

Universal leakage elimination

Mark S. Byrd,¹ Daniel A. Lidar,² Lian-Ao Wu,² and Paolo Zanardi³

¹*Physics Department, Southern Illinois University, Carbondale, Illinois 62901-4401, USA*

²*Chemical Physics Theory Group, Chemistry Department, and Center for Quantum Information and Quantum Control, University of Toronto, 80 St. George St., Toronto, Ontario, Canada M5S 3H6*

³*Institute for Scientific Interchange (ISI), Villa Gualino, Viale Settimio Severo 65, I-10133 Torino, Italy*

(Received 8 September 2004; published 3 May 2005)

“Leakage” errors are particularly serious errors which couple states within a code subspace to states outside of that subspace, thus destroying the error protection benefit afforded by an encoded state. We generalize an earlier method for producing leakage elimination decoupling operations and examine the effects of the leakage eliminating operations on decoherence-free or noiseless subsystems which encode one logical, or protected qubit into three or four qubits. We find that by eliminating a large class of leakage errors, under some circumstances, we can create the conditions for a decoherence-free evolution. In other cases we identify a combined decoherence-free and quantum error correcting code which could eliminate errors in solid-state qubits with anisotropic exchange interaction Hamiltonians and enable universal quantum computing with only these interactions.

DOI: 10.1103/PhysRevA.71.052301

PACS number(s): 03.67.Pp, 03.65.Yz

I. INTRODUCTION

Noise protection for quantum information processing is an important facet of quantum control and the design of quantum devices. In quantum computing, coherent control of a quantum system is required in order to take advantage of quantum computing speedups. A great deal of work has been done, and is still ongoing, to try to achieve multiparticle control for quantum information processing. In order to implement noiseless control of quantum computing systems, several methods of error prevention have been introduced. Quantum error correcting codes (QECCs) [1–6], detect and correct errors; decoherence-free or noiseless subsystems (DFSs) [7–12] avoid noises in quantum systems; and dynamical decoupling controls (DD) [13–27] reduce the errors by averaging or symmetrizing them away. Since none of these has seen the ultimate success of preventing errors in a prototypical quantum computing device, combinations of more than one of these methods have been explored [17,28–37]. One particularly promising example is the combination of dynamical decoupling controls with decoherence-free subsystems [17,19,21,29–32,34,36–38]. This combination can offer several advantages; it can (1) reduce the number of physical qubits required to encode one logical qubit, (2) enable universal control in systems which cannot be completely controlled otherwise, (3) avoid noises, and (4) reduce noises even if they are not eliminated or avoided. Such combinations are very likely to be necessary for the near-term and longer term goals associated with reliable quantum information processing. (For a recent review on error prevention, see [39]).

For those quantum computing proposals which use quantum dots for storing information and the Heisenberg exchange interaction for performing gating interactions, a DFS encoding is promising since it enables universal computing without the need for single qubit gates [11,40–49]. Architecturally, as well as for speed, single qubit gates can be difficult to implement for unencoded (i.e., physical) qubits in

solid-state systems. It has been shown that, for several different types of interactions, and for several different encodings, decoherence-free subspaces provide the ability to perform universal quantum computing without requiring single qubit gates [11,40–49]. It is therefore important to know the conditions for a DFS to exist or, as discussed in this paper, what methods might be used to create a DFS. For those circumstances which do not allow for a DFS implementation alone in order to eliminate all noise in the system, what is (are) the best method(s) for error protection? This clearly depends on the physical system and an analysis of the types of occurring errors will be necessary in order to take advantage of every possible technique for noise suppression, correction and/or avoidance.

In this paper we discuss the elimination of leakage errors. Leakage errors destroy a subspace encoding by coupling states within the encoded subspace of the system Hilbert space with the states which are outside of the code subspace. These are particularly serious errors since they eliminate the usefulness of a subspace encoding. Moreover, they cannot be handled by standard QECC methods under the assumption of a set of operations restricted to act on a subspace [50]. We will first review the bang-bang limit of the method of dynamical decoupling controls, and the algebraic decomposition of the operators on the Hilbert space in Sec. II, in order to make the article more self-contained. We then review, in Sec. III, the definition and construction of leakage elimination operators (LEOs) using canonical gates, and then generalize the construction to gates which are not canonical. In Sec. IV we provide an explicit decomposition of the algebra of operators for the 3-qubit DFS and use it to classify all errors on this DFS. In this section, we also analyze the errors which commonly arise in solid-state implementations of quantum computing proposals, and determine a strategy for eliminating all errors, in addition to leakage errors, which arise from anisotropic exchange errors and cause decoherence. In the following section, Sec. V, we construct a physically realizable LEO which is not made of canonical gates,

using the construction in Sec. III. We then analyze errors in the 4-qubit DFS which arise in solid-state implementations of QC. We summarize our results in the Conclusion.

