Multiparticle entanglement manipulation under positive partial transpose preserving operations

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We consider the transformation of multipartite states in the single-copy setting under positive-partial-transpose-preserving operations (PPT operations) and obtain both qualitative and quantitative results. First, for some pure-state transformations that are impossible under local operations and classical communication (LOCC), we demonstrate that they become possible with a surprisingly large success probability under PPT operations. Furthermore, we clarify the convertibility of arbitrary multipartite pure states under PPT operations and show that a drastic simplification in the classification of pure-state entanglement occurs when the set of operations is switched from LOCC to PPT operations. Indeed, the infinitely many types of LOCC-incomparable entanglement are reduced to only one type under the action of PPT operations. This is a clear manifestation of the increased power afforded by the use of PPT-bound entanglement. In addition, we further enlarge the set of operations to clarify the effect of another type of bound entanglement, multipartite unlockable bound entanglement, and show that a further simplification occurs. As compared to pure states a more complicated situation emerges in mixed-state settings. While single-copy distillation becomes possible under PPT operations for some mixed states it remains impossible for other mixed states.

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I. INTRODUCTION

Constraints and resources are intimately related in physics. If we impose a constraint on a physical setting, then certain tasks become impossible. A resource must be made available to overcome the restrictions imposed by the constraints. By definition such a resource cannot be created employing only the constrained set of operations but it may be manipulated and transformed under these operations. That the amount of resource does not increase under any operation satisfying the constraint emerges then as a fundamental law—for example, in entanglement theory [1,2].

One example of particular importance is the restriction to local quantum operations and classical communication (LOCC). The resources that are implied by this constraint are nonseparable states and in particular pure entangled states such as singlet states, neither of which can be created by LOCC alone. This setting gives rise to a theory of entanglement as a resource under LOCC operations.

Any such theory of entanglement as a resource will generally aim to provide mathematical structures to allow answers to three questions: namely, (1) the characterization of entanglement, (2) the manipulation of entanglement, and (3) the quantification of the entanglement resource [1,2] under the given constraint. Of particular interest is the question of how many inequivalent types of entanglement exist within such a theory. In the limit of infinitely many identically prepared copies of bipartite pure states, entanglement can be interconverted reversibly [3] and it is reasonable to say that there is only one type of pure bipartite entanglement. Even for pure states, the situation changes dramatically when we consider the single-copy setting. It has been shown that the Schmidt rank of bipartite pure states cannot be increased by LOCC [4–8]. At the single-copy level, the convertibility of bipartite entanglement is then characterized by the Schmidt rank [9]. For finite-dimensional systems a state can be converted to another one with finite probability exactly if the Schmidt number of the target state is not larger than that of the initial state. In a tripartite setting the situation is more complicated. Here it is well known, for example, that a GHZ (Greenberger, Horne, and Zeilinger) state cannot be transformed to a *W* state and vice versa [9]. These states are then said to be incomparable. It can be shown that there are two incomparable types of tripartite entanglement in three-qubit systems. The situation is even more complicated in multipartite settings composed of many parties [10–14] or infinite-dimensional bipartite systems [15,16], where there are many (possibly infinitely many) incomparable types of entanglement.

A different setting is presented by the concept of partial time reversal or partial transposition [17]. For two qubits, states that remain positive under partial transposition (denoted as PPT states) are exactly the separable states [18] but for higher dimensions this is generally not the case as there are PPT states that are inseparable [19]. This motivates the definition of the set of positive-partial-transpose-preserving operations (PPT operations), defined as operations that map any PPT state into another PPT state [20]. In this case, the resource are states that are not PPT (denoted as NPT states). In the single-copy setting for pure states, it has been shown that both under PPT operations [21] and with LOCC supported by PPT-bound entanglement [22] the Schmidt number can be increased so that state transformations become possible that are strictly impossible under LOCC. Furthermore, there are mixed-state transformations that are reversible in the asymptotic setting [21]. This suggests that a theory of entanglement under PPT operations might have a much simpler structure than that under the LOCC constraint.

In this paper, we focus attention on the entanglement manipulation under PPT operations in the nonasymptotic, single-copy setting to explore what simplifications occur. We consider PPT-state transformation in multipartite settings and obtain both qualitative and quantitative results. In Sec. II, the general settings and notations of PPT-preserving operations are introduced. In Secs. III and IV, we first demonstrate that the transformations of pure states that are impossible under LOCC become possible with a surprisingly large success probability when employing trace preserving PPT operations. In Sec. V, a rather tractable scheme of trace-nonpreserving PPT operations is introduced and discussed. We will then completely clarify the convertibility of all multipartite pure states under PPT operations in Sec. VI. In Sec. VII we enlarge the set of operations beyond that of PPT operations to consider the effect of multipartite unlockable bound entangled states. In Sec. VIII, we will consider the transformation of a single copy of mixed states into pure entangled states—i.e., the single-copy distillation under PPT operations. A summary and conclusion are given in Sec. IX.

II. BASIC NOTATION

To begin with, let us denote $\mathcal{H}(V)$ $[\mathcal{H}(V')]$ the space of Hermitian operators on the Hilbert space V $[V']$. A superoperator Ψ from V to V' is a linear transformation from $\mathcal{H}(V)$ to $\mathcal{H}(V')$. There is a natural isomorphism [20] which associates with superoperators $\Psi:\mathcal{H}(V)\to\mathcal{H}(V')$ a Hermitian operator $\Omega(\Psi)\in\mathcal{H}(V)\otimes\mathcal{H}(V')$ such that for all $A\in\mathcal{H}(V)$ and $B\in\mathcal{H}(V')$ we have

$$\mathrm{tr}\{\Psi(A)B\}=\mathrm{tr}\{\Omega(\Psi)A\otimes B\}. \tag{1}$$

Maps that are trace nonincreasing then satisfy

$$\mathrm{tr}_{V'}\{\Omega(\Psi)\}\leqslant \mathbb{1}_V, \tag{2}$$

with equality if Ψ is trace preserving. A superoperator Ψ is called positive if for any $A\geqslant 0$ we have $\Psi(A)\geqslant 0$ and it is called completely positive if $\mathbb{1}^W\otimes\Psi\geqslant 0$ for any space W. Following [20] complete positivity (CP) of Ψ can be verified by checking

$$\Omega(\Psi)^{\Gamma_V}\geqslant 0, \tag{3}$$

where Γ_V denotes the partial transposition with respect to V.

An additional concept comes into play when we consider multipartite systems. A CP map on bipartite systems $\Psi:\mathcal{H}(V_A)\otimes\mathcal{H}(V_B)\to\mathcal{H}(V_A')\otimes\mathcal{H}(V_B')$ is called positive partial transpose preserving [20], if we have $\Gamma_A\circ\Psi\circ\Gamma_A\geqslant 0$ $(\Gamma_B\circ\Psi\circ\Gamma_B\geqslant 0)$ for the partial transposition map Γ_A (Γ_B) with respect to party A (B). On the level of the state $\Omega(\Psi)$, this condition reads

$$[\Omega(\Psi)^{\Gamma_V}]^{\Gamma_{V_A}\otimes\Gamma_{V_A'}}\geqslant 0 \quad\text{or}\quad [\Omega(\Psi)^{\Gamma_V}]^{\Gamma_{V_B}\otimes\Gamma_{V_B'}}\geqslant 0,$$

where Γ_{V_A} $(\Gamma_{V_A'})$ denotes partial transposition applied to space V_A (V_A'). In the bipartite case, there are two equivalent choices for the partial transposition. In the tripartite setting, however, there are three different possible partial transpositions that are generally *not* equivalent. A CP map $\Psi:\mathcal{H}(V_A)\otimes\mathcal{H}(V_B)\otimes\mathcal{H}(V_C)\to\mathcal{H}(V_A')\otimes\mathcal{H}(V_B')\otimes\mathcal{H}(V_C')$ will be called PPT in the following if

$$(\Omega(\Psi)^{\Gamma_V})^{\Gamma_{V_i}\otimes\Gamma_{V_i'}}\geqslant 0 \tag{4}$$

for all $i=A$, B, and C.

Let us now consider the transformation of a state $\rho\in\mathcal{H}(V)$ into a state $\sigma\in\mathcal{H}(V')$ with the probability of $p(\rho\to\sigma)$. For this probabilistic transformation, we construct the trace-preserving CP-PPT map with two outcomes: one that gives σ and one that gives some other state. The two parts are given by the CP-PPT maps Ψ and ψ, respectively. The associated Hermitian operators are denoted by Ω and ω. The map Ψ then satisfies $\Psi(\rho)=p(\rho\to\sigma)\sigma$ or

$$\mathrm{tr}\{\Psi(\rho)(\mathbb{1}-\sigma)\}=\mathrm{tr}\{\Omega(\Psi)\rho\otimes(\mathbb{1}-\sigma)\}=0 \tag{5}$$

when σ is a pure state. The success probability is then given by

$$p(\rho\to\sigma)=\mathrm{tr}\{\Psi(\rho)\}=\mathrm{tr}\{\Omega(\Psi)\rho\otimes\mathbb{1}\}=\mathrm{tr}\{\Omega(\Psi)\rho\otimes\sigma\}.$$

The PPT map ψ, on the other hand, does not suffer any constraint other than the condition of trace preservation for $\Psi+\psi$. On the level of states, the trace-preserving condition is

$$\mathrm{tr}_{V'}\{\Omega+\omega\}=\mathbb{1}_V, \tag{6}$$

where, as we will do in the remainder of this paper, we have dropped the Ψ $[\psi]$ in $\Omega(\Psi)$ $[\omega(\psi)]$ for brevity. It should be noted that a rather simple structure can be assumed for ω without loss of generality. Let us consider a map χ which maps arbitrary states in $\mathcal{H}(V)$ onto a maximally mixed state of $\mathbb{1}_{V'}/\dim\{\mathcal{H}(V')\}\in\mathcal{H}(V')$. This map is a trace-preserving CP-PPT map since the corresponding state is $\mathbb{1}_V\otimes\mathbb{1}_{V'}/\dim\{\mathcal{H}(V)\otimes\mathcal{H}(V')\}$. Therefore, a composed map of $\chi\circ\psi$ is a CP-PPT map if ψ is a CP-PPT map. Furthermore, if $\Psi+\psi$ is trace preserving, $\Psi+\chi\circ\psi$ is also trace preserving, and hence the replacement of ψ by $\chi\circ\psi$ does not alter $p(\rho\to\sigma)$. One may then assume $\psi=\chi\circ\psi$ since the output of ψ is arbitrary. On the level of the state, this assumption is

$$\omega=\omega_V\otimes\frac{\mathbb{1}_{V'}}{\dim\{\mathcal{H}(V')\}}. \tag{7}$$

In the subsequent Sections III and IV, we maximize $p(\rho\to\sigma)$ for some important classes of pure states in both bipartite and tripartite settings. In particular, we demonstrate that transformations of pure states that are impossible under LOCC can be achieved under PPT operations with a surprisingly large success probability.

