

## Wigner-Yanase skew information as tests for quantum entanglement

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A Bell-type inequality is proposed in terms of Wigner-Yanase skew information, which is quadratic and involves only one local spin observable at each site. This inequality presents a hierarchic classification of all states of multipartite quantum systems from separable to fully entangled states, which is more powerful than the one presented by quadratic Bell inequalities from two-entangled to fully entangled states. In particular, it is proved that the inequality provides an exact test to distinguish entangled from nonentangled pure states of two qubits. Our inequality sheds considerable light on relationships between quantum entanglement and information theory.

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As is well known, entangled states have become a key concept in quantum mechanics nowadays. On the other hand, from a practical point of view entangled states have found numerous applications in quantum information [1]. A natural question is then how to detect entangled states. The first efficient tool used to detect entangled states is the Bell inequality [2], which was originally designed to rule out various kinds of local hidden variable theories. Precisely, the Bell inequality indicates that certain statistical correlations predicted by quantum mechanics for measurements on two-qubit ensembles cannot be understood within a realistic picture based on the local realism of Einstein, Podolsky, and Rosen [3]. However, Gisin's theorem [4] asserts that all entangled two-qubit pure states violate the Clauser-Horne-Shimony-Holt (CHSH) inequality [5] for some choice of spin observables. There are various Bell-type inequalities used to detect entangled states [6]. In particular, quadratic Bell-type inequalities were derived by Uffink [7] as tests for multipartite entanglement and used to classify all states of  $n$  qubits into  $n-1$  entanglement classes from two-entangled to  $n$ -entangled (fully entangled) states [8]. Although there are some other measures of entangled states [1], the problem of how to classify and quantify entanglement in general is still far from being completely understood today.

In this article, a Bell-type inequality is proposed in terms of Wigner-Yanase skew information, which is shown to be useful for detecting entangled states. In the study of measurement theory [9], Wigner and Yanase [10] introduced the following quantity from an information-theoretic point of view:

$$I(\rho, A) = -\frac{1}{2} \text{tr}[\rho^{1/2}, A]^2, \quad (1)$$

where  $\rho$  is a state and  $A$  an observable of a quantum system. This quantity  $I(\rho, A)$  is called Wigner-Yanase skew information; it is the amount of information on the values of observables not commuting with (being skew to)  $A$ . They proved that this quantity satisfies the requirements that an expression

for the information content should satisfy, among which two basic ones are the following.

(a) When two different ensembles are united, the skew information decreases. Phrased mathematically,  $I(\rho, A)$  is a convex function in  $\rho$ —if  $\rho_1$  and  $\rho_2$  are both density operators, then

$$I(\alpha\rho_1 + \beta\rho_2, A) \leq \alpha I(\rho_1, A) + \beta I(\rho_2, A), \quad (2)$$

where  $\alpha + \beta = 1$  for  $\alpha, \beta \geq 0$ .

(b) The skew information content of the union of two independent systems is the sum of the skew information content of the individuals; namely, let  $\rho_1$  and  $\rho_2$  be two states of the first and second systems, respectively, and  $A_1$  and  $A_2$  the corresponding local observables; then

$$I(\rho_1 \otimes \rho_2, A_1 \otimes 1 + 1 \otimes A_2) = I(\rho_1, A_1) + I(\rho_2, A_2). \quad (3)$$

The equality (3) can be extended to several independent systems.

There are some earlier works concerning the Wigner-Yanase skew information. These constitute the celebrated Wigner-Araki-Yanase theorem, which puts a limitation on the measurement of observables in the presence of a conserved quantity [11], and was further investigated by many authors [12]. Recently, Luo and Zhang [13] reveal that the Wigner-Yanase skew information can be regarded as the quantum analog of the Fisher information in the theory of statistical estimation, and used to establish some results concerning the characterizations of uncertainty relations and measures of entangled states. Here, we present a Bell-type inequality of Wigner-Yanase skew information for separable states of  $n$  particles.

First of all, note that the Wigner-Yanase skew information can be rewritten as

$$I(\rho, A) = \text{tr}\rho A^2 - \text{tr}\rho^{1/2} A \rho^{1/2} A. \quad (4)$$

In particular, if  $\rho = |\psi\rangle\langle\psi|$  is a pure state, then

$$I(|\psi\rangle, A) = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2. \quad (5)$$

From Eqs. (2) and (5) one concludes that

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$$I(\rho, A) \leq 1 \tag{6}$$

for each spin observable  $A(A^2=1)$  in all states  $\rho$ .

