_1 governing the timings that play the role of control parameters in the second step of the algorithm remains the same—that is, $\sigma_1 = \sigma_0$. If we cannot find an appropriate α_1, this means we are situated in a local minimum of G; then, we set $\vec{t}_1 \equiv \vec{t}_0$ and pick a new set of free-varying parameters by simply choosing a new permutation $\sigma_1 \neq \sigma_0$ randomly. This rotation procedure among control parameters allows us to avoid possible local minima of the test function G we want to cancel.

We repeat this sequence of operations as long as needed. At the mth step, we take the supervector $|\mathsf{C}_{m-1}\rangle = \hat{\mathsf{U}}(\vec{t}_{m-1})|\mathsf{C}\rangle$ as the starting point of an elementary step of the iterative algorithm. We calculate $|\Delta\mathsf{C}_{m-1}\rangle = \Sigma\lambda_k^{(m-1)}\hat{\mathbf{E}}_k|\mathsf{C}_{m-1}\rangle$ and find MI^2-dimensional variations vector $\vec{dt}'_{m-1}$ of the MI^2 free parameters (characterized by permutation σ_{m-1}) such that

$$\forall k, \quad \left\langle \mathsf{C} \left| \left(\frac{\partial \hat{\mathsf{U}}^{\dagger}}{\partial \vec{t}'} (\vec{t}_{m-1}) \cdot \vec{dt}'_{m-1} \right) \hat{\mathsf{E}}_k \hat{\mathsf{U}}(\vec{t}_{m-1}) + \hat{\mathsf{U}}^{\dagger}(\vec{t}_{m-1}) \hat{\mathsf{E}}_k \left(\frac{\partial \hat{\mathsf{U}}}{\partial \vec{t}'} (\vec{t}_{m-1}) \cdot \vec{dt}'_{m-1} \right) \right| \mathsf{C} \right\rangle$$
$$= \frac{\left\langle \mathsf{C}_{m-1} + \frac{1}{2}\Delta \mathsf{C}_{m-1} | \hat{\mathsf{E}}_k | \mathsf{C}_{m-1} + \frac{1}{2}\Delta \mathsf{C}_{m-1} \right\rangle - \langle \mathsf{C}_{m-1} | \hat{\mathsf{E}}_k | \mathsf{C}_{m-1} \rangle}{\left\langle \mathsf{C}_{m-1} + \frac{1}{2}\Delta \mathsf{C}_{m-1} | C_{m-1} + \frac{1}{2}\Delta \mathsf{C}_{m-1} \right\rangle}$$

by solving the associated square linear system

$$\hat{S}(\vec{t}_{m-1}) \cdot \vec{dt}_{m-1} = \vec{W}(|\Delta \mathsf{C}_{m-1}\rangle).$$

We complete $\vec{dt}'_{m-1}$ with δn zeros and reorder the timings so as to obtain $\vec{dt}_{m-1}$. Then we take $\vec{t}_m^{\alpha} = \vec{t}_{m-1} + \alpha \vec{dt}_{m-1}$. If there exists an α_m such that $F(\vec{t}_m^{\alpha_m}) < F(\vec{t}_{m-1})$, we set $\vec{t}_m = \vec{t}_m^{\alpha_m}$ as our new time vector and keep the same free parameters for the $(m+1)$th step: the permutation characterizing free-varying timings in the $(m+1)$th step will be the same as in the mth step—that is, $\sigma_m = \sigma_{m-1}$. Otherwise we take $\vec{t}_{\hat{m}} = \vec{t}_{m-1}$as our time vector and randomly pick up MI^2 new free parameters among the n_C timings, by choosing a new permutation σ_m for the $(m+1)$th step.

We have not said anything about the decoding so far. If the two Hamiltonians $\hat{H}_a = \hat{V}_a$ and $\hat{H}_b = \hat{V}_b$ can be reversed (note that we assume $\hat{H}_0 = 0$)—i.e., the sign of $\hat{V}_a$ and $\hat{V}_b$ can be reversed by altering of the control field parameters—the implementation of the decoding matrix is quite easy: it amounts to reversing $\hat{V}_a$ and $\hat{V}_b$ and applying the same control timing sequence backwards. To be more explicit, one starts by applying $-\hat{V}_b$ during timing t_{n_C}, then $-\hat{V}_a$ during $t_{n_C-1,\ldots}$, and finally $-\hat{V}_a$ during t_1. On the contrary, if $\hat{V}_a$ and $\hat{V}_b$ cannot be reversed, one cannot apply this technique. We must use the general nonholonomic control technique, involving N^2 control parameters, to find timings which realize $\hat{C}^{-1}$.

The algorithm we have just described was numerically implemented and has already given satisfying numerical results on a realistic 7-qubit system subject to the action of 21 errors [19]. In the next section, we deal with another real physical system which lends itself particularly well to a demonstration of our method.

To conclude this section, let us emphasize that, to our knowledge, there is only a formal link between our method and the so-called "bang-bang" control schemes [23]. Actually, in this kind of techniques, fast and strong pulses are applied which average the interaction Hamiltonian between the system and its environment to zero. By contrast, our method employs pulses which are designed to code information—that is, to transfer it into a proper subspace, in which errors act orthogonally: decoding and measurement then allow us to recover initial information.