III. CONVERSION OF MAXIMALLY ENTANGLED STATES

For two d-dimensional systems we denote the maximally entangled state by $P_d^+\equiv|\phi_d^+\rangle\langle\phi_d^+|$ where

$$|\phi_d^+\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle.$$

In the single-copy setting, it is known that LOCC cannot increase the Schmidt rank of a pure state [4–8]. Therefore,

$p(P_d^+ \to P_{d'}^+) = 0$ for LOCC transformation whenever $d' > d$.

In the following we proceed with the construction of the CP-PPT maps Ψ and ψ that maximize the success probability for this transformation. For $d' > d$ this amounts to the maximization of

$$p(P_d^+ \to P_{d'}^+) = \mathrm{tr}\{\Omega P_d^+ \otimes P_{d'}^+\} \tag{8}$$

under the constraints

$$\mathrm{tr}\{\Omega P_d^+ \otimes (\mathbb{1} - P_{d'}^+)\} = 0, \quad \mathrm{tr}_{V'}\{\Omega + \omega\} = \mathbb{1},$$

$$(\Omega^{\Gamma_{V_A} \otimes \Gamma'_{V_A}})^{\Gamma_V} \geqslant 0, \quad \Omega^{\Gamma_V} \geqslant 0,$$

$$(\omega^{\Gamma_{V_A} \otimes \Gamma'_{V_A}})^{\Gamma_V} \geqslant 0, \quad \omega^{\Gamma_V} \geqslant 0, \tag{9}$$

where $P_d^+ \in \mathcal{H}(V)$ and $P_{d'}^+ \in \mathcal{H}(V')$. Since both $P_d^+ \otimes P_{d'}^+$ and $P_d^+ \otimes (\mathbb{1} - P_{d'}^+)$ are invariant under the local unitary transformation of $U_1 \otimes U_1^* \otimes U_2 \otimes U_2^*$ with U_1 and U_2 being arbitrary unitary operators, it suffices to consider Ω and ω that are invariant under these local operations: i.e.,

$$\Omega = a_1 P_d^+ \otimes P_{d'}^+ + a_2(\mathbb{1} - P_d^+) \otimes P_{d'}^+ + a_3 P_d^+ \otimes \frac{\mathbb{1} - P_{d'}^+}{d'^2 - 1}$$

$$+ a_4(\mathbb{1} - P_d^+) \otimes \frac{\mathbb{1} - P_{d'}^+}{d'^2 - 1},$$

$$\omega_V = b_1 P_d^+ + b_2(\mathbb{1} - P_d^+).$$

The first two constraints in Eq. (9) yield $a_3 = 0$, $b_1 = 1 - a_1$, and $b_2 = 1 - a_2 - a_4$. These equalities can be used to eliminate b_1 and b_2 in the remaining constraints. The remaining constraints then result in

$$1 \geqslant a_1 \geqslant 0, \quad a_2 \geqslant 0, \quad a_4 \geqslant 0, \quad 1 \geqslant a_2 + a_4,$$

$$(d' + 1)a_1 + (d' + 1)(d - 1)a_2 + (d - 1)a_4 \geqslant 0,$$

$$-(d' + 1)a_1 + (d' + 1)(d + 1)a_2 + (d + 1)a_4 \geqslant 0,$$

$$-(d' - 1)a_1 - (d' - 1)(d - 1)a_2 + (d - 1)a_4 \geqslant 0,$$

$$(d' - 1)a_1 - (d' - 1)(d + 1)a_2 + (d + 1)a_4 \geqslant 0,$$

$$-a_1 - (d - 1)a_2 - (d - 1)a_4 + d \geqslant 0,$$

$$a_1 - (d + 1)a_2 - (d + 1)a_4 + d \geqslant 0.$$

The constraints in the first row arise from $\omega^{\Gamma_V} \geqslant 0$ and $\Omega^{\Gamma_V} \geqslant 0$. The last two rows are due to $(\omega^{\Gamma_{V_A} \otimes \Gamma'_{V_A}})^{\Gamma_V} \geqslant 0$ and the remaining for inequalities arising from $(\Omega^{\Gamma_{V_A} \otimes \Gamma'_{V_A}})^{\Gamma_V} \geqslant 0$. The maximization of $p(P_d^+ \to P_{d'}^+) = a_1$ under these constraints is a linear program and we can identify the optimal solution as $a_1 = d(d-1)/(dd' + d' - 2d)$, $a_2 = 0$, and $a_4 = d(d'-1)/(dd' + d' - 2d)$. Consequently, for $d' > d$ the optimal probability for the transformation of P_d^+ into $P_{d'}^+$, thereby increasing the Schmidt rank, under PPT operations is given by

$$p(P_d^+ \to P_{d'}^+) = \frac{d(d-1)}{dd' + d' - 2d}. \tag{10}$$

We emphasize that this success probability is nonzero even when $d' > d \geqslant 2$, while it is strictly zero for the LOCC transformation.

IV. CONVERSION FROM THE GHZ TO *W* STATE

In the tripartite setting, it is well known [9] that the success probability $p(\mathrm{GHZ} \to W) = 0$ for the LOCC transformation from a single copy of

$$|\mathrm{GHZ}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

to

$$|W\rangle = \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}}.$$

In the following we will demonstrate that this is not the case when we consider PPT operations. To this end, we maximize

$$p(\rho_{\mathrm{GHZ}} \to \rho_W) = \mathrm{tr}\{\Omega \rho_{\mathrm{GHZ}} \otimes \rho_W\} \tag{11}$$

under the constraints for $i = A, B, C$,

$$\mathrm{tr}\{\Omega \rho_{\mathrm{GHZ}} \otimes (\mathbb{1} - \rho_W)\} = 0, \quad \mathrm{tr}_{V'}\{\Omega + \omega\} = \mathbb{1},$$

$$(\Omega^{\Gamma_{V_i} \otimes \Gamma'_{V_i}})^{\Gamma_V} \geqslant 0, \quad \Omega^{\Gamma_V} \geqslant 0,$$

$$(\omega^{\Gamma_{V_i} \otimes \Gamma'_{V_i}})^{\Gamma_V} \geqslant 0, \quad \omega^{\Gamma_V} \geqslant 0,$$

where $\rho_{\mathrm{GHZ}} = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| \in \mathcal{H}(V)$ and $\rho_W = |W\rangle\langle W| \in \mathcal{H}(V')$.

The solution of the problem is greatly aided by the use of a number of symmetries. Indeed, both the states $\rho_{\mathrm{GHZ}} \otimes (\mathbb{1} - \rho_W)$ and $\rho_{\mathrm{GHZ}} \otimes \rho_W$ are invariant under the local operations

$$(a)\ X \otimes X \otimes X \otimes \mathbb{1} \otimes \mathbb{1} \otimes\ ,$$

$$(b)\ Z \otimes Z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes\ ,$$

$$(c)\ \mathbb{1} \otimes Z \otimes Z \otimes \mathbb{1} \otimes \mathbb{1} \otimes\ ,$$

$$(d)\ \otimes \mathbb{1} \otimes \mathbb{1} \otimes Z \otimes Z \otimes Z,$$

$$(e)\ P_1 \otimes P_1 \otimes P_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes\ ,$$

$$(f)\ \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes P_2 \otimes P_2 \otimes P_2,$$

where $P_1 = |0\rangle\langle 0| + |1\rangle\langle 1| e^{2\pi i/3}$ and $P_2 = e^{\pi i/2}|0\rangle\langle 0| + |1\rangle\langle 1| e^{\pi i}$. These local symmetries are supplemented by the nonlocal joint permutation symmetry

$$(g)\ \mathcal{P}(123) \times \mathcal{P}(456),$$

where $\mathcal{P}$ represents an arbitrary index permutation. The symmetries (a)–(g) allow for a considerable simplification of Ω and ω. Indeed, the symmetries (b), (c), and (e) ensure that the matrix elements $\Omega_{i_1j_1k_1l_1m_1n_1,i_2j_2k_2l_2m_2n_2}$ can only be nonzero if the indices satisfy simultaneously $i_1=i_2$, $j_1=j_2$, and $k_1=k_2$ or $i_1=1-i_2$, $j_1=1-j_2$, and $k_1=1-k_2$. The symmetry (g) yields

$$\Omega_{abcdef,ghijkl} = \Omega_{\mathcal{P}(abc)\mathcal{P}(def),\mathcal{P}(ghi)\mathcal{P}(jkl)} \tag{12}$$

for any index permutation $\mathcal{P}$. Symmetry (a) yields

$$\Omega_{000l_1m_1n_1,000l_2m_2n_2} = \Omega_{111l_1m_1n_1,111l_2m_2n_2}, \tag{13}$$

$$\Omega_{001l_1m_1n_1,001l_2m_2n_2} = \Omega_{110l_1m_1n_1,110l_2m_2n_2}, \tag{14}$$

$$\Omega_{000l_1m_1n_1,111l_2m_2n_2} = \Omega_{111l_1m_1n_1,000l_2m_2n_2}. \tag{15}$$