Now, consider a system of  $n$  particles and assume that a local spin observable is measured on each particle, and hence there correspond  $n$  local spin observables  $A_1, \dots, A_n$ . Since separable states are written as a convex sum over the states  $\rho_1 \otimes \dots \otimes \rho_n$ , and by Eqs. (3) and (6), one has

$$I(\rho_1 \otimes \dots \otimes \rho_n, A_1 + \dots + A_n) = \sum_{j=1}^n I(\rho_j, A_j) \leq n,$$

it is concluded from Eq. (2) that

$$I(\rho, A_1 + \dots + A_n) \leq n \tag{7}$$

for all separable states  $\rho$  of the  $n$  particles. (We write  $A_1$ , etc., as shorthand for  $A_1 \otimes 1 \otimes \dots \otimes 1$ .) The inequality (7) is a Bell-type inequality in terms of Wigner-Yanase skew information, which can be violated by entangled states. We will show that this inequality not only distinguishes, for systems of  $n$  particles, between fully entangled states and states in which at most  $n-1$  particles are entangled, similar to quadratic Bell inequalities [7], but also presents an  $n$ -hierarchical classification of all states of  $n$ -partite quantum systems from one-entangled (separable) to  $n$ -entangled (fully entangled) states, which is more powerful than the one presented by quadratic Bell inequalities giving  $n-1$  entanglement classes from two-entangled to fully entangled states [8]. Roughly speaking, the more entangled the larger the violation value of Eq. (7), which means it is easier to measure an observable not commuting with local spin observables on entangled states, thus corresponding to larger skew information. This directly uncovers the information-theoretic meaning of entangled states.

We would like to mention that, contrary to the usual Bell-type inequalities possessing two local spin observables or more at each site [14], Eq. (7) involves only one local spin observable at each site. On the other hand, our inequality, similarly to quadratic Bell-type inequalities [7], is not linear but quadratic. However, we will show that Gisin's theorem holds true for Eq. (7), that is, it provides an exact test to distinguish entangled from nonentangled pure states of two qubits. This shows that our inequality is stronger than quadratic Bell-type inequalities, which provide tests for entanglement for systems of only three particles or more, and not for  $n=2$ .

Since  $I(\rho, A)$  is a convex function in  $\rho$ , the maximal violation of Eq. (7) for entangled states will occur in pure states. Note that for local spin observables  $A_1, \dots, A_n$ ,

$$\langle \psi | (A_1 + \dots + A_n)^2 | \psi \rangle = \sum_{j,k} \langle \psi | A_j A_k | \psi \rangle \leq n^2$$

( $A_j A_k = 1 \otimes \dots \otimes A_j \otimes \dots \otimes A_k \otimes \dots \otimes 1$  in short) we conclude from Eq. (5) that

$$I(\rho, A_1 + \dots + A_n) \leq n^2 \tag{8}$$

for all (entangled) states  $\rho$  of the  $n$  particles. Equality in Eq. (8) can be attained even for systems of  $n$  spin- $\frac{1}{2}$  particles by the Greenberger-Horne-Zeilinger (GHZ) state [15]

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0 \dots 0\rangle + |1 \dots 1\rangle).$$

Indeed, choosing  $A_j = \sigma_z^j$  ( $j=1, \dots, n$ ) one has that

$$\sigma_z^j \sigma_z^k |\text{GHZ}\rangle = |\text{GHZ}\rangle, \quad j, k = 1, \dots, n,$$

$$\langle \text{GHZ} | \sigma_z^1 + \dots + \sigma_z^n | \text{GHZ} \rangle = 0.$$