II. UNIVERSAL LEAKAGE ELIMINATION

In this section we briefly review dynamical decoupling controls, symmetrization and the results of [37]. For a more detailed discussion of dynamical decoupling controls, the reader is referred to [51], for the group-theoretical underpinnings see [29], for an empirical approach see [19,31], for a geometrical approach see [23] or [39,51] for recent reviews. We then provide a general formula for producing a leakage elimination operator (LEO).

A. Dynamical decoupling controls

Dynamical decoupling controls are control pulses which are used to average away noises in a quantum system. When hard, fast pulses are used, these are commonly referred to as bang-bang controls. Here we review decoupling controls in the bang-bang limit.

Consider a general Hamiltonian of the form

$$H = H_S + H_B + H_{SB},$$

where H_S acts only on the system, H_B acts only on the bath, and $H_I = H_{SB} = \sum_{\gamma} S_{\gamma} \otimes B_{\gamma}$ couples the system to the bath. Let us now implement control operations, U_i , periodically with the system undergoing free unitary evolution (by H) for a time Δt between control operations. If we assume that the free evolution is negligible during the time the control is “on” (this assumes “strong” control Hamiltonians are available), then we obtain an effective unitary evolution for the combined system bath given by:

$$U_{\text{eff}} \approx \prod_{i=0}^{N-1} U_i \exp[-iH\Delta t] U_i^{\dagger}. \quad (1)$$

In the limit of large N and small Δt , such that $N\Delta t = t < \infty$, we can approximate U_{eff} , by a first-order Magnus expansion and use an effective Hamiltonian to describe this evolution,

$$H_{\text{eff}} \approx \frac{1}{N} \sum_{i=0}^{N-1} U_i H U_i^{\dagger}. \quad (2)$$

In ideal circumstances (as $N \rightarrow \infty$), we can eliminate H_{SB} completely so as to decouple the system and bath. However, *in this paper we combine decoupling operations with an encoding, therefore we only require that H_{SB} be modified. This drastically reduces the demands on a physical system and on our controls.*

One should note that “strong” and “fast” are relative to system-bath interactions, notions which have been thoroughly quantified in [52]. In addition, we need not require strong pulses in some cases [51,53] and in other cases, the fast requirement can be relatively easily satisfied [54,55]. In this paper, we will consider dynamical decoupling controls which assume hard, fast pulses, but we note that appropriate controls may be available which can serve as decoupling

pulses without the necessity of the “bang-bang” limit.

As a final remark on decoupling operations, we state the following theorem [31,32] which follows from [29].

Theorem II.1. Dynamical decoupling with respect to the set of logical operations of an encoded qubit can be used to completely decouple the dynamics of the encoded subspace from the bath.

This theorem is important for the following reasons. First, the number of pulses required to eliminate noise on physical qubits can be quite taxing on physical resources. If we restrict ourselves to logical operations, we can reduce the number of required pulses dramatically. Second, in many cases, if we use logical operations to remove errors, we are restricting to those operations which are available in experiment. Often an encoding is chosen for its universality considerations. In other words, many codes are chosen so that universal quantum computing can be performed on a subspace even if it cannot be performed on the entire Hilbert space. Those operations which achieve universal control can also be used for complete decoupling.

B. Algebraic decomposition

In order to discuss the effects of the dynamical decoupling operations on encoded qubits, we will briefly review the decomposition of the algebra [10,56] which can describe all error prevention schemes [57].

The interaction algebra, denoted \mathcal{A} , is generated by the set $\{H_S, S_{\gamma}\}$. This algebra is, in general, reducible, and closed under Hermitian conjugation (meaning $\mathcal{A}^{\dagger} = \mathcal{A}$). This algebra is a subalgebra of the full set of endomorphisms of the total Hilbert space \mathcal{H} , $\mathcal{E}(\mathcal{H})$ which are linear operators on \mathcal{H} . The irreducible components of this algebra are described by the decomposition

$$\mathcal{A} = \bigoplus_{J \in \mathcal{J}} I_{n_J} \otimes M(d_J, \mathbb{C}), \quad (3)$$

where the J , a shorthand for all relevant representation indices, label the irreducible representations and the $M(d_J, \mathbb{C})$ are $d_J \times d_J$ complex matrices. This representation is a direct sum decomposition (block diagonal) with n_J labelling the states of the system in the corresponding Hilbert space decomposition

$$\mathcal{H} \cong \bigoplus_{J \in \mathcal{J}} \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}. \quad (4)$$