Presenting all nonzero matrix elements of $\Omega_{abcdef,ghijkl}$ for $(abc,ghi)=(000,000)$, $(abc,ghi)=(001,001)$, and $(abc,ghi)=(000,111)$ fixes all other matrix elements by virtue of the symmetries, Eqs. (12)–(15), and the Hermiticity of Ω. To obtain a trial solution we chose

$$\Omega_{000000,000000} = \Omega_{001000,001000} = -\Omega_{000000,111000} = b_1,$$

$$\Omega_{000001,000001} = \Omega_{000001,000010} = \Omega_{000001,000100} = b_2,$$

$$\Omega_{000011,000011} = \Omega_{000011,000101} = \Omega_{000011,000110} = b_4,$$

$$\Omega_{001001,001001} = -\Omega_{001001,001010} = -\Omega_{001001,001100} = b_2,$$

$$\Omega_{001010,001010} = \Omega_{001010,001100} = \Omega_{001100,001100} = b_2,$$

$$\Omega_{001011,001011} = \Omega_{001011,001101} = -\Omega_{001011,001110} = -b_4,$$

$$\Omega_{001101,001101} = -\Omega_{001101,001110} = \Omega_{001110,001110} = b_4,$$

$$\Omega_{001111,001111} = 3\Omega_{000111,000111} = -3\Omega_{000111,111111} = 3b_6,$$

$$\Omega_{000001,111001} = \Omega_{000001,111010} = \Omega_{000001,111100} = b_2,$$

$$\Omega_{000010,111010} = \Omega_{000010,111100} = \Omega_{000100,111100} = b_2,$$

$$\Omega_{000011,111011} = \Omega_{000011,111101} = \Omega_{000011,111110} = -b_4,$$

$$\Omega_{000101,111101} = \Omega_{000101,111110} = \Omega_{000110,111110} = -b_4.$$

Likewise, the nonzero matrix elements of ω_V can be constructed from

$$(\omega_V)_{000,000} = 1 - b_6 - 3b_4 - 3b_2 - b_1,$$

$$(\omega_V)_{001,001} = (\omega_V)_{000,111},$$

$$(\omega_V)_{000,111} = b_6 + 3b_4 - 3b_2 + b_1,$$

where we chose

$$b_1 = \frac{1+\sqrt{1-4x^2}}{6}, \quad b_2 = \frac{x}{3}, \quad b_4 = \frac{b_2^2}{b_1}, \quad b_6 = \frac{9b_4^2}{3x},$$

$$x = \frac{1}{8}[-2 + (18-6\sqrt{3})^{1/3} + (18+6\sqrt{3})^{1/3}].$$

A lengthy but elementary calculation (preferably executed employing a program capable of symbolic manipulations) then confirms that this trial solution satisfies all the constraints and yields the success probability

$$\mathrm{tr}\{\Omega\rho_{\mathrm{GHZ}} \otimes \rho_W\} = 6b_2. \tag{16}$$

We then consider the dual problem, of the primal problem Eq. (11) [23]. Every feasible point of the dual problem provides an upper bound on the solution of the primal problem, Eq. (11). The above result of Eq. (16) is then proved to be optimal as shown in Appendix A.

As a consequence, the optimal probability for the transformation of a GHZ to a W state under PPT operations is given by

$$p(\mathrm{GHZ} \to W) = 6b_2 \approx 0.75436\ldots, \tag{17}$$

which is more than 75%. This very high success probability is somewhat surprising, since the success probability for the LOCC transformation is strictly zero. Note that this result also implies that a GHZ state can be transformed into a W state employing LOCC supplemented by PPT-bound entanglement.

V. TRACE-NONPRESERVING CP-PPT MAPS

In the previous two sections we have demonstrated explicitly that the success probability for the transformation between pure states can in some cases be improved significantly by employing PPT operations instead of LOCC operations. Obtaining the optimal success probabilities is a hard task, however, especially in the multipartite setting. In the following we will consider the slightly more tractable setting of trace-nonpreserving PPT maps. In this setting we also optimize a CP-PPT map Ψ or equivalently the associated state Ω, but the trace-preserving condition of Eq. (6) is replaced by the trace-nonincreasing condition of

$$\mathrm{tr}_{V'}\{\Omega\} \leq \mathbb{1}_V. \tag{18}$$

As a result, the completion ψ of the map Ψ is a CP map but it is *not* necessarily a PPT map. This will generally allow one to find success probabilities for state transformations that are larger than those obtained under trace-preserving PPT operations. It is important to note, however, that any transformation that possesses a nonvanishing success probability under *trace-nonpreserving* CP-PPT maps will also have a nonvanishing success probability under *trace-preserving* CP-PPT maps. To see this, let $\Omega(\Psi)$ be the state corresponding to a trace-nonpreserving CP-PPT map Ψ. Since the completion ψ is not necessarily a PPT map, $\omega(\psi)^{\Gamma_V}$ is sometimes a NPT state. However, if we consider the states of $\Omega'(\Psi') = \epsilon\Omega(\Psi)$ and $\omega'(\psi') = \epsilon\omega(\psi) + (1-\epsilon)\mathbb{1}\otimes\mathbb{1}/\dim\{\mathcal{H}(V')\}$, the state $(\omega')^{\Gamma_V}$ becomes a PPT state for a nonzero value of $1 \geq \epsilon > 0$. Both $(\Omega')^{\Gamma_V}$ and $(\omega')^{\Gamma_V}$ are PPT states satisfying the trace-preserving condition of Eq. (6), and Ψ' accomplishes the same transformation as Ψ, albeit with a smaller success probability. In this way, one can always construct a trace-preserving CP-PPT map from the trace-nonpreserving CP-PPT map giving the same transformation.

The optimal probability in the trace-nonpreserving scheme for the transformation of maximally entangled states ($d' > d$) can be obtained in the same fashion as in Sec. III. Employing the notation of Sec. III we obtain the constraints

$$1 \geqslant a_1 \geqslant 0, \quad a_2 \geqslant 0, \quad a_4 \geqslant 0, \quad 1 \geqslant a_2 + a_4,$$

$$(d'+1)a_1 + (d'+1)(d-1)a_2 + (d-1)a_4 \geqslant 0,$$

$$-(d'+1)a_1 + (d'+1)(d+1)a_2 + (d+1)a_4 \geqslant 0,$$

$$-(d'-1)a_1 - (d'-1)(d-1)a_2 + (d-1)a_4 \geqslant 0,$$

$$(d'-1)a_1 - (d'-1)(d+1)a_2 + (d+1)a_4 \geqslant 0,$$

under which the success probability, given by a_1, has to be maximized. The result is

$$p(P_d^+ \to P_{d'}^+) = \frac{d-1}{d'-1}, \tag{19}$$

whose PPT map Ψ is, on the level of the state Ω,

$$\Omega = \frac{d-1}{d'-1} P_d^+ \otimes P_{d'}^+ + \frac{1}{d'^2-1}(\mathbb{1} - P_d^+) \otimes (\mathbb{1} - P_{d'}^+). \tag{20}$$

It is noteworthy that the probability of Eq. (19) can be written as a ratio of the negativity of the initial and target state: i.e.,

$$p(P_d^+ \to P_{d'}^+) = \frac{N(P_d^+)}{N(P_{d'}^+)},$$

where $N(\sigma) = (\mathrm{tr}|\sigma^\Gamma| - 1)/2$ [21,24]. This somewhat fascinating expression resembles the case of the LOCC transformation of pure states, where the optimal probability agrees with a ratio of a LOCC monotone such that it is the partial summation of squared Schmidt coefficients [6]. Although the monotonicity of the negativity in *trace-nonpreserving* PPT operations has not been proved yet (in *trace-preserving* PPT operations with a single outcome the negativity is a monotone [21]), the tractable expression of Eq. (19) is likely to be explained as a ratio of some monotone function.

In the tripartite setting, the optimization of the success probability is still a hard task even in this trace-nonpreserving scheme. The result of the optimization for the transformation of GHZ$\to W$ is

$$p(\mathrm{GHZ} \to W) = \frac{4}{5}, \tag{21}$$

and for the transformation of $W \to$GHZ we have

$$p(W \to \mathrm{GHZ}) = \frac{1}{3}. \tag{22}$$

The proofs of these two results are described in Appendixes B and C. This result implies that the transformation of $W \to$GHZ is also possible by trace-preserving PPT operations, although the optimal probability may be smaller than 1/3. Therefore, PPT operations can interconvert even the LOCC-incomparable pure states. In the next section, we will completely clarify the convertibility by PPT operations for all multipartite pure states in the single-copy setting.