Hence, by Eq. (5) one has

$$I(|\text{GHZ}\rangle, \sigma_z^1 + \dots + \sigma_z^n) = n^2. \tag{9}$$

In particular, for the familiar case of two spin- $\frac{1}{2}$  particles, the maximal violation of Eq. (7) occurs for Bell states with the violation value 4, which is larger than the maximal violation value  $2\sqrt{2}$  of the CHSH inequality [16]. Moreover, for a pure state  $|\psi\rangle$  of two qubits we can write by the Schmidt decomposition theorem

$$|\psi\rangle = p|\phi_1\rangle|\chi_1\rangle + q|\phi_2\rangle|\chi_2\rangle, \tag{10}$$

where  $p$  and  $q$  are two nonnegative numbers satisfying  $p^2 + q^2 = 1$  and  $\{|\phi_1\rangle, |\phi_2\rangle\}$  and  $\{|\chi_1\rangle, |\chi_2\rangle\}$  are orthonormal bases in the two-dimensional Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$  of the two particles, respectively. Now, choose a representation for the first particle such that  $|\phi_1\rangle = |0\rangle_1$ ,  $|\phi_2\rangle = |1\rangle_1$  in the  $z_1$  direction, and similarly,  $|\chi_1\rangle = |0\rangle_2$ ,  $|\chi_2\rangle = |1\rangle_2$  in the  $z_2$  direction. Then, a simple computation yields that

$$I(|\psi\rangle, \sigma_x^1 + \sigma_x^2) = 2 + 4pq. \tag{11}$$

This shows that  $|\psi\rangle$  is entangled ( $pq > 0$ ) if and only if it violates Eq. (7), that is, Gisin's theorem holds true for the inequality (7) of two qubits. Therefore, Wigner-Yanase skew information provides an exact test to distinguish entangled from nonentangled pure states of two qubits.

In the following, we show how to detect multipartite entanglement using Wigner-Yanase skew information. Consider a system of  $n$  particles. Recall that a state of  $n$  particles is fully entangled [17], if it cannot be written as  $\rho_{(k)} \otimes \rho_{(n-k)}$  or mixtures of these states, where  $\rho_{(k)}$  ( $1 \leq k \leq n-1$ ) is a (entangled or not) state of  $k$  particles among the  $n$  particles. For example, states of three particles that are not of the form  $\rho_1 \otimes \rho_{23}$ ,  $\rho_2 \otimes \rho_{13}$ ,  $\rho_{12} \otimes \rho_3$ , or mixtures of these states are fully entangled states. In particular,  $|\text{GHZ}\rangle$  are fully entangled states of multiqubit systems. Generally, for  $1 \leq k \leq n$ , a state of the  $n$  particles is at most  $k$ -entangled, if it is of the form  $\rho_{(k_1)} \otimes \dots \otimes \rho_{(k_m)}$  or mixtures of these states, where  $1 \leq k_j \leq k$ ,  $\sum_j k_j = n$ , and  $\rho_{(k_j)}$  is a (entangled or not) state of  $k_j$  particles among the  $n$  particles. We denote by  $\text{ES}_k$  all states that are at most  $k$ -entangled. Then,

$$\text{ES}_1 \subset \text{ES}_2 \subset \dots \subset \text{ES}_n. \tag{12}$$

Clearly,  $\text{ES}_1$  is the set of all separable states,  $\text{ES}_n$  are all states of the  $n$  particles, and  $\text{ES}_n - \text{ES}_{n-1}$  are all fully entangled states. Thus, for  $2 \leq k \leq n$ ,  $k$ -entangled states are all states in  $\text{ES}_k - \text{ES}_{k-1}$ , one-entangled states are separable states, and  $n$ -entangled states are just fully entangled states.

Given a state of the form  $\rho_{(k_1)} \otimes \dots \otimes \rho_{(k_m)}$ , for local spin observables  $A_1, \dots, A_n$ , by Eqs. (3) and (8) we have

$$I(\rho_{(k_1)} \otimes \cdots \otimes \rho_{(k_m)}, A_1 + \cdots + A_n) \\ = I(\rho_{(k_1)}, A_{(k_1)}) + \cdots + I(\rho_{(k_m)}, A_{(k_m)}) \leq k_1^2 + \cdots + k_m^2,$$

where  $A_{(k_j)}$  is the sum of the corresponding local spin observables among  $A_1, \dots, A_n$  associated with  $\rho_{(k_j)}$ . Then, by the convexity (a), we obtain

$$I(\rho, A_1 + \cdots + A_n) \leq \left[ \frac{n}{k} \right] k^2 + \left( n - \left[ \frac{n}{k} \right] k \right)^2 \quad (13)$$

for all states  $\rho \in \text{ES}_k$ , where  $[x]$  denotes the largest integer less than or equal to  $x$ . Here, the right-hand side of Eq. (13),