Each factor \mathbb{C}^{n_J} corresponds to a noiseless subsystem. The commutant of \mathcal{A} , denoted \mathcal{A}' , is in $\mathcal{E}(\mathcal{H})$ and is defined as

$$\mathcal{A}' = \{X \in \mathcal{E}(\mathcal{H}) | [X, \mathcal{A}] = 0\}. \quad (5)$$

The existence of a decoherence-free, or noiseless, subsystem is equivalent to

$$\mathcal{A}' = \bigoplus_{J \in \mathcal{J}} M(n_J, \mathbb{C}) \otimes I_{d_J} \neq \mathbb{C}I. \quad (6)$$

This implies a nontrivial group of symmetries of the commutant. The unitary part of \mathcal{A}' , $U(\mathcal{A}')$, is the set of unitary symmetries of the error algebra \mathcal{A} .

Note that DFSs, QECCs and topological codes can all be described by this same algebraic decomposition [57]. We can

therefore generically discuss encoded qubits in the context of quantum error prevention without regard to the type of encoding although we will primarily direct our attention to DFSs.

III. LEAKAGE AND LEAKAGE ELIMINATION OPERATORS (LEOs)

Qubits can be either a subspace of a larger system Hilbert space or an encoded subspace of a larger Hilbert space. The idealized, isolated two-level system never occurs in nature, when all energy scales are taken into account. We therefore seek to eliminate, or reduce the difference between an idealized qubit and the approximate two-level systems available in experiments. Whether these are two physical states in a larger Hilbert space, or a state which is encoded into some set of states through a nontrivial transformation, we will discuss a generalized notion of a code and encoded subspace. This encoded subspace, or codespace, will be denoted \mathcal{C} . The orthogonal complement of the codespace, also a subspace of the system Hilbert space, will be denoted \mathcal{C}^\perp . Our objective will be to eliminate the coupling between \mathcal{C} , and \mathcal{C}^\perp . We refer to such errors as leakage errors. However, unlike considerations of leakage introduced during logical, or gating operations [5,47,58–62], we will consider residual errors and errors introduced by system-bath couplings as in [37].

Let us first consider the simple case of a physical or encoded qubit. In this case we would like to eliminate the leakage from a two-level system within an N -level system. We will choose an ordered basis for the N -level system Hilbert space $\{|j\rangle\}_{j=0}^{N-1}$ such that the code \mathcal{C} will be spanned by the first two levels. The algebra of operations on the system Hilbert space can then be classified in the following way,

$$E = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad E^\perp = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, \quad L = \begin{pmatrix} 0 & D \\ F & 0 \end{pmatrix}, \quad (7)$$

where B and C are 2×2 and $(N-2) \times (N-2)$ blocks respectively, and D, F are $2 \times (N-2)$ and $(N-2) \times 2$ blocks respectively. Operators of the type E represent logical operations, i.e., they act entirely within the code subspace. E^\perp operations act only on \mathcal{C}^\perp and thus have no effect on the qubit subspace [63]. Finally, L represents the leakage operators. This decomposition, for physical or encoded qubits is quite general and the operators B act only on the logical qubit labels.

Generally, modifying the Hamiltonian (more specifically, the system-bath interaction Hamiltonian) through the use of dynamical decoupling controls can change the conditions under which quantum information is protected against errors. Clearly, this is accomplished by modifying the set $\{H_S, S_\gamma\}$, which modifies the interaction algebra \mathcal{A} and thus the irreducible components of \mathcal{A} . For example, we can create DFSs where none were possible without such a modification by inducing a symmetry [17,19,21,29–32,34,36–38]. We could also eliminate correlated errors, thus changing the requirements for a QECC. As described in Sec. II A, dynamical decoupling controls can be seen as a projection onto a subspace of the space of operators acting on the system Hilbert space.

More to the point of this paper, we can eliminate some of the components of the interaction algebra by eliminating some of the S_γ (or some combination thereof). In our case, we seek to eliminate leakage. We therefore give a general classification of elements of the algebra according to their effect on the code space. The elements of the interaction algebra $\{A_i\} = \mathcal{A}$, will be classified as $\{E_{ij}\} \subset \mathcal{A}$, which act on the code, $\{E_i^\perp\} \subset \mathcal{A}$ which affect the orthogonal subspace, and leakage errors $\{L_{ij}\} \subset \mathcal{A}$ which couple the elements of the code \mathcal{C} with states in the orthogonal subspace \mathcal{C}^\perp .