VI. CONVERTIBILITY OF PURE STATES

In this section we will consider the transformation between single copies of N-partite pure states under PPT operations. By definition, PPT operations map PPT states to PPT states. As a consequence, transformations such as $|\phi_{AB}^+\rangle \otimes |0_C\rangle \to |\mathrm{GHZ}\rangle$ or $|\phi_{AB}^+\rangle \otimes |0_C\rangle \to |0_A\rangle \otimes |\phi_{BC}^+\rangle$ are impossible, since they are not PPT preserving with respect to party C. Therefore, let us first assume for the transformation of $|\psi\rangle \to |\phi\rangle$ that both $|\psi\rangle$ and $|\psi\rangle$ are "genuinely" entangled over all N parties. This assumption means that

$$(|\psi\rangle\langle\psi|)^{\Gamma_i} \ngeq 0 \text{ and } (|\phi\rangle\langle\phi|)^{\Gamma_i} \ngeq 0, \tag{23}$$

for all possible bipartite partitioning of i. For example, $i = A,B,C$ in a tripartite setting, and $i = A,B,C,D,AB,AC,AD$ in a four-partite setting. As discussed in the previous section, it suffices to consider trace-nonpreserving CP-PPT maps Ψ in order to check the convertibility under trace-preserving PPT operations. Therefore, we will construct an Ω satisfying the constraints

$$\mathrm{tr}\{\Omega |\psi\rangle\langle\psi| \otimes (\mathbb{1} - |\phi\rangle\langle\phi|)\} = 0,$$

$$\Omega^{\Gamma_V} \geqslant 0, \quad (\Omega^{\Gamma_{V_i} \otimes \Gamma'_{V_i}})^{\Gamma_V} \geqslant 0, \tag{24}$$

where $|\psi\rangle \in \mathcal{H}(V)$, $|\phi\rangle \in \mathcal{H}(V')$, and i stands for any possible bipartite partitioning as explained below Eq. (23). We have omitted the trace-nonincreasing condition, because we are not interested in the explicit value of the success probability but only whether it is zero or not. In view of Eq. (20), a suitable trial form is

$$\Omega = x|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| + (\mathbb{1} - |\psi\rangle\langle\psi|) \otimes (\mathbb{1} - |\phi\rangle\langle\phi|), \tag{25}$$

for which the first two constraints in Eq. (24) are satisfied when $x \geqslant 0$. Furthermore, due to the assumption of Eq. (23), the last constraint $(\Omega^{\Gamma_i \otimes \Gamma'_i})^{\Gamma_V} \geqslant 0$ is also satisfied for an appropriate value of $x = x_0 > 0$ as shown in [22]. As a result, for $x = x_0$ we have

$$\mathrm{tr}\{\Omega(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|)\} = x_0 > 0, \tag{26}$$

so that for arbitrary pairs of genuine N-partite entangled states of $|\psi\rangle$ and $|\phi\rangle$ we can always find an Ω such that $p(|\psi\rangle \to |\phi\rangle) > 0$. As a consequence, all genuine N-partite pure entangled states are interconvertible by PPT operations. In this way, the classification of N-partite entanglement is drastically simplified when we consider PPT operations.

Let us next investigate the convertibility between an N-partite state $|\psi^{(N)}\rangle$ and an $(N-1)$-partite state $|\phi^{(N-1)}\rangle$. It is obvious that $|\phi^{(N-1)}\rangle \to |\psi^{(N)}\rangle$ is impossible because such a transformation is not PPT preserving. Likewise the transformation of $|\psi^{(N)}\rangle \to |\phi^{(N-1)}\rangle$ is impossible if the set of en-

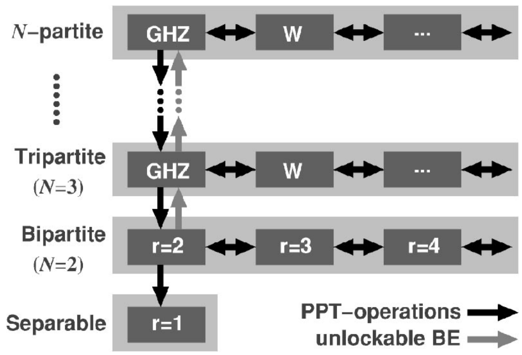

FIG. 1. The classification and convertibility of multipartite pure entangled states under PPT operations. r denotes the Schmidt rank of bipartite entanglement, and the set of entangled parties in $(N-1)$-partite entanglement is assumed to be a subset of the set of entangled parties in N-partite entanglement. There is only one type of N-partite entanglement under PPT operations. Furthermore, the convertibility with the support of unlockable bound entanglement (BE) is also shown (see also [29]).

tangled parties in $|\phi^{(N-1)}\rangle$ is not a subset of the set of entangled parties in $|\psi^{(N)}\rangle$ (e.g., $|\psi^{(3)}_{ABC}\rangle \to |\phi^{(2)}_{AD}\rangle$ is impossible). Otherwise, the transformation is possible because an N-partite GHZ state can be transformed to an $(N-1)$-partite GHZ state by LOCC, and hence the sequential transformation of $|\psi^{(N)}\rangle \to |\mathrm{GHZ}^{(N)}\rangle \to |\mathrm{GHZ}^{(N-1)}\rangle \to |\phi^{(N-1)}\rangle$ is possible. The classification and convertibility of arbitrary multipartite pure entangled states under PPT operations are summarized in Fig. 1.

It is important to note here that the power of PPT operations, by which N-partite pure entangled states become interconvertible as discussed above, immediately implies that the same holds for LOCC supported by PPT-bound entanglement. This is due to the fact that any PPT transformation can be accomplished (with smaller but nonzero probability) by LOCC supported by the additional resource of PPT states [25] (see the note of [26]). Indeed,

$$\Psi(\rho) = \mathrm{tr}_V\{\Omega(\Psi)^{\Gamma_V}(\rho^{\Gamma_V} \otimes \mathbb{1})\}, \tag{27}$$

and the state of $\Omega(\Psi)^{\Gamma_V} \geq 0$, which is a PPT state if Ψ is a CP-PPT map due to $[\Omega(\Psi)^{\Gamma_i \otimes \Gamma'_i}]^{\Gamma_V} \geq 0$, is utilized and consumed in the LOCC implementation of $\Psi(\rho)$ [25]. If a CP-PPT map Ψ can accomplish a transformation that is impossible under LOCC alone, then $\Omega(\Psi)^{\Gamma_V}$ must be entangled (otherwise the transformation can also be accomplished by LOCC because LOCC can generate any separable state), and therefore the PPT state $\Omega(\Psi)^{\Gamma_V}$ is a PPT-bound entangled state [19]. Consequently, one can conclude that a transformation such as GHZ$\leftrightarrow W$ can be accomplished by LOCC with the consumption of PPT-bound entangled states. Much attention has been paid to bound entanglement to clarify its properties, and several applications of bound entanglement have been reported [28–39]. As shown above, PPT-bound entanglement enables the LOCC implementation of large classes of entanglement transformations that are impossible by LOCC alone.

VII. UNLOCKABLE STATES AND CONVERSION OF PURE STATES

As mentioned in the previous section, the transformation

$$|\phi^+_{AB}\rangle \otimes |0_C\rangle \to |\mathrm{GHZ}_{ABC}\rangle \tag{28}$$

cannot be achieved even when PPT operations are employed and therefore cannot be achieved by LOCC supported by PPT-bound entanglement. However, it has been shown that a GHZ state can be distilled from a tripartite NPT-bound entangled state if A and B perform a global operation on the state [29]. Such NPT-bound entangled states are called unlockable states because bound entanglement is unlocked by the global operation [30,40,41]. The global operation of A and B can be accomplished by LOCC consuming $|\phi^+_{AB}\rangle$, and consequently the transformation of Eq. (28) is possible when LOCC is supported by the unlockable bound entanglement [29]. Likewise, unlockable states which can be utilized for the LOCC transformation from an $(N-1)$-partite GHZ state to an N-partite GHZ state have been shown in [29]. In this section we consider this type of transformation using a certain general scheme.

To this end, we generalize PPT operations by relaxing the PPT-preserving condition with respect to C, $(\Omega^{\Gamma_{V_C}\otimes\Gamma'_{V_C}})^{\Gamma_V} \geq 0$, which is responsible for the impossibility of the transformation of Eq. (28). We will therefore construct a map Ψ whose associated Hermitian operator Ω satisfies

$$\mathrm{tr}\{\Omega P^+_{AB} \otimes (\mathbb{1} - \rho_{\mathrm{GHZ}})\} = 0, \quad \Omega^{\Gamma_V} \geq 0,$$

$$(\Omega^{\Gamma_{V_A}\otimes\Gamma'_{V_A}})^{\Gamma_V} \geq 0, \quad (\Omega^{\Gamma_{V_B}\otimes\Gamma'_{V_B}})^{\Gamma_V} \geq 0, \tag{29}$$

where $P^+_{AB} = |\phi^+_{AB}\rangle\langle\phi^+_{AB}| \in \mathcal{H}(V)$ and $\rho_{\mathrm{GHZ}} \in \mathcal{H}(V')$. As a trial form for Ω, we adopt again Eq. (25): i.e.,

$$\Omega = xP^+_{AB} \otimes \rho_{\mathrm{GHZ}} + (\mathbb{1} - P^+_{AB}) \otimes (\mathbb{1} - \rho_{\mathrm{GHZ}}). \tag{30}$$