IV. COHERENCE PROTECTION APPLIED TO THE RUBIDIUM ATOM

The goal of this section is to apply our method to a real physical system. As we shall see below, the chosen system, a rubidium isotope, due to its structure, lends itself particularly well to a straightforward implementation of our technique and allows us to illustrate its different steps quite simply: to be more specific, following the scheme we presented in the previous sections, we show that it is possible to protect one qubit of information encoded on the two spin states of the ground level 5*s* of the radioactive isotope ^{78}Rb against the action of $M=6$ error-inducing Hamiltonians $\hat{E}_m$. For numerical calculations we considered three magnetic Hamiltonians

$$\{\hat{E}_k^{\beta} \propto \hat{L}_k + 2\hat{S}_k, k = x,y,z\}$$

and three electric Hamiltonians of second order:

$$\{\hat{E}_{k,l}^{\varepsilon} \propto \hat{r}_k^2 - \hat{r}_l^2, k,l = x,y,z, k < l\}.$$

In the following, we propose a detailed physical setting which achieves the desired protection operation: in particular, we provide characteristic values of control fields and pulse timings. These different calculated parameters relate to a single isolated atom. As we shall see at the end of this section, when dealing with an ensemble of atoms, serious experimental drawbacks emerge which prevent us from actually implementing our application. Nevertheless, the example considered shows the operationality of our method which is able, in a given physical situation, to provide a precise frame for its implementation.

Before presenting the details of the proposed implementation, let us motivate the choice of the rubidium atom. Alkali-metal atoms like Rb are very interesting for our purpose because of their hydrogenlike behavior. Such an atom is the compound of an information subsystem—i.e., the spin part of the wave function, and an ancilla—i.e., the orbital part of the quantum state. As we shall see, it is easy to increase the dimensionality of the ancilla by simply pumping the atom towards a shell of higher orbital angular momentum L.

We chose ^{78}Rb among all alkali-metal systems because of its spectroscopic characterics (Fig. 2) [24,25]. In particular, ^{78}Rb has no hyperfine structure (its nuclear spin is 0) which ensures that the ground level 5*s* is degenerate: this is neces-

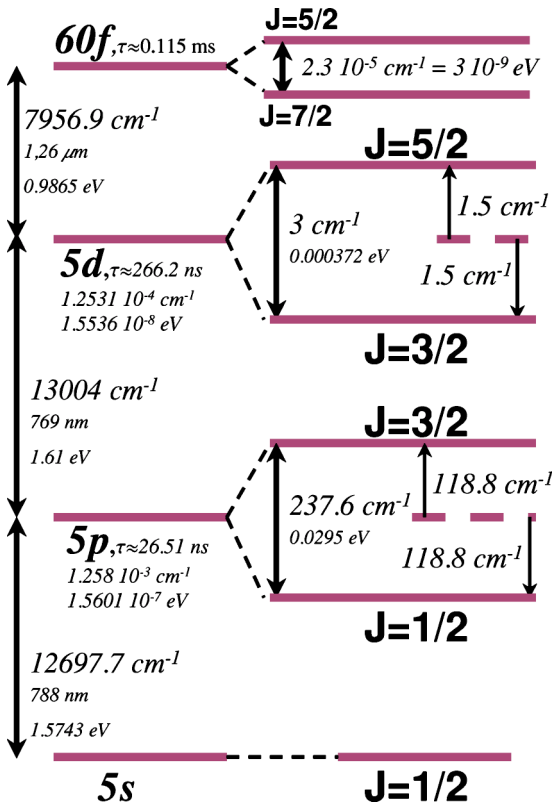

FIG. 2. Spectrum of ^{78}Rb: The useful part of the spectrum of rubidium is represented.

sary for the projection scheme as we shall see below. Moreover, it has a long enough lifetime ($\tau \simeq 17.66$ min) for the proposed experiment.

Let us now review each step of our method in detail. As mentioned above, the information we want to protect is initially encoded on the two spin states $|\nu_1\rangle = |5s, j=\frac{1}{2}, m_j=-\frac{1}{2}\rangle$ and $|\nu_2\rangle = |5s, j=\frac{1}{2}, m_j=\frac{1}{2}\rangle$ of the ground level $5s$ of the atom: these two states span the information space $\mathcal{H}_I = \mathrm{Span}[|\nu_1\rangle, |\nu_2\rangle]$ whose dimension is in that case $I=2$. The first step of our scheme consists in adding an ancilla $\mathcal{A}$ to the information system. The role of $\mathcal{A}$ is played by the orbital part of the wave function. In the ground state ($L=0$), its dimension is $A=2L+1=1$ (roughly speaking, there is no ancilla). If we want to protect one qubit of information against $M=6$ error-inducing Hamiltonians, we have to increase the dimensionality of the ancilla up to $A=M+1=7$ [Eq. (7)]: this can be achieved by pumping the atom up to a shell nf ($L=3$). We choose the highly excited Rydberg state $60f$ so as to make the fine structure as weak as possible (the splitting for $60f$ is approximately 10^{-5} cm^{-1} [24]). We shall first consider the fine structure is negligible so that the $N=I\times A=2\times 7=14$ basis vectors of the total Hilbert space $\mathcal{H}=\mathcal{H}_I\otimes\mathcal{H}_A$ are almost perfectly degenerate; the validity of this approximation will be discussed at the end of this section. To be more specific, the pumping is done in such a way that

$$|\nu_1\rangle \rightarrow |\gamma_1\rangle = \left|60f, j=\frac{5}{2}, m_j=-\frac{3}{2}\right\rangle,$$

$$|\nu_2\rangle \rightarrow |\gamma_2\rangle = \left|60f, j=\frac{5}{2}, m_j=-\frac{1}{2}\right\rangle.$$