From the general decomposition of the algebra in Sec. II B, we should note the following facts. One can always decompose the algebra into a set which acts on the system Hilbert space and a set which acts on the Hilbert space of the environment. These sets can be taken to be Hermitian operators with complex coefficients. The set of operators may then always be expressed as some linear combination of tensor products of the two sets with complex coefficients. This implies that, although the symmetries appropriate for a DFS may not always exist, we may always use a DFS-compatible basis. This will be important in Secs. IV and V where we will discuss a basis for which a code can be constructed which will protect against errors, even when no symmetry in the operator algebra exists initially.

A. Canonical LEOs

In the simplest case of a “parity-kick” bang-bang control [13,15,18,56], the decoupling sequence produces the effective evolution:

$$H_{eff} \approx \frac{1}{2} \sum_{i=0}^1 U_i H U_i^\dagger, \quad (8)$$

i.e., there is only one nontrivial decoupling pulse ($U_0 \equiv I$) [64]. We will restrict our attention to parity-kick pulses due to time constraints which restricts the number of pulses that can be applied in many physical systems.

Abstractly, we can state the consequences of the parity-kick pulse sequence as follows. Given any subspace $\mathcal{C} \subset \mathcal{H}$, there is a canonically associated \mathbb{Z}_2 group (the cyclic group of order two). This group is generated by the operator $R_L := \exp(i\pi\Pi_{\mathcal{C}})$, where $\Pi_{\mathcal{C}}$ is the projector onto the code space \mathcal{C} . In the language of \mathbb{Z}_2 graded spaces, this operator is a parity operator, i.e., $R_L^2 = I$, inducing a \mathbb{Z}_2 grading of the state space. This means that \mathcal{H} splits as a direct sum, $\mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$, of two orthogonal subspaces: the odd (even) sector $\mathcal{H}^{(0)} = \mathcal{C}$ ($\mathcal{H}^{(1)}$). This grading can be lifted to the operator algebra over \mathcal{H} turning this (Lie) algebra into a super- or \mathbb{Z}_2 -graded Lie algebra. Operators commuting (anticommuting) with R_L are referred to as even (odd). Let $X \in \mathcal{E}(\mathcal{H})$; the even sector of the algebra is given by $\{X | [R_L, X] = 0\}$ (i.e., $R_L X R_L^\dagger = X$) and the odd sector of the algebra is given by $\{X | \{R_L, X\} = 0\}$ (i.e., $R_L X R_L^\dagger = -X$).

As an example, let us suppose that all S_γ are in the odd sector of the algebra. According to the pulse sequence, Eq. (8), any $H_{SB} = S_\gamma \otimes B_\gamma$, odd, in the system-bath Hamiltonian will be removed after a complete set of operations. Therefore when all S_γ are odd, complete decoupling is achievable using only R_L .

Now consider a *leakage-elimination operator* (LEO) as in [37]

$$R_L = e^{i\phi} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad (9)$$

where the blocks have the same dimensions as in Eq. (7) and $\exp(i\phi)$ is an overall phase factor. This operator anticommutes with the leakage operators $\{R_L, L\}=0$, while $[R_L, E]=[R_L, E^\perp]=0$. Clearly such a sequence exactly produces the grading of the algebra described above. R_L is a LEO since it follows that the following (parity-kick) sequence eliminates the leakage errors:

$$\lim_{n \rightarrow \infty} (e^{-iHt/n} R_L^\dagger e^{-iHt/n} R_L)^n = e^{-iH_E t} e^{-iH^\perp t}, \quad (10)$$

where H_E (H^\perp) corresponds to part of the error algebra which affects only \mathcal{C} (\mathcal{C}^\perp). To physically implement this, in practice one takes $n=1$ and makes the time t very small compared to the bath correlation time as discussed in Sec. II A. Equation (10) then holds to order t^2 , and implies that one intersperses periods of free evolution for time t with R_L, R_L^\dagger applied as strong pulses. The term $e^{-iH^\perp t}$ in Eq. (10) has no effect on the qubit subspace. The term $e^{-iH_E t}$ may result in logical errors, which will have to be treated by other methods, e.g., concatenation with a QECC [28,65,66], or additional BB pulses [16,23,31]. Therefore, in order to eliminate *leakage*, we seek a LEO for a given encoding, which is obtainable from a controllable system Hamiltonian H_S acting for a time τ , i.e., $R_L = \exp(-iH_S \tau)$.

In [37], several examples were given of physical systems which, formally, have logical operations which are also naturally projective (i.e., they act as projections onto the code subspace). Such operations were termed canonical. As mentioned above, in some situations, the physically available (and controllable) interactions do not include operations which are also projections. We will provide one important example in Sec. V, the four-qubit DFS in a solid state system which uses Heisenberg exchange interactions for gating operations. In this case it is highly desirable to have a more general method for producing the appropriate LEO which does not include a projection. Next we will provide a generalized LEO which circumvents the need for canonical gating operations, before discussing the LEOs for the three- and four-qubit DFSs.