As mentioned in the previous section,

$$(P^+_{AB})^{\Gamma_A} \not\geq 0 \text{ and } \rho^{\Gamma_A}_{\mathrm{GHZ}} \not\geq 0$$

ensure the existence of $x_0 > 0$ such that $(\Omega^{\Gamma_{V_A}\otimes\Gamma'_{V_A}})^{\Gamma_V} \geq 0$ for $0 < x \leq x_0$ (indeed, we have $x_0 = 3$) and likewise with respect to B. We can now check easily that all constraints in Eq. (29) are satisfied for $x=3$, yielding a nonzero success probability $p(|\phi^+_{AB}\rangle \to |\mathrm{GHZ}\rangle) > 0$ since $\mathrm{tr}\{\Omega(P^+_{AB} \otimes \rho_{\mathrm{GHZ}})\} > 0$. Consequently, the transformation of $|\phi^+_{AB}\rangle \to |\mathrm{GHZ}\rangle$ is possible under the set of operations that maps NPT-BE states with respect to party C into itself, as expected. Employing symmetries of $P^+_{AB} \otimes \rho_{\mathrm{GHZ}}$, the optimized success probability in the trace-nonpreserving scheme is then obtained as

$$p(|\phi^+_{AB}\rangle \to |\mathrm{GHZ}\rangle) = \frac{3}{5}, \tag{31}$$

and, on the level of states, the map Ψ realizing this success probability is given by

$$\Omega = \frac{3}{5}P_{AB}^{+} \otimes \rho_{\mathrm{GHZ}} + \frac{1}{5}(\mathbb{1} - P_{AB}^{+}) \otimes (\mathbb{1} - \rho_{\mathrm{GHZ}} - \rho_{001} - \rho_{110}), \tag{32}$$

where $\rho_{001}=|001\rangle\langle 001|$ and $\rho_{110}=|110\rangle\langle 110|$. It can be confirmed that the state Ω^{Γ_V} is an unlockable state as follows. Due to the constraints of $(\Omega^{\Gamma_{V_A}\otimes\Gamma'_{V_A}})^{\Gamma_V}\geq 0$ and $(\Omega^{\Gamma_{V_B}\otimes\Gamma'_{V_B}})^{\Gamma_V}\geq 0$, the mixed state of Ω^{Γ_V} is undistillable by LOCC, because LOCC is PPT preserving and no tripartite and bipartite *pure* entangled states exist that are PPT with respect to both A and B. However, a GHZ state can be distilled from Ω^{Γ_V} of Eq. (30) or Eq. (32) if A and B perform global operations that distinguish P_{AB}^{+} and $\mathbb{1}-P_{AB}^{+}$.

Similarly, the map Ψ, whose associated state is

$$\Omega = 3\rho_{\mathrm{GHZ}}^{(N)} \otimes \rho_{\mathrm{GHZ}}^{(N')} + (\mathbb{1} - \rho_{\mathrm{GHZ}}^{(N)}) \otimes (\mathbb{1} - \rho_{\mathrm{GHZ}}^{(N')}), \tag{33}$$

can transform an N-partite GHZ state ($\rho_{\mathrm{GHZ}}^{(N)}$) to an N'-partite GHZ state, and furthermore the state Ω^{Γ_V} is an unlockable state if $N' > N \geq 2$ [42]. As shown in the previous section, all genuine N-partite entangled states are interconverted by PPT maps. The composition of the PPT maps and the map given in Eq. (33) is again a map whose associated state is an unlockable state. This implies that all pure entangled states can be interconverted independently of the number of parties (N) when a single copy of an appropriate unlockable bound entangled state is available as a resource. In this way, the consumption of unlockable bound entanglement allows one to overcome the LOCC constraint between pure states with different sets of entangled parties, while the consumption of PPT-bound entanglement overcomes the LOCC constraint between pure states with the same set of entangled parties (Fig. 1).

VIII. SINGLE-COPY DISTILLATION

So far, we have concentrated our attention on the discussion of transformations between pure states. In this section, we will now consider the transformation of a single copy of a mixed state ρ into a maximally entangled state $P_{d'}^{+}$: i.e., the single-copy distillation from a mixed state employing PPT operations.

Let us consider the antisymmetric Werner state which is defined as

$$\sigma_d^a = \frac{2}{d^2 - d}P_d^a = \frac{2}{d^2 - d}\sum_{j>i} |\psi_{ij}^{-}\rangle\langle\psi_{ij}^{-}|, \tag{34}$$

where P_d^a is the projector onto the antisymmetric subspace of $\mathbb{C}^d\otimes\mathbb{C}^d$, and $|\psi_{ij}^{-}\rangle=(|ij\rangle-|ji\rangle)/\sqrt{2}$. For the transformation of $\sigma_d^s\to P_{d'}^{+}$, we can construct CP-PPT maps of Ψ and its CP-PPT completion ψ employing the twirling symmetries of the two states. The result of the optimization is, on the level of the state Ω (the state ω is given by $\omega_V=\mathbb{1}-\mathrm{tr}_{V'}\,\Omega$),

$$\Omega = \frac{2}{dd' + d' - 2d}\left[P_d^a \otimes P_{d'}^{+} + (d' - 1)P_d^s \otimes \frac{\mathbb{1} - P_{d'}^{+}}{d'^2 - 1}\right]$$

for $d' \geq d \geq 2$ and

$$\Omega = \frac{2}{d(d' - 1)}\left[P_d^a + \frac{(d - d')}{(d + 1)d'}P_d^s\right] \otimes P_{d'}^{+} + \frac{2(d' + 1)}{(d + 1)d'}P_d^s \otimes \frac{\mathbb{1} - P_{d'}^{+}}{d'^2 - 1}$$

for $2\leq d'\leq d$, where P_d^s is the projector onto the symmetric subspace of $\mathbb{C}^d\otimes\mathbb{C}^d$. The optimal success probability under trace-preserving CP-PPT operations is then given by

$$p(\sigma_d^a \to P_{d'}^{+}) = \begin{cases} \dfrac{2}{dd' + d' - 2d} & \text{for } d' > d \geq 2, \\[2ex] \dfrac{2}{d(d' - 1)} & \text{for } 2 \leq d' \leq d. \end{cases} \tag{35}$$

Therefore, the success probability is nonzero for $d' \geq 2$.

On the other hand, the success probability for the same transformation under LOCC operations alone is strictly zero whenever $d' > 2$. This can be proved as follows: The $|\psi_{ij}^{-}\rangle$ in Eq. (34) are maximally entangled states on $\mathbb{C}^2\otimes\mathbb{C}^2$. Therefore, each $|\psi_{ij}^{-}\rangle$ can be prepared from P_2^{+} by local unitary transformations only. As σ_d^a is an equal mixture of all possible $|\psi_{ij}^{-}\rangle$, σ_d^a can be prepared from a single copy of P_2^{+} by LOCC, and hence the transformation of $P_2^{+}\to\sigma_d^a$ has a finite success probability. If we furthermore assume that for $d' > 2$ the transformation $\sigma_d^a\to P_{d'}^{+}$ has a finite success probability under LOCC, then this implies that $P_2^{+}\to\sigma_d^a\to P_{d'}^{+}$ also has a finite success probability under LOCC. This contradicts that the Schmidt rank cannot be increased by LOCC. Therefore, the result of Eq. (35) implies that the success probability of the single-copy distillation is also significantly improved when PPT operations are considered.

It should be noted that the transformation of $\sigma_d^a\to P_2^{+}$ is possible under LOCC. Indeed, the local projection $P\otimes P$ to σ_d^a, where $P=|0\rangle\langle 0|+|1\rangle\langle 1|$, can accomplish this. Furthermore, $P_2^{+}\to P_{d'}^{+}$ is possible under PPT operations, which enables the sequential transformation of $\sigma_d^a\to P_2^{+}\to P_{d'}^{+}$. Therefore, the feasibility of $p(\sigma_d^a\to P_{d'}^{+})$ can be regarded as being a consequence of the feasibility of $p(P_2^{+}\to P_{d'}^{+})$ under PPT operations. Note, however, that Eqs. (10) and (35) for $d' > 2$ imply that we have

$$p(\sigma_d^a \to P_{d'}^{+}) > p(\sigma_d^a \to P_2^{+})p(P_2^{+} \to P_{d'}^{+}). \tag{36}$$

Hence the direct transformation is accomplished with a higher success probability than that for the corresponding sequential transformation.

The discussion above demonstrates that PPT operations can improve the success probability of the single-copy distillation for some mixed states. One may perhaps expect that single-copy distillation becomes possible for all NPT mixed states when we consider PPT operations. This, however, is not the case. As shown in [43] (see also [44]), LOCC cannot distill any pure entangled state from a single copy of mixed states ρ on $\mathbb{C}^d\otimes\mathbb{C}^d$ if $\mathrm{rank}(\rho)\geq d^2-2$. For such high-rank mixed states, PPT operations cannot distill any pure entangled state either. The proof of this statement is given in Appendix D.

This highlights the fact that LOCC-state manipulation suffers certain restrictions that PPT operations cannot relax. Indeed, the convertibility of some mixed states (into pure entangled states) at the single-copy level and, therefore, the convertibility of mixed states under PPT operations remain much more involved than the convertibility of pure states.

IX. SUMMARY

In this paper we have considered the transformation of single copies of multiparticle entanglement under sets of operations that are larger than the class of local operations and classical communication. In particular, we considered probabilistic-state transformations under positive-partial-transpose-preserving maps (PPT maps). We demonstrated that transformations that are strictly impossible under LOCC can have a finite success probability under trace-preserving PPT maps. For specific examples the optimal success probabilities are determined. Surprisingly large values are obtained, for example, for the transformation from the GHZ to *W* state which under trace-preserving PPT maps has a success probability of more than 75% while it is strictly forbidden under LOCC. Furthermore, we completely clarified the convertibility of arbitrary multipartite pure states under PPT operations. As a remarkable result, we showed that all *N*-partite pure entangled states are interconvertible under PPT operations at the single-copy level, and therefore infinitely many different types of entanglement under LOCC are merged into only one type. In this way, a drastic simplification in the classification of pure-state entanglement occurs when the constrained set of operations is changed from LOCC to PPT operations. It should be emphasized that despite such drastic simplification in the single-copy settings, the theory of entanglement under PPT operations possesses the desirable properties that PPT operations alone cannot create pure-state entanglement and that the amount of bipartite pure-state entanglement is uniquely determined in asymptotic settings [45].