In other words, using the terminology of the previous sections, the information initially stored in $\mathcal{H}_I$ is transferred into

$$\mathcal{C} = \mathrm{Span}\left[|\gamma_1\rangle = \left|60f, j=\frac{5}{2}, m_j=-\frac{3}{2}\right\rangle,\right.$$

$$\left.|\gamma_2\rangle = \left|60f, j=\frac{5}{2}, m_j=-\frac{1}{2}\right\rangle\right].$$

The choice of the subspace $\mathcal{C}$ may appear arbitrary at this stage, but it will be justified later by the practical feasibility of the projection process onto $\mathcal{C}$. Let us note that $\mathcal{C}$ is an "entangled" subspace whose basis vectors $\{|\gamma_i\rangle\}_{i=1,2}$ are general entangled states of the spin and orbital parts: this means (Sec. II) that the projection step will not consist in a simple measurement of the ancilla but will involve a more intricate process we shall describe in detail later.

Practically, the pumping can be achieved as follows. One applies three lasers to the atom: the first laser is right polarized and slightly detuned from the transition $(5s \leftrightarrow 5p)$ whereas the second and third lasers are left polarized and slightly detuned from the transitions $(5p_{3/2} \leftrightarrow 5d_{3/2})$ and $(5d_{3/2} \leftrightarrow 60f)$, respectively. The detunings forbid real one-photon processes: the atom can only absorb three photons simultaneously and is thereby excited from the ground level $5s$ to the Rydberg level $60f$. By using selection rules, one can construct the allowed paths represented in Fig. 3: these paths only couple $|\nu_1\rangle$ and $|\nu_2\rangle$ to $|\gamma_1\rangle$ and $|\gamma_2\rangle$, respectively.

The second step consists in encoding the information by the nonholonomic control technique: to impose the coding matrix on the system, we submit the atom to $n_C=34$ control pulses of timings $\{t_i\}_{i=1,\ldots,34}$, during which two different combinations of magnetic and Raman electric Hamiltonians are alternately applied (see Fig. 4). To be more explicit, during odd-numbered pulses (A-type pulses) we apply a constant magnetic field

$$\vec{B} = \begin{pmatrix} B_x = 710^{-3}T \\ B_y = 8.210^{-3}T \\ B_z = -6.810^{-3}T \end{pmatrix},$$

which is associated with the Zeeman Hamiltonian $\hat{W}_Z$, and two sinusoidal electric laser fields

$$\vec{E}_a(t) = \mathrm{Re}[\underline{\vec{E}}_a e^{-i\omega_R t}], \quad \vec{E}'_a(t) = \mathrm{Re}[\underline{\vec{E}}'_a e^{-i\omega'_R t}],$$

$$\underline{\vec{E}}_a = \begin{vmatrix} E_{x,a} \\ E_{y,a}e^{-i\varphi_{y,a}} \\ 0 \end{vmatrix}, \quad \underline{\vec{E}}'_a = \begin{vmatrix} E'_{x,a} \\ E'_{y,a}e^{-i\varphi'_{y,a}} \\ 0 \end{vmatrix},$$

whose frequencies ω_R and ω'_R are, respectively, slightly detuned from the two transitions $\left(60f \leftrightarrow 5d, j=\frac{3}{2}\right)$ and $\left(60f \leftrightarrow 5d, j=\frac{5}{2}\right)$ (detunings δ and δ'). The characteristic values of these fields are

$$E_{x,a} = E'_{x,a} = 8.5 \times 10^5 \text{ V m}^{-1},$$

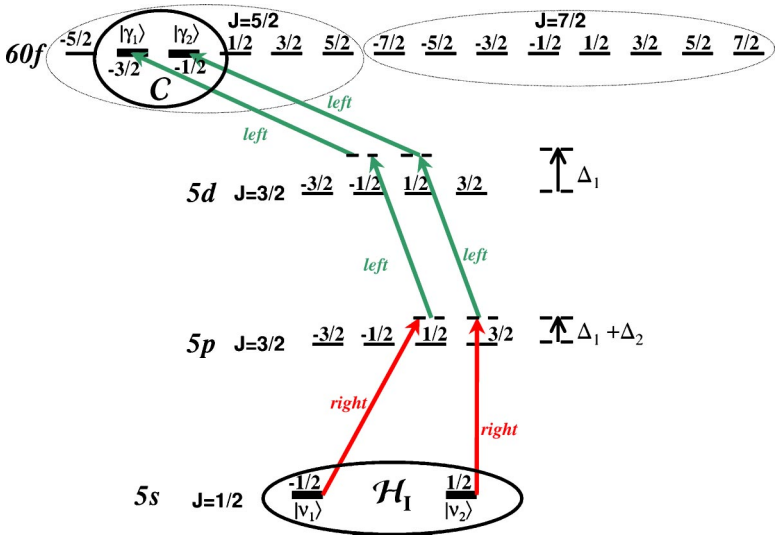

FIG. 3. Ancilla adding by pumping. Photon polarization and involved sub-Zeeman levels are represented. The fine structure of the Rydberg level $60f$ is not resolvable.