B. Generalized LEO

Generally, when a canonical logical operation is experimentally available, we can construct a LEO using the methods from [37] where the projector Π_C (defined there) is redundant. In fact, an operator of the following form serves as a LEO,

$$R_L = \exp(-i\pi\sigma_L), \quad (11)$$

where σ_L is any operation such that $\sigma_L = \sigma_L^\dagger$, $\sigma_L^2 = I$ on the code space, and $\sigma_L|\psi\rangle = 0$ for any $|\psi\rangle \in \mathcal{C}^\perp$. The primary example is when σ_L is a canonical logical operation. Such is the case for the 3-qubit DFS which uses Heisenberg ex-

change for logical gating operations, as will be shown later (see also [37]).

A more general characterization of a LEO is the following. Let the Hamiltonian for a LEO be given by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad (12)$$

where H_1 acts on the code subspace and H_2 on the orthogonal subspace. If H_1 is diagonal with even (odd) integers as the diagonal elements and H_2 is diagonal with odd (even) integers as the diagonal elements, then one may write the LEO as

$$R_L = U \exp(-i\pi H) U^\dagger, \quad (13)$$

where $U = U_1 \oplus U_2$ is a direct sum (block diagonal). *In this case H is not projective since it has nonzero eigenvalues when acting on the subspace orthogonal to the code.* The effective LEO, however, is unchanged, i.e., the form Eq. (9) is obtained, which again produces a \mathbb{Z}_2 grading of the algebra and thus eliminates leakage errors as desired. Such is the case for the four-qubit DFS example in Sec. V.

C. Leakage elimination to/from a subspace

Given the form of the operators that cause leakage, Eq. (7) and the form of the leakage elimination operator Eq. (9), we in fact have the choice to eliminate leakage between \mathcal{C} and \mathcal{C}^\perp either by acting on \mathcal{C} , or by action on \mathcal{C}^\perp .

The advantage of the first, is that, in principle, we may use theorem II.1 [31,32], or the methods of [19], to eliminate all errors on the encoded state space, even the logical errors. This requires (in many cases) acting on the codespace with logical operations. If, however, we do not have the experimental capabilities to implement the operations quickly enough for the given bath, or if the operations are imperfect, such operations may cause more errors in the system [52]. However, we may choose to apply the alternative of operating on \mathcal{C}^\perp . If the states are properly confined to the code space, then \mathcal{C}^\perp states should not be occupied, and decoupling pulses will have no effect. If the \mathcal{C}^\perp states are becoming populated, then the decoupling pulses applied to \mathcal{C}^\perp will eliminate leakage. As long as the decoupling pulses are properly constrained to act purely on \mathcal{C}^\perp then this results in increased tolerance to other pulse imperfections, in the sense that states in \mathcal{C} are unaffected. As we will see, in the case of the three- and four-qubit DFSs, a large class of errors are leakage type errors. Therefore decoupling with respect to \mathcal{C}^\perp could serve as an effective error suppression method in those cases where direct action on the codespace is inadvisable.

IV. LEOs AND THE 3-QUBIT DFS

The three qubit DFS encodes one logical qubit using a subsystem of three physical qubits. It is the most efficient way in which to protect a single logical qubit from collective errors of any type (bit-flip, phase-flip, or both) [10]. It has been shown that the exchange interaction is sufficient for implementing a universal set of gating operations on this system while preserving the DFS [11,40]. In this section we

will consider the effects of a LEO on this DFS and examine ways in which to protect information which is encoded in this subsystem.

A. 3-qubit DFS

We will represent the three-qubit DFS encoded qubit in the following way, where $|0\rangle=|1/2\rangle$, $|1\rangle=|-1/2\rangle$ are the two states of a single spin-1/2 (note that this convention is opposite to that of Ref. [11]):

$$\left(\begin{array}{c} (|010\rangle - |100\rangle)/\sqrt{2} \\ (|011\rangle - |101\rangle)/\sqrt{2} \\ (2|001\rangle - |010\rangle - |100\rangle)/\sqrt{6} \\ (-2|110\rangle + |011\rangle + |101\rangle)/\sqrt{6} \\ |000\rangle \\ (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3} \\ (|011\rangle + |101\rangle + |110\rangle)/\sqrt{3} \\ |111\rangle \end{array} \right) \left. \begin{array}{l} \right\} |0_L\rangle \\ \left. \begin{array}{l} \right\} |1_L\rangle \\ \left. \begin{array}{l} \right\} \mathcal{C}^\perp \end{array} \right. \end{array} \quad (14)$$