$$E_k = \left[ \frac{n}{k} \right] k^2 + \left( n - \left[ \frac{n}{k} \right] k \right)^2,$$

can be obtained by induction [18]. Clearly,

$$E_1 = n < E_2 < \cdots < E_n = n^2. \quad (14)$$

Also,  $E_2 = 2n$  for  $n$  even or  $2n-1$  for  $n$  odd and  $E_{n-1} = (n-1)^2 + 1$ . For example, in the cases of  $n=2, 3, 4$ , and  $5$ , we have (1) for  $n=2$ ,  $E_1=2$  and  $E_2=4$ ; (2) for  $n=3$ ,  $E_1=3$ ,  $E_2=5$ , and  $E_3=9$ ; (3) for  $n=4$ ,  $E_1=4$ ,  $E_2=8$ ,  $E_3=10$ ,  $E_4=16$ ; and (4) for  $n=5$ ,  $E_1=5$ ,  $E_2=9$ ,  $E_3=13$ ,  $E_4=17$ ,  $E_5=25$ . Moreover,  $E_k$  can be attained in Eq. (13) by  $k$ -entangled states. Indeed, choosing  $\rho = \rho_{(k_1)} \otimes \cdots \otimes \rho_{(k_m)}$  such that  $m = [n/k] + 1$ ,  $k_j = k$  for  $j=1, \dots, [n/k]$ ,  $\rho_{(k_j)}$  is the GHZ state of the corresponding  $k$  particles, and  $\rho_{(k_m)}$  is the GHZ state of the remaining particles, by the additivity (b) and Eq. (9) we have

$$I(\rho, \sigma_z^1 + \cdots + \sigma_z^n) = E_k.$$

Note that the inequality (13) is only a necessary condition that states are at most  $k$ -entangled. Later, we will present a concrete example showing that there are fully entangled states satisfying Eq. (7).

Now, we turn to the problem of how to classify states of  $n$  particles into  $n$  entanglement classes by using Wigner-Yanase skew information. To this end, we define

$$I(\rho) = \sup I(\rho, A_1 + \cdots + A_n), \quad (15)$$

where the supremum is taken over all local spin observables  $A_1, \dots, A_n$ . According to the physical meaning of Wigner-Yanase skew information, the quantity  $I(\rho)$  is the most amount of information on the values of observables not commuting with (being skew to) local spin observables we can obtain on the state  $\rho$ . Hence,  $I(\rho)$  can be regarded as the nonlocal skew information of  $\rho$ . By Eq. (13) we find that the larger the nonlocal skew information, the more entangled or the less separable is the state. This means that we can obtain more information of some nonlocal observables on entangled states than on separable ones. In this sense, we may regard entangled states as an information-theoretic concept. In particular, by using the nonlocal skew information of states of multipartite systems we obtain a classification of them as follows.

*Classification theorem.* The states of  $n$  particles can be classified into  $n$  entanglement classes. For  $k=1, \dots, n$ , the states in the  $k$ th entanglement class satisfy

$$I(\rho) \leq E_k. \quad (16)$$

Therefore, the larger the nonlocal skew information, the more entangled or the less separable is the state in the sense that a larger maximal violation of the inequality (13) is attainable for this class of states.

As shown in Eq. (13),  $k$ -entangled states are all in the  $k$ th entanglement class. In particular, if

$$I(\rho) > (n-1)^2 + 1,$$

then  $\rho$  is fully entangled. For example, Eq. (9) shows that  $|\text{GHZ}\rangle$  is a fully entangled state. Hence, by using Wigner-Yanase skew information, we give a hierarchic classification of multipartite entangled states.

For each separable pure state  $|\psi_1\rangle \cdots |\psi_n\rangle$ , it is easy to check that  $I(|\psi_1\rangle \cdots |\psi_n\rangle) = n$ . For mixed states, we illustrate the following states of  $n$  qubits [17]:

$$\rho_\lambda = \lambda |\text{GHZ}\rangle \langle \text{GHZ}| + \frac{1-\lambda}{2^n} I, \quad (17)$$

where  $0 \leq \lambda \leq 1$  and  $I$  is the identity on the  $n$  qubits. Since

$$\rho_\lambda^{1/2} = f(\lambda) |\text{GHZ}\rangle \langle \text{GHZ}| + \sqrt{\frac{1-\lambda}{2^n}} I,$$