$$E_{y,a} = E'_{y,a} = 5.2 \times 10^6 \text{ V m}^{-1},$$

$$\varphi_{y,a} = \varphi'_{y,a} = 2.3,$$

$$\hbar\omega_R = 0.986324 \text{ eV} = 7955.14 \text{ cm}^{-1},$$

$$\delta = -0.000010 \text{ eV} = -0.080654 \text{ cm}^{-1},$$

$$\hbar\omega'_R = 0.986676 \text{ eV} = 7958.14 \text{ cm}^{-1},$$

$$\delta' = 0.000010 \text{ eV} = 0.080654 \text{ cm}^{-1}.$$

The intensity of the laser beams are typically of the order of 2×10^8 W cm^{-1}. The Raman Hamiltonian associated with these fields is denoted by $\hat{W}_{R,A}$. The total perturbation is $\hat{V}_a = \hat{W}_Z + \hat{W}_{R,A}$. During even-numbered pulses (B-type pulses), we apply the same magnetic field as for A-type pulses, which is experimentally convenient, and two sinusoidal electric laser fields

$$\vec{E}_b(t) = \text{Re}[\underline{\vec{E}}_b e^{-i\omega_R t}], \quad \vec{E}_b(t) = \text{Re}[\underline{\vec{E}}'_b e^{-i\omega'_R t}],$$

where

$$\underline{\vec{E}}_b = \begin{vmatrix} E_{x,b} \\ E_{y,b}e^{-i\varphi_{y,b}} \\ 0 \end{vmatrix}, \quad \underline{\vec{E}}'_b = \begin{vmatrix} E'_{x,b} \\ E'_{y,b}e^{-i\varphi'_{y,b}} \\ 0 \end{vmatrix},$$

whose frequencies are the same as above and whose characteristics values are

$$E_{x,b} = E'_{x,b} = -5.2 \times 10^6 \text{ V m}^{-1},$$

$$E_{y,b} = E'_{y,b} = 8.5 \times 10^5 \text{ V m}^{-1},$$

$$\varphi_{y,a} = \varphi'_{y,a} = 2.3.$$

The Raman Hamiltonian associated with these fields is denoted by $\hat{W}_{R,B}$. The corresponding perturbation is $\hat{V}_b = \hat{W}_Z + \hat{W}_{R,B}$. Therefore, since the fine structure of the level $60f$ is neglected, the unperturbed Hamiltonian $\hat{H}_0$ is taken to be 0, and the total Hamiltonian has the form $\hat{H}_A = \hat{V}_a$ during A pulses, $\hat{H}_B = \hat{V}_b$ during B pulses. The 34 different timings have been calculated so that

$$\hat{U}(t_1, \ldots, t_{34}) = e^{-i\hat{H}_B t_{n_C}} e^{-i\hat{H}_A t_{n_C-1}} \cdots e^{-i\hat{H}_A t_1} = \hat{C}$$

meets conditions (5). At the end of the coding step the information is transferred into the code space $\tilde{\mathcal{C}} = \hat{C}\mathcal{C}$, encoded on the codewords $\{|\tilde{\gamma}_i\rangle = \hat{C}|\gamma_i\rangle\}_{i=1,2}$.

As can be easily checked from Fig. 4 the total duration of a control period ($\simeq$125 ns) is approximately 10^3 times shorter than the lifetime of the $60f$ Rydberg state which is approximately 0.115 ms as can be calculated from [24]. The different pulse timings range between 2.9 and 7.4 ns, which are feasible.

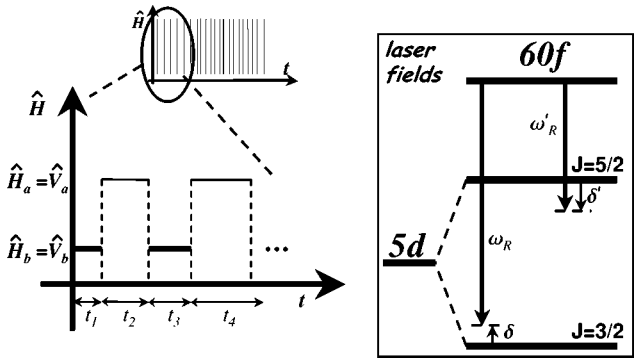

FIG. 4. Coding step through the nonholonomic control technique. The two Hamiltonians $\hat{H}_a$ and $\hat{H}_b$ are alternately applied to the system during pulses of timings $\{t_i(\text{ns})\} = \{3.9763, 6.4748, 4.2274, 3.6259, 2.8717, 3.6281, 7.2263, 6.4260, 4.8070, 5.0394, 6.5242, 4.8890, 4.2400, 7.3834, 4.8653, 5.4799, 4.5341, 4.3099, 6.2959, 3.7346, 6.5293, 6.8586, 6.0749, 5.1213, 4.6806, 3.4985, 3.9909, 4.6701, 4.5168, 6.4702, 4.7787, 5.3476, 3.4567, 3.8009\}$. The frequencies of the laser fields involved in the encoding step are represented on the spectrum of the rubidium atom. The fine structure of the Rydberg level $60f$ is not represented.