The above results can be regarded as an application of PPT-bound entanglement. In multipartite settings, however, another type of bound entanglement called unlockable bound entanglement exists. Motivated by this, we enlarged the class of PPT operations to consider the effects of unlockable bound entanglement. As a result we showed that all pure entangled states become interconvertible independent of the number of parties, and therefore a further drastic simplification in the classification of pure states occurs when LOCC is supported by unlockable bound entanglement.

Finally, we considered one aspect of mixed-state entanglement transformations: namely, the single-copy distillation by PPT operations. We demonstrated that PPT operations can distill a pure entangled state from a single copy of some mixed states with finite success probability, while the success probability under LOCC is strictly zero. However, we also proved that PPT operations cannot distill pure entangled states from mixed states with very high rank. Therefore, certain restrictions of entanglement manipulation of mixed states under LOCC persist under PPT maps, and the classification of mixed states under PPT operations in the single-copy settings is not as simple as that in the pure-state case.

It is important to further clarify how the structure of theory of entanglement is simplified under PPT operations especially in the mixed-state settings and in asymptotic settings, as this might enable a unified and systematic understanding of characteristics of quantum entanglement as a resource.

ACKNOWLEDGMENTS

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APPENDIX A: OPTIMALITY OF THE CONVERSION FROM THE GHZ TO *W* STATE

In this appendix, we prove the optimality of Eq. (16), the probability for the transformation from the GHZ to *W* state. To this end, we consider the dual problem of the primal problem, Eq. (11) [23]. The Lagrange function for the minimization problem in Eq. (11) is given by

$$\begin{aligned} L = & -\mathrm{tr}\{\Omega\rho_{\mathrm{GHZ}}\otimes\mathbb{1}\} - \sum_{i=A,B,C}(\mathrm{tr}\{\lambda_i^{\Gamma_i}\Omega\} + \mathrm{tr}\{\mu_i^{\Gamma_i}\omega\}) \\ & + \mathrm{tr}\{\lambda_e(\mathrm{tr}_{V'}(\Omega+\omega)-\mathbb{1})\} + \nu\mathrm{tr}\{\Omega\rho_{\mathrm{GHZ}}\otimes(\mathbb{1}-\rho_W)\} \\ & + \mathrm{tr}\{\lambda_p(\mathrm{tr}_{V'}\Omega-\mathbb{1})\} + \mathrm{tr}\{\lambda_{ep}(\mathrm{tr}_{V'}\omega-\mathbb{1})\}, \end{aligned}$$

where $\lambda_p,\lambda_{ep},\lambda_A,\lambda_B,\lambda_C,\mu_A,\mu_B,\mu_C \geqslant 0$. This Lagrange function has to be minimized over all $\Omega,\omega\geqslant 0$. This is feasible only if

$$0 \geqslant \sum_{i=A,B,C}\lambda_i^{\Gamma_i} + (\rho_{\mathrm{GHZ}}-\lambda_p-\lambda_e)\otimes\mathbb{1} - \nu\rho_{\mathrm{GHZ}}\otimes(\mathbb{1}-\rho_W),$$

$$0 \geqslant -(\lambda_e+\lambda_{ep})\otimes\mathbb{1} + \mu_A^{\Gamma_A} + \mu_B^{\Gamma_B} + \mu_C^{\Gamma_C},$$

in which case we obtain the dual function

$$g(\lambda_p,\lambda_{ep},\lambda_e,\nu) = -\mathrm{tr}\{\lambda_{ep}+\lambda_e+\lambda_p\}. \tag{A1}$$

Every feasible point of the dual problem provides an upper bound on the solution of the primal problem, Eq. (11). With the symmetries shown in Sec. IV, the Lagrange dual problem of primal problem, Eq. (11), is

$$\min \mathrm{tr}\{\lambda_{ep}+\lambda_e+\lambda_p\} \tag{A2}$$

under the constraints

$$(\rho_{\mathrm{GHZ}}-\lambda_p-\lambda_e)\otimes\mathbb{1} - \nu\rho_{\mathrm{GHZ}}\otimes(\mathbb{1}-\rho_W) + \sum_{i=A,B,C}\lambda_i^{\Gamma_{V_i}} \leqslant 0,$$

$$\lambda_A,\lambda_B,\lambda_C,\mu_A,\mu_B,\mu_C,\lambda_p,\lambda_{ep} \geqslant 0,$$

$$-(\lambda_e+\lambda_{ep})\otimes \mathbb{1}+\mu_A^{\Gamma_{V_A}}+\mu_B^{\Gamma_{V_B}}+\mu_C^{\Gamma_{V_C}}\leqslant 0.$$

To prove the optimality of Eq. (16), it suffices to provide a trial solution for the dual problem that matches the value Eq. (16). To this end, we chose $\nu=\frac{8}{3}$, $\lambda_{pe}=\lambda_p=0$, and $(\lambda_e)_{i,j}=0$ except for

$$(\lambda_e)_{i,i}=b_2,\quad (\lambda_e)_{1,8}=(\lambda_e)_{8,1}=-3b_2.$$

Furthermore,

$$\begin{aligned}(\mu_A)_{i,i}&=(\mu_A)_{i+40,i+40}=-(\mu_A)_{i,i+40}=-(\mu_A)_{i+40,i}=(\mu_B)_{i+8,i+8}\\&=(\mu_B)_{i+32,i+32}=-(\mu_B)_{i+8,i+32}=-(\mu_B)_{i+32,i+8}\\&=(\mu_C)_{i+16,i+16}=(\mu_C)_{i+24,i+24}=-(\mu_C)_{i+16,i+24}\\&=-(\mu_C)_{i+24,i+16}=b_2\end{aligned}$$

for $i=9,\ldots,16$. Finally, one chooses the matrices $\lambda_A^{\Gamma_{V_A}}$, $\lambda_B^{\Gamma_{V_B}}$ and $\lambda_C^{\Gamma_{V_C}}$. As $\lambda_B^{\Gamma_{V_B}}$ and $\lambda_C^{\Gamma_{V_C}}$ can be obtained from $\lambda_A^{\Gamma_{V_A}}$ by cyclic permutations, we only need to specify $\lambda_A^{\Gamma_{V_A}}$. For $i,j=1,\ldots,8$ we have

$$(\lambda_A^{\Gamma_{V_A}})_{i,j}=(\lambda_A^{\Gamma_{V_A}})_{56+i,56+j}=X_{i,j},$$

$$(\lambda_A^{\Gamma_{V_A}})_{i,56+j}=(\lambda_A^{\Gamma_{V_A}})_{56+i,j}=Y_{i,j},$$

$$(\lambda_A^{\Gamma_{V_A}})_{i+8,j+8}=(\lambda_A^{\Gamma_{V_A}})_{48+i,48+j}=\delta_{i,j},$$

where the nonzero elements of X and Y are given by

$$X_{1,1}=1,\quad X_{4,4}=X_{6,6}=X_{6,4}=X_{4,6}=25/16,$$

$$X_{2,3}=X_{2,5}=X_{3,2}=X_{5,2}=-5/4,$$

$$Y_{1,1}=Y_{4,4}=Y_{6,6}=-1/3,$$

$$Y_{2,2}=-Y_{7,7}=-1,$$

$$Y_{2,3}=Y_{2,5}=Y_{3,2}=Y_{3,3}=Y_{3,5}=-Y_{4,6}=-2/3,$$

$$Y_{5,2}=Y_{5,3}=-Y_{5,5}=-Y_{6,4}=-Y_{8,8}=-2/3,$$

$$Y_{6,7}=Y_{7,6}=7/80,$$

$$Y_{7,4}=Y_{4,7}=(-42+\sqrt{159559})/1200.$$

A direct calculation, ideally employing a software capable of symbolic manipulations, now shows that these values determine a feasible point of the dual problem. The dual function for the above choice yields the value $6b_2$—i.e., the same as for the primal problem which establishes the optimality of the solution for the primal problem.

APPENDIX B: FROM GHZ TO *W* EMPLOYING TRACE-NONPRESERVING PPT MAPS

In this appendix we determine the optimal success probability for the transformation of a GHZ state to a W state under trace-nonpreserving CP-PPT maps. This problem is equivalent to the maximization of

$$\mathrm{tr}\{\Psi(\rho_{\mathrm{GHZ}})\}=\mathrm{tr}\{\Omega\rho_{\mathrm{GHZ}}\otimes \mathbb{1}\} \tag{B1}$$

under the constraints

$$\mathrm{tr}\{\Omega\rho_{\mathrm{GHZ}}\otimes(\mathbb{1}-\rho_W)\}=0,$$

$$\Omega^{\Gamma_V}\geqslant 0,\quad \mathrm{tr}_{V'}\{\Omega(\Psi)\}\leqslant \mathbb{1},$$

$$(\Omega^{\Gamma_A\otimes\Gamma_A'})^{\Gamma_V}\geqslant 0,\quad (\Omega^{\Gamma_B\otimes\Gamma_B'})^{\Gamma_V}\geqslant 0,\quad (\Omega^{\Gamma_C\otimes\Gamma_C'})^{\Gamma_V}\geqslant 0. \tag{B2}$$