This notation means that $|0_L\rangle = \alpha_0(|010\rangle - |100\rangle)/\sqrt{2} + \beta_0(|011\rangle - |101\rangle)/\sqrt{2}$ (arbitrary superposition), and likewise $|1_L\rangle = \alpha_1(2|001\rangle - |010\rangle - |100\rangle)/\sqrt{6} + \beta_1(-2|110\rangle + |011\rangle + |101\rangle)/\sqrt{6}$. The logical zero ($|0_L\rangle$) and logical one ($|1_L\rangle$) comprise the code subspace \mathcal{C} . These states belong to the two $J=1/2$ irreducible representations (irreps) of $SU(2)$. The coefficients are then the Wigner-Clebsch-Gordan coefficients [67] and the last 4 states comprise a $J=3/2$ representation of $SU(2)$. The two $J=1/2$ irreps can be distinguished by a degeneracy label $\lambda=0,1$. Thus a basis state in the eight-dimensional Hilbert space is fully identified by the three quantum numbers $|J, \lambda, \mu\rangle$, where μ is the z component of the total spin J . In this notation we can write

$$|0_L\rangle = \alpha_0|1/2, 0, 1/2\rangle + \beta_0|1/2, 0, -1/2\rangle$$

and

$$|1_L\rangle = \alpha_1|1/2, 1, 1/2\rangle + \beta_1|1/2, 1, -1/2\rangle$$

B. Gating operations for the 3-qubit DFS and LEOs

Physical gates were given in Refs. [11,40,42] and shown to be compatible with the DFS. The gates derived in Ref. [11] are generated by the Heisenberg exchange interaction between pairs of physical qubits:

$$E_{ij} = \frac{1}{2}(I + \vec{\sigma}_i \cdot \vec{\sigma}_j), \quad (15)$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. As written E_{ij} is the exchange operation between qubits i and j , i.e., $E_{ij}|\phi\rangle_i|\psi\rangle_j = |\psi\rangle_i|\phi\rangle_j$. The logical “X” operation is given by [11]

$$\bar{X} = \frac{1}{\sqrt{3}}(E_{23} - E_{13}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I_2, \quad (16)$$

where we have labeled the rows and columns by the basis elements $\{|J=1/2, \lambda=0\rangle, |J=1/2, \lambda=1\rangle, |J=3/2\rangle\}$, and I_2 is the 2×2 identity matrix which accounts for the fact that the action of E_{ij} is independent of the μ label (z component of J) of the basis states. The logical “Z” operation is given by [68]

$$\bar{Z} = \frac{1}{3}(E_{13} + E_{23} - 2E_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I_2, \quad (17)$$

and \bar{Y} can be obtained from these two by commutation.

Recall that any operator can be decomposed in terms of a linear combination of traceless, Hermitian matrices (plus the identity) with complex coefficients. The exponentiation of the set of traceless, Hermitian operators with real coefficients will give the set of unitary transformations on the Hilbert space. Thus a Hermitian basis can be used to decompose the error operators and Hamiltonians on the set of quantum states.

Here we will give a complete set of 64 operators on the space of three qubits (63 traceless and the identity) and identify the logical operators, collective operators, and leakage operators. This enables the identification of various types of noise which can occur on the DFS and their effect on the code. The decomposition of the errors in terms of basis elements and the identification of the types of errors (leakage, collective, logical) on the code, will be useful for identifying the type of error correction procedure which should be used to correct errors affecting the code.

We note that it is also possible to eliminate all noises which are not collective, thus producing the conditions for a DFS [19,21,29,34]. Our primary concern will be leakage errors and a leakage elimination operator (LEO). For this purpose, we note that, given the form of Eqs. (16) and (17), *the logical operations for the 3-qubit DFS formed from physically available exchange interactions are canonical, and thus provide an experimentally available LEO.*

C. The DFS basis

As we saw above, collective errors on the DFS set of states act to mix the two states within the $|0_L\rangle$ subspace (and simultaneously those within the $|1_L\rangle$ subspace), but do not mix the two subspaces with each other. Thus we use the labels 0_L and 1_L to identify the two-state subsystem used for storing quantum information. This set of states is related to the standard set (computational basis set, $|000\rangle, \dots, |111\rangle$) through a transformation we will denote U_{dfs} , and which can be read off directly from Eq. (14):

$$U_{dfs}^{(3)} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

Circuits for implementing logic gates on the 3-qubit DFS were given in Refs. [11,40,42]. Here we take a different approach. Rather than examining the DFS in the physical basis, as in Refs. [11,40,42], we examine the DFS in the DFS basis where the code and the effects of logical operators and leakage errors are more transparent. This DFS basis will be denoted with a tilde. This distinguishes a set of operators acting on the code from the physical gating operators. The two sets of operators are related by the DFS transformation $U_{dfs}OU_{dfs}^\dagger = \tilde{O}$. In the DFS-basis operators take the form