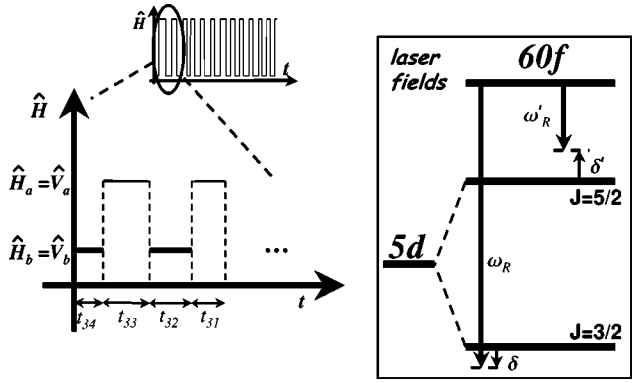

FIG. 5. Decoding step by the nonholonomic control technique. We reverse the magnetic field and the detunings of the electric fields, as represented on the spectrum of the rubidium atom, and apply the same control sequence as for coding (same timings) backwards. The fine structure of the level $60f$ is not represented.

After a short time, the information stored in the system acquires a small erroneous component due to the action of the error Hamiltonians, which is orthogonal to the code space $\tilde{\mathcal{C}}$. Then, we apply the decoding matrix $\hat{C}^{-1}$ to the atom as suggested at the end of Sec. III. We reverse $\vec{B}$ and the detunings δ and δ', and leave all the other values unchanged (this amounts to taking the opposite of Hamiltonians $\hat{H}_A$ and $\hat{H}_B$), and apply the same sequence of control pulses backwards: we start with an A pulse whose timing is t_{n_C}, then apply a B pulse during t_{n_C-1}, etc. (see Fig. 5). The decoding step yields an erroneous state whose projection onto $\mathcal{C}$ is the initial information state.

In the last step the erroneous state vector is projected onto the subspace $\mathcal{C}$ to recover the initial information. Projection is a nonunitary process which cannot be achieved through a Hamiltonian process, but requires the introduction of irreversibility. To this end, we make use of a path which is symmetric with the pumping step and consists in two stimulated and one spontaneous emissions. To be more explicit, we apply two left circularly polarized lasers (see Fig. 6) slightly detuned from the transitions $(60f \leftrightarrow 5d, j=\frac{3}{2})$ and $(5d, j=\frac{3}{2} \leftrightarrow 5p, j=\frac{3}{2})$. Due to these laser fields, the atom is likely to fall towards the ground state and emit two stimulated and one spontaneous photons.

Using the selection rules, one can infer that, if a circularly right-polarized spontaneous photon is emitted, the only states to be coupled to the ground level are $|\gamma_1\rangle$ and $|\gamma_2\rangle$ to $|\nu_1\rangle$ and $|\nu_2\rangle$, respectively (see Fig. 6). This means that the emission of a right-polarized spontaneous photon brings the "correct" part of the state vector back into $\mathcal{H}_I = \mathrm{Span}[|\nu_1\rangle, |\nu_2\rangle]$. On the contrary, the other cases—"left polarized," "linearly polarized spontaneous photon," or "no photon at all"—do not lead to the right projection process.

The "left-polarized—photon" and "no-photon-emitted" cases are quite unlikely: indeed the probability that they occur is proportional to the square of the error amplitude—that is, to the square of the Zeno interval T, which is very short. The "linearly-polarized-photon" case is quite annoying because it mixes the two paths $|\gamma_1\rangle \rightarrow |\nu_1\rangle$ and $|\gamma_2\rangle \rightarrow |\nu_2\rangle$. This parasitic process and its relative probability must be suppressed, with respect to the process followed by the "right-polarized"-photon emission. This can be done by launching the ^{78}Rb atom, previously cooled, into a Fabry-Perot cavity, in an atomic fountain manner (fine tuning of the lasers driving the $60f$-$5d$ and $5d$-$5p$ transitions will be necessary to avoid reflection of the external laser radiation from the cavity). The decay rate for the three-photon transition $|\gamma_i\rangle \rightarrow |\nu_i\rangle$ is

$$\Gamma_{\gamma_i \nu_i} = 2\pi \left| \frac{d_{\gamma_i \lambda_j} E_1}{\hbar \Delta_1} \right|^2 \left| \frac{d_{\lambda_j \mu_k} E_2}{\hbar(\Delta_1 + \Delta_2)} \right|^2 \overline{2\pi\hbar c k_s |\vec{d}_{\mu_k \nu_i} \vec{e}_R^{\,*}|^2 \varrho(\vec{k}_s)},$$

where $\vec{k}_s$ is the wave vector of the spontaneously emitted photon, $\vec{e}_R$ is the left-polarized-photon polarization unit vector, $\varrho(\vec{k}_s)$ is the density of states (normalized to the cavity volume) for the cavity field at $\vec{k}_s$, and the overbar denotes averaging over the directions of $\vec{k}_s$. The transition dipole moments are denoted by d_{ab}: during the projective process the states coupled to $|\gamma_1\rangle$ and $|\gamma_2\rangle$ are, respectively,