This problem possesses the same symmetries (a)–(g) presented in Sec. IV. Following the same arguments as in Sec. IV most matrix elements of Ω vanish. In the following we will present those nonvanishing matrix elements that are sufficient to reconstruct all the remaining nonzero elements of the trial solution from the symmetries of the problem. With $b_1=0.8/3=2b_2=4b_3$ we find

$$\Omega_{001000,001000}=2\Omega_{001111,001111}=b_1,$$

$$\Omega_{001001,001001}=\Omega_{001010,001010}=\Omega_{001100,001100}=b_2,$$

$$\Omega_{001001,001010}=\Omega_{001001,001100}=-\Omega_{001010,001100}=-b_2,$$

$$\Omega_{001011,001011}=\Omega_{001101,001101}=\Omega_{001110,001110}=b_3,$$

$$\Omega_{001011,001101}=-\Omega_{001011,001110}=-\Omega_{001101,001110}=b_3,$$

$$\Omega_{000000,000000}=8\Omega_{000111,000111}=b_1,$$

$$\Omega_{000001,000001}=\Omega_{000010,000010}=\Omega_{000100,000100}=b_2,$$

$$\Omega_{000001,000010}=\Omega_{000001,000100}=\Omega_{000010,000100}=b_2,$$

$$\Omega_{000011,000011}=\Omega_{000101,000101}=\Omega_{000110,000110}=b_3,$$

$$\Omega_{000011,000101}=\Omega_{000011,000110}=\Omega_{000101,000110}=b_3,$$

$$\Omega_{000000,111000}=8\Omega_{000111,111111}=-b_1,$$

$$\Omega_{000001,111001}=\Omega_{000010,111010}=\Omega_{000100,111100}=b_2,$$

$$\Omega_{000001,111010}=\Omega_{000001,111100}=\Omega_{000010,111100}=b_2,$$

$$\Omega_{000011,111011}=\Omega_{000101,111101}=\Omega_{000110,111110}=-b_3,$$

$$\Omega_{000011,111101}=\Omega_{000011,111110}=\Omega_{000101,111110}=-b_3.$$

Now an elementary but lengthy calculation shows that the chosen parameters define a feasible point of the problem and yield a success probability of $\mathrm{tr}\{\Omega\rho_{\mathrm{GHZ}}\otimes\mathbb{1}\}=0.8$.

To prove the optimality of this result we now consider the dual problem. The Lagrange function for the minimization problem in Eq. (B1) is given by

$$L(\Omega,\lambda_A,\lambda_B,\lambda_C,\lambda_p,\nu) = \mathrm{tr}\{\nu\rho_{\mathrm{GHZ}}\otimes(\mathbb{1}-\rho_W)\} + \mathrm{tr}\{\lambda_p\} - \mathrm{tr}\{\Omega(\rho_{\mathrm{GHZ}}-\lambda_p)\otimes\mathbb{1} + \lambda_A^{\Gamma_A} + \lambda_B^{\Gamma_B} + \lambda_C^{\Gamma_C}\},$$

where λ_p, $\lambda_A^{\Gamma_A}$, $\lambda_B^{\Gamma_B}$, $\lambda_C^{\Gamma_C}\geqslant 0$. The Lagrange function has to be minimized over all $\Omega\geqslant 0$. This is feasible only if

$$\rho_{\mathrm{GHZ}}\otimes\mathbb{1} + \lambda_A^{\Gamma_A} + \lambda_B^{\Gamma_B} + \lambda_C^{\Gamma_C} - \lambda_p\otimes\mathbb{1} - \nu\rho_{\mathrm{GHZ}}\otimes(\mathbb{1}-\rho_W) \leqslant 0, \tag{B3}$$

in which case we obtain the dual function

$$g(\lambda_A,\lambda_B,\lambda_C,\lambda_p,\nu) = -\mathrm{tr}\{\lambda_p\}. \tag{B4}$$

Maximizing this function under the constraints λ_A, λ_B, λ_C, $\lambda_p\geqslant 0$ and Eq. (B3) yields upper bounds on the success probabilities of the primal problem. The following trial solution yields $-\mathrm{tr}\{\lambda_p\}=-0.8$, satisfying all the constraints and matching the value of the primal optimum, thereby proving its optimality. For simplicity we only give the nonzero matrix elements

$$\lambda_{p1,1} = \lambda_{p8,8} = -\lambda_{p1,8} = -\lambda_{p8,1} = 0.4,$$

$$\lambda_{A1,1} = -\lambda_{A1,4} = -\lambda_{A1,6} = \lambda_{A4,4} = \lambda_{A6,6} = \lambda_{A8,8} = \lambda_{A4,6} = 4\lambda_{A5,5} = -2\lambda_{A5,8} = 0.8/3,$$

$$\lambda_{A57,57} = -\lambda_{A57,60} = -\lambda_{A57,62} = \lambda_{A60,60} = \lambda_{A62,62} = \lambda_{A64,64} = \lambda_{A60,62} = 4\lambda_{A61,61} = -2\lambda_{A61,64} = 0.8/3,$$

$$\nu = 1.8.$$

The elements of λ_B and λ_C are obtained from λ_A by cyclic permutation of the parties A, B, and C so that, for example, $\lambda_{A5,5}=\lambda_{B2,2}$. Direct calculation no shows that this trial solution is feasible for the dual problem and yields the value $g=-0.8$ which is identical to that obtained from the trial solution for the primal problem. This completes the proof of optimality.

APPENDIX C: FROM *W* TO GHZ EMPLOYING TRACE-NONPRESERVING PPT MAPS

The optimization of the success probability for the transformation from W to GHZ proceed along very similar lines as those given in the previous appendix. Mathematically the problem is formulated as

$$\mathrm{tr}\{\Psi(\rho_W)\} = \mathrm{tr}\{\Omega\rho_W\otimes\mathbb{1}\} \tag{C1}$$

under the constraints

$$\mathrm{tr}\{\Omega\rho_W\otimes(\mathbb{1}-\rho_{\mathrm{GHZ}})\} = 0,$$

$$\mathrm{tr}_{V'}\{\Omega\} \leqslant \mathbb{1}, \quad \Omega^{\Gamma_V}\geqslant 0,$$

$$(\Omega^{\Gamma_A\otimes\Gamma_A'})^{\Gamma_V}\geqslant 0, \quad (\Omega^{\Gamma_B\otimes\Gamma_B'})^{\Gamma_V}\geqslant 0, \quad (\Omega^{\Gamma_C\otimes\Gamma_C'})^{\Gamma_V}\geqslant 0.$$

Symmetries analogous to those presented in the previous sections hold. Following the arguments analogous to those in Sec. IV most matrix elements of Ω vanish. In the following we will present those nonvanishing matrix elements that are sufficient to reconstruct all the remaining nonzero elements of the trial solution from the symmetries of the problem. With $b_1=3b_2/4=3b_3=1/6$,

$$\Omega_{000001,000001} = \Omega_{111001,111001} = 11/90,$$

$$\Omega_{001001,001001} = 4\Omega_{010001,010001} = 4\Omega_{100001,100001} = b_2,$$

$$\Omega_{001001,010001} = \Omega_{001001,100001} = -2\Omega_{010001,100001} = -b_2/2,$$

$$\Omega_{011001,011001} = \Omega_{101001,101001} = 2\Omega_{110001,110001} = b_1,$$

$$\Omega_{011001,101001} = -2\Omega_{011001,110001} = -2\Omega_{101001,110001} = b_3,$$

$$\Omega_{000000,000000} = \Omega_{111000,111000} = b_2/2,$$

$$\Omega_{001000,001000} = \Omega_{010000,010000} = \Omega_{100000,100000} = b_3,$$

$$\Omega_{001000,010000} = \Omega_{001000,100000} = \Omega_{010000,100000} = b_3,$$

$$\Omega_{011000,011000} = \Omega_{101000,101000} = \Omega_{110000,110000} = b_3/2,$$

$$\Omega_{011000,101000} = \Omega_{011000,110000} = \Omega_{101000,110000} = b_3/2,$$

$$\Omega_{000000,000111} = -7\Omega_{111000,111111}/10 = -7/90,$$

$$\Omega_{001000,001111} = \Omega_{010000,010111} = \Omega_{100000,100111} = b_3,$$

$$\Omega_{001000,010111} = \Omega_{001000,100111} = \Omega_{010000,100111} = b_3,$$

$$\Omega_{011000,011111} = \Omega_{101000,101111} = \Omega_{110000,110111} = b_3/2,$$

$$\Omega_{011000,101111} = \Omega_{011000,110111} = \Omega_{101000,110111} = b_3/2.$$

With this trial solution we find $\mathrm{tr}\{\Omega\rho_W\otimes\rho_{\mathrm{GHZ}}\}=\frac{1}{3}$.

To prove the optimality of this result we now consider the dual problem. The Lagrange function for the minimization problem in Eq. (C1) is given by

$$L(\Omega,\lambda_A,\lambda_B,\lambda_C,\lambda_p,\nu) = -\mathrm{tr}\lambda_p - \mathrm{tr}\sum_{i=A,B,C}\lambda_i^{\Gamma_i\otimes\Gamma_i'} - \mathrm{tr}\{\Omega((\rho_W - \lambda_p)\otimes\mathbb{1} - \nu\rho_W\otimes(\mathbb{1}-\rho_{\mathrm{GHZ}})\ \}, \tag{C2}$$

where λ_p, λ_A, λ_B, $\lambda_C\geqslant 0$. This Lagrange function has to be minimized over all $\Omega\geqslant 0$ which is feasible only if

$$(\rho_W-\lambda_p)\otimes\mathbb{1} - \nu\rho_W\otimes(\mathbb{1}-\rho_{\mathrm{GHZ}}) + \sum_{i=A,B,C}\lambda_i^{\Gamma_i\otimes\Gamma_i'} \leqslant 0, \tag{C3}$$

in which case we obtain the dual function

$$g(\lambda_A,\lambda_B,\lambda_C,\lambda_p,\nu) = -\mathrm{tr}\{\lambda_p\}. \tag{C4}$$

Now we need to maximize this function under the constraints λ_A, λ_B, λ_C, $\lambda_p\geqslant 0$ and Eq. (C3). Each trial solution gives an upper bound on the success probability of the primal problem. It turns out that we can approach the $-\mathrm{tr}\{\lambda_p\}=-\frac{1}{3}$ arbitrarily closely.