$$\tilde{O} = \begin{pmatrix} \tilde{O}_{0,0}^{1/2,1/2} & \tilde{O}_{0,1}^{1/2,1/2} & \tilde{O}_{0,0}^{1/2,3/2} \\ \tilde{O}_{1,0}^{1/2,1/2} & \tilde{O}_{1,1}^{1/2,1/2} & \tilde{O}_{1,0}^{1/2,3/2} \\ \tilde{O}_{0,0}^{3/2,1/2} & \tilde{O}_{0,1}^{3/2,1/2} & \tilde{O}_{0,0}^{3/2,3/2} \end{pmatrix}, \quad (19)$$

where $\tilde{O}_{\lambda_1, \lambda_2}^{J_1, J_2}$ is a $(2J_1+1) \times (2J_2+1)$ -dimensional block coupling $|J_1, \lambda_1\rangle$ states with $|J_2, \lambda_2\rangle$ states. In the DFS basis, states are represented, using the notation of Sec. IV A, as vectors of the form $(\alpha_0, \beta_0, \alpha_1, \beta_1, \gamma_{-3/2}, \gamma_{-1/2}, \gamma_{1/2}, \gamma_{3/2})^t$, where the α, β coefficients belong to the $J=1/2$ subsystems, and where the γ coefficients belong to the $J=3/2$ subspace.

The advantage of the DFS basis is this: In the DFS-basis every operator can be decomposed into a tensor product of the form $\tilde{O}_\mu \otimes \tilde{O}_\lambda \otimes \tilde{O}_J$, where each operator acts on the corresponding quantum number in a state $|J, \lambda, \mu\rangle$.

D. Decomposition of the algebra

In [37], leakage errors between \mathcal{C} and \mathcal{C}^\perp were treated. Here we will carry this analysis further and investigate the types of errors which may arise according to the algebraic decomposition and their effect on the code. Note that the matrices in Eq. (7) are 8×8 matrices. Out of these, only D and F represent leakage processes, so that there are a total of 32 independent such errors. In the DFS-basis these leakage errors between the $J=1/2$ and $J=3/2$ subspaces have a simple representation. In terms of Eq. (19) they appear as

$$\tilde{O}_{\text{leak}} = \begin{pmatrix} & & \tilde{O}_{0,0}^{1/2,3/2} \\ & & \tilde{O}_{1,0}^{1/2,3/2} \\ \tilde{O}_{0,0}^{3/2,1/2} & \tilde{O}_{0,1}^{3/2,1/2} & \end{pmatrix}. \quad (20)$$

The nonzero, off-diagonal blocks are $(2 \times [2(1/2)+1]) \times ([2(3/2)+1]) = 4 \times 4$ matrices, while \tilde{O}_{leak} is 8×8 . It is then clear that we can construct an operator basis for the leakage errors using the DFS basis as follows:

$$\tilde{X} \otimes \tilde{O}_\lambda \otimes \tilde{O}_J, \text{ or } \tilde{Y} \otimes \tilde{O}_\lambda \otimes \tilde{O}_J, \quad (21)$$

where $\tilde{O}_\lambda, \tilde{O}_J \in \{\tilde{I}, \tilde{X}, \tilde{Y}, \tilde{Z}\}$, i.e., each $\tilde{O}_i, i=\lambda, \mu$ is a Pauli matrix, or the identity matrix, in the DFS basis. The role of \tilde{X} and \tilde{Y} (which act on the J factor) is to put the 4×4 matrix $\tilde{O}_\lambda \otimes \tilde{O}_J$ on the off-diagonal, as in Eq. (20).

Similarly, a logical operator takes the form

$$\tilde{O}_{\text{logic}} = \begin{pmatrix} \tilde{O}_{0,0}^{1/2,1/2} & \tilde{O}_{0,1}^{1/2,1/2} \\ \tilde{O}_{1,0}^{1/2,1/2} & \tilde{O}_{1,1}^{1/2,1/2} \end{pmatrix}, \quad (22)$$

(a nonzero 4×4 block) and, e.g., the logical σ_x appears as

$$\tilde{\sigma}_x = \begin{pmatrix} & I \\ I & \end{pmatrix}. \quad (23)$$

Thus the logical basis elements for the 3-qubit DFS code are represented simply by

$$\tilde{\sigma}_x = \frac{1}{2}(\tilde{I} + \tilde{Z}) \otimes \tilde{X} \otimes \tilde{I},$$

$$\tilde{\sigma}_y = \frac{1}{2}(\tilde{I} + \tilde{Z}) \otimes \tilde{Y} \otimes \tilde{I},$$

$$\tilde{\sigma}_z = \frac{1}{2}(\tilde{I} + \tilde{Z}) \otimes \tilde{Z} \otimes \tilde{I}. \quad (24)$$