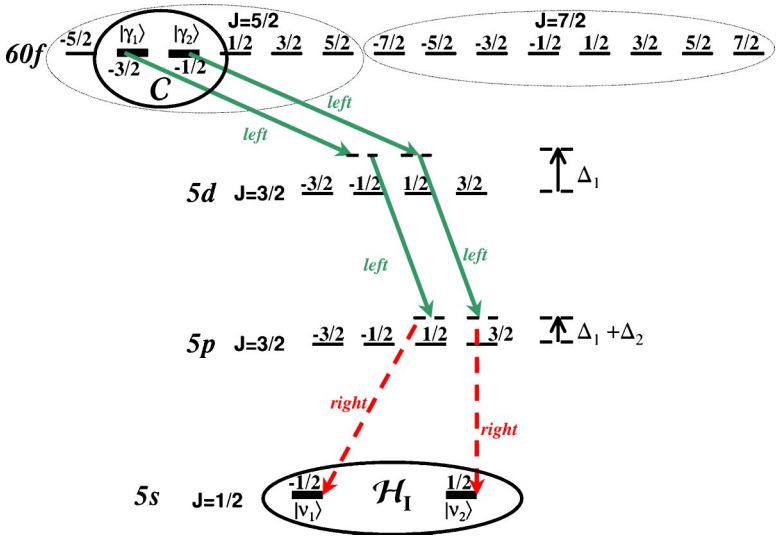

FIG. 6. Projection path. The lasers involved are marked by solid arrows; the spontaneous photon is represented by a dashed arrow. The different polarizations are specified. The fine structure of the level $60f$ is not represented.

$$\left\{|\lambda_1\rangle = \left|5d, j=\frac{3}{2}, m_j=-\frac{1}{2}\right\rangle, |\lambda_2\rangle = \left|5p, j=\frac{3}{2}, m_j=+\frac{1}{2}\right\rangle\right\},$$

$$\left\{|\mu_1\rangle = \left|5d, j=\frac{3}{2}, m_j=\frac{1}{2}\right\rangle, |\mu_2\rangle = \left|5p, j=\frac{3}{2}, m_j=\frac{3}{2}\right\rangle\right\}.$$

The enhancement (by the presence of cavity) of the density of states for the modes propagating paraxially to the z axis ensures that

$$\Gamma_{\gamma_1\nu_1}, \Gamma_{\gamma_2\nu_2} \gg \left|\frac{d_{\gamma_i\lambda_j}E_1}{\hbar\Delta_1}\right|^2 \left|\frac{d_{\lambda_j\mu_k}E_2}{\hbar(\Delta_1+\Delta_2)}\right|^2 \gamma,$$

where γ is the decay rate of $|5p, j=\frac{3}{2}, m_j=+\frac{1}{2}\rangle$ into $|5s, j=\frac{1}{2}, m_j=+\frac{1}{2}\rangle$, so that the undesired process followed by the π-photon emission is relatively less important than it was in free space. For the density matrix elements ρ_{ab} the following system of equations can be written ($i=0,1$):

$$\dot{\rho}_{\gamma_i\gamma_i} = -\Gamma_{\gamma_i\nu_i}\dot{\rho}_{\gamma_i\gamma_i},$$

$$\dot{\rho}_{\nu_i\nu_i} = \Gamma_{\gamma_i\nu_i}\dot{\rho}_{\gamma_i\gamma_i},$$

$$\dot{\rho}_{\gamma_1\gamma_2} = -\frac{1}{2}(\Gamma_{\gamma_1\nu_1}+\Gamma_{\gamma_2\nu_2})\rho_{\gamma_1\gamma_2},$$

$$\dot{\rho}_{\nu_1\nu_2} = \sqrt{\Gamma_{\gamma_1\nu_1}\Gamma_{\gamma_2\nu_2}}\rho_{\gamma_1\gamma_2}.$$

To avoid dephasing which would corrupt the information, the coherence matrix element $\rho_{\gamma_1\gamma_2}$ must be transferred with the maximum efficiency into $\rho_{\nu_1\nu_2}$: the efficiency

$$\eta = \frac{2\sqrt{\Gamma_{\gamma_1\nu_1}\Gamma_{\gamma_2\nu_2}}}{\Gamma_{\gamma_1\nu_1}+\Gamma_{\gamma_2\nu_2}}$$

is thus crucial. According to the Wigner-Eckart theorem,

$$\frac{\Gamma_{\gamma_1\nu_1}}{\Gamma_{\gamma_2\nu_2}} = \left(\frac{C^{5/2\ -3/2}_{3/2\ -1/2\ 1\ -1}C^{3/2\ -1/2}_{3/2\ 1/2\ 1\ -1}C^{3/2\ 1/2}_{1/2\ -1/2\ 1\ 1}}{C^{5/2\ -1/2}_{3/2\ 1/2\ 1\ -1}C^{3/2\ 1/2}_{3/2\ 3/2\ 1\ -1}C^{3/2\ 3/2}_{1/2\ 1/2\ 1\ 1}}\right)^2,$$

where on the right-hand side the ratio of the products of the Clebsch-Gordan coefficients corresponding to the transitions stands. These coefficients, which can be found in [26], lead to $\eta = 12\sqrt{2}/17 \approx 0.998\,27$. In other words, the probability of error during the Zeno projection stage due to the small difference of the Clebsch-Gordan coefficient products for the two paths is equal to or less than $1-\eta \approx 0.001\,73$ [equality is reached if the initial state is $(|0\rangle \pm |1\rangle)/\sqrt{2}$]. Note that the states $60f$, $5d$, and $5p$ have finite lifetimes τ_k (see Fig. 2). Thus the transition rates $\Gamma_{\gamma_i\nu_i}$ must be much larger than

$$1/\tau_{60f},$$

$$\left|\frac{d_{\gamma_i\lambda_j}E_1}{\hbar\Delta_1}\right|^2 \Big/ \tau_{5d},$$

and

$$\left|\frac{d_{\gamma_i\lambda_j}E_1}{\hbar\Delta_1}\right|^2 \left|\frac{d_{\lambda_j\mu_k}E_2}{\hbar(\Delta_1+\Delta_2)}\right|^2 \Big/ \tau_{5p},$$

in order to diminish errors caused by the decay of these unstable states.