where  $f(\lambda) = \sqrt{\lambda + (1-\lambda)/2^n} - \sqrt{(1-\lambda)/2^n}$ , we have

$$I(\rho_\lambda, A) = \left( \lambda - 2f(\lambda) \sqrt{\frac{1-\lambda}{2^n}} \right) \langle \text{GHZ} | A^2 | \text{GHZ} \rangle \\ - f(\lambda)^2 \langle \text{GHZ} | A | \text{GHZ} \rangle^2 \leq \left( \lambda - 2f(\lambda) \sqrt{\frac{1-\lambda}{2^n}} \right) n^2,$$

for all  $A = A_1 + \cdots + A_n$  of local spin observables. Taking  $A_j = \sigma_z^j$  for  $j=1, \dots, n$ , we have

$$I(\rho_\lambda) = \left[ \lambda - 2 \sqrt{\frac{1-\lambda}{2^n}} \left( \sqrt{\lambda + \frac{1-\lambda}{2^n}} - \sqrt{\frac{1-\lambda}{2^n}} \right) \right] n^2. \quad (18)$$

It is shown in [17] that  $\rho_\lambda$  is separable if and only if  $0 \leq \lambda \leq 1/(1+2^{n-1})$ , and fully entangled otherwise. The lower bound of values  $\lambda$  for which our criterion can be used to detect the (full) entanglement of  $\rho_\lambda$  is

$$\lambda_n = \frac{1}{n} \left( 1 - \frac{1}{2^{n-1}} \right) + \sqrt{\frac{1}{2^{n-2}n} \left( 1 - \frac{1}{n} + \frac{1}{2^n} \right)}, \quad (19)$$

that is,  $I(\rho_\lambda) > n$  provided  $\lambda > \lambda_n$ . We have (i)  $\lambda_2 = (1 + \sqrt{5})/4$  and  $\lambda_3 = (3 + \sqrt{17})/12$ ; (ii)  $\lambda_4 = 7/16 < 1/2$  and  $\lambda_5 = (15 + \sqrt{129})/80 < 1/3$ ; (iii)  $\lambda_6 = (31 + \sqrt{321})/192 < 1/3$  and  $\lambda_7 = (63 + \sqrt{769})/448 < 1/4$ . Moreover, we have the following estimates:

$$\frac{1}{n-1} < \lambda_n < \frac{1}{n-2}, \quad 8 \leq n \leq 12; \quad (20)$$

TABLE I. Each row indicates the maximum values under three different classes possibly attained by MK, BI<sub>2</sub>, and WY, respectively. Note that the difference values of WY two-entangled from separable and fully entangled from two-entangled states are 2 and 4, respectively, which are both larger than the associated ones  $\sqrt{2}-1$  and  $2-\sqrt{2}$  of MK.

	ES <sub>1</sub>	ES <sub>2</sub>	ES <sub>3</sub>
MK	1	$\sqrt{2}$	2
BI <sub>2</sub>	8	8	16
WY	3	5	9

$$\frac{1}{n} < \lambda_n < \frac{1}{n-1}, \quad n > 12. \quad (21)$$

On the other hand, it is easy to check that the lower bound of values  $\lambda$  is  $1/2^{n/2-1}$  for the (full) entanglement of  $\rho_\lambda$  obtained by the quadratic Bell inequality [7], as well as  $1/2^{(n-1)/2}$  by the standard Mermin-Klyshko inequality [6,14]. Although the lower bounds of values derived by the quadratic Bell and Mermin-Klyshko inequalities, respectively, are both better than the one of our inequality, our criterion, which requires only one measurement setting per site, is more appealing from an experimental point of view.

Setting  $\lambda_0 = 1/(1+2^{n-1})$ , one has

$$I(\rho_{\lambda_0}) = \frac{2-\sqrt{3}}{1+2^{n-1}} n^2.$$

Since  $I(\rho_{\lambda_0}) \rightarrow 0$  as  $n \rightarrow \infty$  and by Eq. (18),  $I(\rho_\lambda)$  is continuous in  $\lambda$ , we find that there are some fully entangled states  $\rho_\lambda$  with  $\lambda > 1/(1+2^{n-1})$  such that  $I(\rho_\lambda) \rightarrow 0$  as  $n \rightarrow \infty$ . This means that the nonlocal skew information of some fully entangled states can be arbitrarily small and hence smaller than that of separable states. Since  $I(\rho)$  can be regarded as a measure of nonlocality of quantum states, therefore, from the information-theoretic viewpoint there is a gap between nonlocality and quantum entanglement.