We begin by determining all nonzero matrix elements of λ_p in terms of $\lambda_{p2,2}$ so that

$$\lambda_{p3,3} = \lambda_{p5,5} = \lambda_{p2,2},$$

$$\lambda_{p2,3} = \lambda_{p2,5} = \lambda_{p3,5} = -\lambda_{p2,2}/2.$$

Furthermore, we completely determine the matrices λ_A, λ_B, and λ_C. To this end we give all the nonzero values of λ_A as the other matrices are uniquely determined through cyclic permutations from λ_A:

$$\lambda_{A18,23} = -\frac{1}{5} = \lambda_{A34,39},$$

$$\lambda_{A18,39} = -\frac{3}{10} = \lambda_{A34,23}$$

and

$$\lambda_{A17,17} = \lambda_{A33,33} = -\lambda_{A17,33} = -\lambda_{A33,17} = 0.1/9,$$

$$\lambda_{A18,18} = \lambda_{A34,34} = 0.3; \lambda_{A18,34} = \lambda_{A34,18} = 0.2,$$

$$4\lambda_{A19,19} = \lambda_{A35,35} = 2\lambda_{A19,35} = 2\lambda_{A35,19} = 0.4/9,$$

$$\lambda_{A20,20} = 4\lambda_{A36,36} = 2\lambda_{A20,36} = 2\lambda_{A36,20} = 0.4/9,$$

$$\lambda_{A21,21} = 4\lambda_{A37,37} = 2\lambda_{A21,37} = 2\lambda_{A37,21} = 0.4/9,$$

$$4\lambda_{A32,32} = \lambda_{A38,38} = 2\lambda_{A32,38} = 2\lambda_{A38,32} = 0.4/9,$$

$$\lambda_{A23,23} = \lambda_{A39,39} = 0.3; \lambda_{A23,39} = \lambda_{A39,23} = 0.2,$$

$$\lambda_{A24,24} = \lambda_{A40,40} = -\lambda_{A24,40} = -\lambda_{A40,24} = 0.1/9.$$

The elements of λ_B and λ_C are obtained from λ_A by cyclic permutation of the parties A, B, and C so that, for example, $\lambda_{A5,5}=\lambda_{B2,2}$. A direct calculation shows that the constraints λ_A, λ_B, λ_C, $\lambda_p \geq 0$ are satisfied with these choices. Now we need to verify whether the constraint

$$(\rho_W - \lambda_p) \otimes \mathbb{1} - \nu\rho_W \otimes (\mathbb{1} - \rho_{GHZ}) + \sum_{i=A,B,C} \lambda_i^{\Gamma_i \otimes \Gamma'_i} \leq 0 \tag{C5}$$

can be verified as well. Note that we still have the free parameters $\lambda_{p2,2}$ and ν. A lengthy computation (preferably employing Mathematica) shows that the left-hand side of the constraint has six distinct nonzero eigenvalues: namely,

$$\mu_1 = 2 - \nu, \quad \mu_2 = \frac{1}{90}(13 - 135\lambda_{p2,2}),$$

$$\mu_3 = \frac{1}{30}(-2 - 45\lambda_{p2,2}), \quad \mu_4 = \frac{1}{30}(4 - 45\lambda_{p2,2}),$$

$$\mu_\pm = \frac{1}{60}[47 - 30\nu - 45\lambda_{p2,2} \pm \sqrt{1569 - 2220\nu + 3330\lambda_{p2,2} + (45\lambda_{p2,2} - 30\nu)^2}].$$

Clearly, for $\nu \geq 2$ and $\lambda_{p2,2} \geq \frac{13}{135}$ the first four eigenvalues are nonpositive. Now we can verify by direct inspection that for any choice of $\lambda_{p2,2} > 1/9$ there is a choice of $\nu > 2$ such that the two eigenvalues $\mu_\pm$ are negative so that also the constraint, Eq. (C5), is satisfied. Therefore, for any value of $-\mathrm{tr}\{\lambda_p\} < -\frac{1}{3}$ we can satisfy the constraints. This shows that the primal problem which achieves a success probability $p = 1/3$ is optimal.

APPENDIX D: SINGLE-COPY DISTILLATION FROM HIGH-RANK MIXED STATES

In this appendix, we prove that PPT perations cannot distill any pure entangled states from a single copy of ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$ when $\mathrm{rank}(\rho) \geq d^2 - 2$. To this end, it suffices to show that the success probability $p(\rho \to P_{d'}^+)$ under PPT operations (Ψ) in the trace-nonpreserving scheme is strictly zero, where $\rho \in \mathcal{H}(V)$ and $P_{d'}^+ \in \mathcal{H}(V')$. Since both $\rho \otimes P_{d'}^+$ and $\rho \otimes (\mathbb{1} - P_{d'}^+)$ are invariant under the local unitary transformation of $\mathbb{1} \otimes \mathbb{1} \otimes U \otimes U^*$, it suffices to consider Ω invariant under these local operations: i.e.,

$$\Omega = A \otimes P_{d'}^+ + B \otimes \frac{\mathbb{1} - P_{d'}^+}{d'^2 - 1}, \tag{D1}$$

with A and B being matrices on $\mathcal{H}(V)$. The success probability is then

$$p(\rho \to P_{d'}^+) = \mathrm{tr}\{\Omega\rho \otimes P_{d'}^+\} = \mathrm{tr}\{A\rho\}, \tag{D2}$$

and constraints for Ω are

$$\mathrm{tr}\{\Omega\rho \otimes (\mathbb{1} - P_{d'}^+)\} = \mathrm{tr}\{B\rho\} = 0,$$

$$A \geq 0, \quad B \geq 0, \quad \mathbb{1} \geq A + B,$$

$$\frac{1}{d' - 1} B^{\Gamma_A} \geq A^{\Gamma_A} \geq -\frac{1}{d' + 1} B^{\Gamma_A}.$$

Since $B \geq 0$ and $\mathrm{tr}\, B\rho = 0$, the support space of B must be contained in the kernel space of ρ, and hence $\mathrm{rank}(B) \leq 2$ when $\mathrm{rank}(\rho) \geq d^2 - 2$. On the other hand, $B^{\Gamma_A} \geq 0$ must hold from

$$\frac{1}{d' - 1} B^{\Gamma_A} \geq -\frac{1}{d' + 1} B^{\Gamma_A},$$

and B must be a separable state (leaving out normalization) since $\mathrm{rank}(B) \leq d$ [47]. Therefore, by using appropriate local basis, B can be written as

$$B = y|11\rangle\langle 11| + z|ef\rangle\langle ef|, \tag{D3}$$

where y and z are non-negative values and

$$|ef\rangle = (\cos u|1\rangle + \sin u|2\rangle) \otimes (\cos v|1\rangle + \sin v|2\rangle) \quad \text{(D4)}$$

is a product vector. In this choice of local basis, $B^{\Gamma_A}=B$. Let P be the projector on the support space of B^{Γ_A} and $Q\equiv I-P$. The condition of

$$\frac{1}{d'-1}B^{\Gamma_A} \geqslant A^{\Gamma_A} \geqslant -\frac{1}{d'+1}B^{\Gamma_A}$$

implies that $\pm QA^{\Gamma_A}Q\geqslant 0$, and hence $QA^{\Gamma_A}Q=0$ must hold. Furthermore, $A^{\Gamma_A}+[1/(d'+1)]B^{\Gamma_A}$ must be a positive operator, for which $Q(A^{\Gamma_A}+[1/(d'+1)]B^{\Gamma_A})Q=0$ also holds. Therefore, support space of $A^{\Gamma_A}+[1/(d'+1)]B^{\Gamma_A}$ must be P, and hence the support space of A^{Γ_A} must be contained in the support space of B^{Γ_A}. As a result, $\text{rank}(A^{\Gamma_A})\leqslant\text{rank}(B^{\Gamma_A})\leqslant 2$. Furthermore, A^{Γ_A} must be written in the form of

$$A^{\Gamma_A} = r|11\rangle\langle 11| + s|11\rangle\langle ef| + s^*|ef\rangle\langle 11| + t|ef\rangle\langle ef|,$$

and A is then given by

$$A = r|11\rangle\langle 11| + s|e1\rangle\langle 1f| + s^*|1f\rangle\langle e1| + t|ef\rangle\langle ef|.$$

Therefore, A must be essentially a two-qubit state (leaving out normalization) since $A\geqslant 0$ must hold. If the two-qubit state A is entangled, $\text{rank}(A^{\Gamma_A})$ must be 4 [48,49], which contradicts that $\text{rank}(A^{\Gamma_A})\leqslant 2$. Therefore, A and A^{Γ_A} must be written in a separable form.

In the case where $\sin u \sin v \neq 0$, the support space of A^{Γ_A}, which is spanned by $|11\rangle$ and $|ef\rangle$, contains only two product vectors ($|11\rangle$ and $|ef\rangle$ itself) [50], and hence A^{Γ_A} must be written as

$$A^{\Gamma_A} = r|11\rangle\langle 11| + t|ef\rangle\langle ef|, \quad \text{(D5)}$$

and $A=A^{\Gamma_A}$. As a result, the support space of A is contained in the support space of B, and hence $p(\rho\to P^+_{d'})=\text{tr}\,A\rho=0$ as $\text{tr}\,B\rho=0$. In the case where $\sin u \sin v=0$, $|e\rangle=|1\rangle$ or $|f\rangle=|1\rangle$ holds. As a result, A is spanned by $\{|11\rangle,|1f\rangle\}$ (or $\{|11\rangle,|e1\rangle\}$) which is a kernel of ρ, and hence $p(\rho\to P^+_{d'})=\text{tr}A\rho=0$.

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