To complete the projection step, one has to transfer the atom in its coherent superposition back to the $60f$ state: this is achieved by the same pumping sequence as in the first step. The mismatch of the Clebsch-Gordan coefficient products will cause again the error probability $1-\eta$. The information is then restored with very high probability and the system is ready to undergo a new protection cycle.

From the beginning of this section we have neglected the fine structure splitting of the level $60f$, which is approximately $2\times10^{-5}\ \mathrm{cm}^{-1}$ and corresponds to a period $\tau_f \sim 1.5\ \mu$s. To conclude this section, let us now take it into account and see its effect on each step of our scheme.

Obviously the pumping and projection steps will not be affected by the fine structure, since the information-carrying vectors $\{|\gamma_1\rangle, |\gamma_2\rangle\}$ belong to the same multiplet ($J=5/2$).

The coding and decoding steps are neither modified by the existence of the fine structure. Indeed, since the typical period of the fine structure Hamiltonian, $\tau_f \sim 1.5\ \mu$s, is more than 10 times longer than the total duration of the coding or decoding steps, it is legitimate to neglect its effect.

The influence of the fine structure on the free evolution period during which errors are likely to occur is more complicated to study in the general case. Yet two simple limiting regimes can be considered. If the spectrum of the coupling functions $f_m(t)$'s is very narrow [i.e., if the variation time scale of the $f_m(t)$'s is much longer than τ_f], one can show that our scheme applies directly as though there were no fine structure, provided the error Hamiltonians $\{\hat{E}_m\}$ are replaced by $\{\hat{E}_m^{(0)}\}$, where $\hat{E}_m^{(0)}$ is obtained from $\hat{E}_m$ by simply setting to zero the rectangular submatrices which couple the two multiplets ($J=5/2, 7/2$). The second limiting regime corresponds to a very broad spectrum for the $f_m(t)$'s (variation time scale much shorter than τ_f): in that case, one can show that our scheme applies provided one chooses a Zeno interval T multiple of τ_f.

In all this section, we implicitly supposed that the rubidium atom was alone, but in actual experiments, one usually works with an ensemble of atoms: this generates serious experimental drawbacks which we deal with now. Rydberg atoms are sensitive to the Doppler effect: nevertheless, in the case of cold atoms, this is negligible. But the most dramatic effect is due to interactions between atoms such as dipolar forces [27]. In a standard magneto-optical trap containing about 1000 atoms in Rydberg states ($n \simeq 60$), the typical energy of these interactions is 1 MHz, corresponding to a dephasing time of 1 ms [28]. As different atoms see different environments and are thus subject to different interactions, it will be impossible to properly code and thus protect the information stored in the different atoms of the ensemble. Beyond these problems, we nevertheless want to emphasize the demonstrative value of our example: the system considered here (rubidium isotope in a Rydberg state), though not completely satisfactory from an experimental point of view, is

indeed quite practical for a straightforward demonstration of our scheme, since the information-carrying subsystem and ancilla are clearly identified and every step is "simply" achieved. Other systems must be found, which will be addressed in future publications; however, the application considered here has already suggested the physical relevance and applicability of our method.

V. CONCLUSIONS

In this paper, an original scheme has been presented which allows us to protect the quantum coherence stored in a information system $\mathcal{I}$ against the action of a set of M given error-inducing Hamiltonians $\hat{E}_k$.

The information initially stored in the Hilbert space $\mathcal{H}_I$ of the information system is transferred into a subspace $\mathcal{C}$ of the Hilbert space $\mathcal{H}=\mathcal{H}_I\otimes\mathcal{H}_A$ of the compound system $\mathcal{I}\otimes\mathcal{A}$ formed through adding an auxiliary system $\mathcal{A}$ called an ancilla to the main system. A multidimensional generalization of the QZE has been presented which makes it possible to protect such a subspace against the action of the $\hat{E}_k$'s, provided the dimension A of the ancilla meets the Hamming bound $A \geqslant M+1$. The information is thus encoded in another subspace $\tilde{\mathcal{C}}$, called the "code space," through the application of the coding matrix $\hat{C}$: in this appropriate subspace, the error-inducing Hamiltonians $\hat{E}_k$ act orthogonally. After a short time, the information thus contains a small orthogonal erroneous component due to the action of the $\hat{E}_k$; it is then decoded by application of $\hat{C}^{-1}$ and restored by an appropriate physical measurement which projects the state vector onto $\mathcal{C}$ with very high probability. The repetition of this sequence as long as needed protects the information stored in the system.

A physical achievement of the coding and decoding steps have been proposed which employs the nonholonomic control technique. The different algorithmic tools needed to implement our scheme have been presented.

Finally, an application has been proposed which makes use of the rubidium atom. One qubit of information is encoded in the spin states of the atom whereas the orbital part plays the role of the ancilla. A realistic physical setting has been proposed: in particular, a projection process based on the spontaneous emission has been suggested.