As follows, we apply our theorem to the classification of three-qubit states. As stated before, there are three types of three-qubit states: one-entangled (totally separable) states denoted by ES<sub>1</sub>, two-entangled states ES<sub>2</sub>, and three-entangled (fully entangled) states ES<sub>3</sub>. The results for the Mermin-Klyshko (MK) inequalities [6], the quadratic Bell inequality (BI<sub>2</sub>) [7], and our inequality of Wigner-Yanase (WY) information are summarized in Table I.

We now turn to our entanglement criterion to see how good it is to test the entanglement of generalized GHZ states of three qubits

$$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle,$$

where  $\alpha, \beta > 0$ , and  $\alpha^2 + \beta^2 = 1$ . Write  $A_j = a_{j1}\sigma_x + a_{j2}\sigma_y + a_{j3}\sigma_z$ , where  $\vec{a}_j = (a_{j1}, a_{j2}, a_{j3}) \in \mathbb{R}^3$  are all unit vectors. Note that

$$I(|\psi\rangle, A_1 + A_2 + A_3) = 3 + 2(a_{13}a_{23} + a_{13}a_{33} + a_{23}a_{33}) - (\alpha^2 - \beta^2)^2(a_{13} + a_{23} + a_{33})^2.$$

Hence, we have

$$I(|\psi\rangle) = 3 + 3[2 - 3(\alpha^2 - \beta^2)^2]. \quad (22)$$

It is concluded that whenever

$$\alpha, \beta > \sqrt{\frac{1}{2} \left( 1 - \sqrt{\frac{2}{3}} \right)}, \quad (23)$$

our criterion can detect the entanglement of generalized GHZ states, which is more efficient than the MK inequality with the critical value  $\sqrt{(1-1/\sqrt{2})/2}$ .

In summary, a Bell-type inequality is proposed in terms of Wigner-Yanase skew information, which presents an  $n$ -hierarchical classification of all states of  $n$ -partite quantum systems from separable to fully entangled states. The inequality is not linear but quadratic; however, contrary to quadratic Bell inequalities [7], it involves only one local spin observable at each site and provides an exact test to distinguish entangled from nonentangled pure states of two qubits. Our  $n$ -hierarchical classification for  $n$ -partite entangled states by using Wigner-Yanase skew information is more powerful than the classification in terms of quadratic Bell inequalities [8], which only classifies all states of  $n$  particles into  $n-1$  entanglement classes from two-entangled to  $n$ -entangled states and cannot be used to distinguish two-entangled from separable states. We have defined the quantity  $I(\rho)$  of multipartite states, which is the most amount of information on the values of observables not commuting with (being skew to) local spin observables we can obtain on the state  $\rho$ .  $I(\rho)$  is the nonlocal skew information of entangled states, as we can obtain more information of some nonlocal observables on entangled states than on separable ones.

Nowadays, it has been recognized that most physical processes in nature can be formulated in terms of processing of information, and information may be central to understanding quantum theory [19]. The notion of the Wigner-Yanase skew information quantifies the amount of information on the values of observables being skew to other ones [11,12], and has been used to establish some results concerning the characterizations of uncertainty relations and measures of entangled states [13]. Our results furthermore shed considerable light on relationships between quantum entanglement and information theory in terms of Bell-type inequalities of the Wigner-Yanase skew information. We expect that, similarly to the usual Bell inequalities [6], the Bell-type inequality of the Wigner-Yanase skew information and its various ramifications will play an important role in quantum information [1].

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- [18] Consider the function  $f = x_1^2 + \dots + x_n^2$  on the convex set  $C_k = \{(x_1, \dots, x_n) : 0 \leq x_j \leq k, \sum_j x_j = n\}$ . Since  $f$  is strictly convex, it attains the maximum at an extreme point of the convex set  $C_k$ . Then, without loss of generality, assuming  $x_1 \geq x_2 \geq \dots \geq x_n$ , we have that  $x_1 = k$ . Inductively, we obtain Eq. (13).
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