ACKNOWLEDGMENTS

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EXPLICIT DERIVATION OF THE CODE SUBSPACE

In this appendix we deal with a particular physical situation in which the code subspace $\tilde{\mathcal{C}}$ can be explicitly derived. We consider an atom with zero nuclear spin on the level characterized by the orbital angular momentum L. The electronic spin of the atom is $S=1/2$. The natural basis wave functions are $|L,M_L;S,M_S\rangle$. A qubit of information is encoded on the two states

$$|\tilde{\gamma}_i\rangle = |JLSM_{J_i}\rangle = \sum_{M_L,M_S} C^{JM_{J_i}}_{LM_LSM_S}|L,M_L;S,M_S\rangle, \quad i=1,2,$$

where $C^{JM_J}_{LM_LSM_S}$ is the Clebsch-Gordan coefficient. In the scheme we proposed for a rubidium atom, $L=3$, $J=5/2$, $M_{J_1}=-3/2$, $M_{J_2}=-1/2$—that is,

$$|\gamma_1\rangle = \left|60f, j=\frac{5}{2}, m_j=-\frac{3}{2}\right\rangle,$$

$$|\gamma_2\rangle = \left|60f, j=\frac{5}{2}, m_j=-\frac{1}{2}\right\rangle.$$

We want to protect this information against the action of six independent error-inducing Hamiltonians $\hat{E}_k$, three magnetic and three electric interaction Hamiltonians. We shall see that the code space $\tilde{\mathcal{C}}=\mathrm{Span}[|\tilde{\gamma}_1\rangle,|\tilde{\gamma}_2\rangle]$ can be simply built from physical considerations on the action of the Hamiltonians $\hat{E}_k$.

Let us first consider magnetic errors. The interaction Hamiltonian of the atom with the magnetic field $\vec{B}$ directed along the kth axis ($k=x,y,z$) is

$$\hat{E}^{\beta}_k = \mu_B B_k(\hat{L}_k + 2\hat{S}_k),$$

μ_B being the Bohr magneton. Remembering that $\hat{L}_x=(\hat{L}_+ + \hat{L}_-)/2$, $\hat{L}_y=(\hat{L}_+ - \hat{L}_-)/2i$, where $\hat{L}_+$ ($\hat{L}_-$) is the operator increasing (lowering) the z projection of the orbital angular momentum, and similar relations for the spin operators, one can conclude that a pair of the states with definite z projections of orbital and spin angular momenta is a good basis for the code subspace if the difference of the of their quantum number $M_L \equiv L_z$ is greater than or equal to 2. To use this option, one needs to consider the error caused by a magnetic field oriented along z. The states with definite L_z, S_z are the eigenstates of the Hamiltonian $\hat{E}^{\beta}_k$. A general superposition of two such states must not be rotated in the Hilbert space under the action of $\hat{E}^{\beta}_z$. This means that the eigenvalues must be equal to each other. Thus the states

$$|\tilde{\gamma}^{\beta}_1\rangle = |L,M_L;S,M_S=+1/2\rangle,$$

$$|\tilde{\gamma}^{\beta}_2\rangle = |L,M_L+2;S,M_S=-1/2\rangle,$$

with $M \leqslant L-2$, constitute a good code basis for protecting one qubit against the action of the $\hat{E}^{\beta}_k$.

We shall now consider errors caused by quasistatic electric fields. The static Stark shift of a level with zero fine splitting—i.e., a highly excited Rydberg level, like $60f$ in rubidium, caused by the electric field $\vec{\mathcal{E}}$ oriented along z—is given by $\mathrm{const}-b\mathcal{E}^2M_L^2$. The value b characterizes the polarizability of the atom in the given state. Omitting the irrelevant constant, we may represent the Hamiltonian of the atom-field interaction (with respect to the particular manifold of sublevels of the given atomic state) by the operator

$$\hat{E}_k^{\varepsilon} = -b\mathcal{E}_k^2 \hat{L}_k^2, \quad k = x,y,z. \tag{A1}$$

Note that, since the fine splitting is zero, the spin variables are unaffected by the Stark effect. Rewriting the operators $\hat{E}_{x,y}^{\varepsilon}$ in terms of $\hat{L}_z$,

$$\hat{E}_x^{\varepsilon} = -b\mathcal{E}_x^2\left[\frac{1}{4}\hat{L}_+^2 + \frac{1}{4}\hat{L}_-^2 + \frac{1}{2}L(L+1) - \frac{1}{2}\hat{L}_z^2\right],$$

$$\hat{E}_y^{\varepsilon} = -b\mathcal{E}_y^2\left[-\frac{1}{4}\hat{L}_+^2 - \frac{1}{4}\hat{L}_-^2 + \frac{1}{2}L(L+1) - \frac{1}{2}\hat{L}_z^2\right],$$

one can conclude that the basis of the coding space can be a pair of states of opposite S_z and opposite L_z (so that $|L_z|$ is the same for both of these states). Indeed, these states are not mixed by the Hamiltonian (A1), which does not cause spin flips. The error vector is always orthogonal to any their superposition, as can be easily seen. Among various code subspaces protecting against electric errors there is one that protects against magnetic errors, too. The basis vectors of this subspace are

$$|\tilde{\gamma}_1^{\beta,\varepsilon}\rangle = |L, M_L = -1; S, M_S = +1/2\rangle,$$

$$|\tilde{\gamma}_2^{\beta,\varepsilon}\rangle = |L, M_L = +1; S, M_S = -1/2\rangle.$$

It may happen that for singlet electronic states of atoms with nonzero nuclear spin, whose nuclear magnetic moment μ_{nucl} is incommensurate with μ_B, one cannot apply this explicit derivation of the code space for the correction of *both* the "electric" and "magnetic" errors. One has then to look for a more complex coding transformation through the algorithm we presented in Sec. III.

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