The factor $(\tilde{I} + \tilde{Z})/2$ in these tensor products acts as a projection onto the code space. Therefore these operations act as ordinary Pauli matrices on \mathcal{C} , or the $0_L - 1_L$ block of the code and are canonical operators in our sense (note that they also preserve the μ factors of $|0_L\rangle, |1_L\rangle$). As we have seen in the previous section, these can be implemented using Heisenberg exchange interactions between qubits. Thus a *canonical LEO is experimentally available in systems which use Heisenberg exchange operations for gating and are encoded in the 3-qubit DFS*.

The DFS logical states $|\Psi_L\rangle = a|0_L\rangle + b|1_L\rangle$ are, by construction, invariant under collective errors. The (unnormalized) generators of collective errors are $S_\alpha = \sum \sigma_i^\alpha$, $\alpha=x, y, z$. To express these operators in the DFS basis, we transform by U_{dfs} :

$$S_X = (\tilde{I} + \tilde{Z})\tilde{I}\tilde{X} + \sqrt{3}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{X} + (\tilde{I} - \tilde{Z})\tilde{X}\tilde{X} + (\tilde{I} - \tilde{Z})\tilde{Y}\tilde{Y}, \quad (25)$$

$$S_Y = (\tilde{I} + \tilde{Z})\tilde{I}\tilde{Y} + \sqrt{3}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{Y} + (\tilde{I} - \tilde{Z})\tilde{X}\tilde{Y} + (\tilde{I} - \tilde{Z})\tilde{Y}\tilde{X}, \quad (26)$$

$$S_Z = (\tilde{I} + \tilde{Z})\tilde{I}\tilde{Z} + (\tilde{I} - \tilde{Z})\tilde{I}\tilde{Z} + (\tilde{I} - \tilde{Z})\tilde{Z}\tilde{I} = \tilde{I}\tilde{Z} + (\tilde{I} - \tilde{Z})\tilde{Z}\tilde{I}. \quad (27)$$

This may appear to be a less convenient form than the form of the operators in the physical basis, but in fact the interpretation is quite simple. For example, consider the term $(\tilde{I} + \tilde{Z})\tilde{I}\tilde{X}$ appearing in S_X ; in the DFS basis it is represented by

$$\begin{pmatrix} X & | & | \\ \hline & & X \\ \hline & & \end{pmatrix} \quad (28)$$

i.e., it corresponds to an identical action on the two $J=1/2$ subsystems, which is the signature of a collective error. Of course, the terms $(\tilde{I} + \tilde{Z})\tilde{I}\tilde{Y}$, $(\tilde{I} + \tilde{Z})\tilde{I}\tilde{Z}$ have a similar interpretation. The remaining terms appearing in Eqs. (25)–(27) are all projections on the orthogonal subspace \mathcal{C}^\perp , since they contain the factor $(\tilde{I} - \tilde{Z})$. Thus they do not cause any errors on the DFS.

These observations facilitate the completion of the basis. Consider next the basis for operations on the orthogonal subspace \mathcal{C}^\perp which are analogous to the logical operations on \mathcal{C} . These can be determined from the set (24) by simply replacing the projector $(\tilde{I} + \tilde{Z})$ by $(\tilde{I} - \tilde{Z})$. However, in order to keep all basis elements linearly independent, in particular orthogonal to S_Z , we must modify $\bar{\sigma}_z^\perp$:

$$\begin{aligned} \bar{\sigma}_x^\perp &= (\tilde{I} - \tilde{Z})\tilde{X}\tilde{I}, \\ \bar{\sigma}_y^\perp &= (\tilde{I} - \tilde{Z})\tilde{Y}\tilde{I}, \\ \bar{\sigma}_z^\perp &= 2\tilde{I}\tilde{Z} - (\tilde{I} - \tilde{Z})\tilde{Z}\tilde{I}. \end{aligned} \quad (29)$$

We now complete the 64 element orthogonal basis for the algebra [69]. Consider first the collective X error, Eq. (25). The following four elements, appearing in S_X , and whose interpretation as errors that either leave the DFS invariant or annihilate it was discussed above, span a 4-dimensional subspace of the algebra:

$$(\tilde{I} + \tilde{Z})\tilde{I}\tilde{X}, (\tilde{I} - \tilde{Z})\tilde{I}\tilde{X}, (\tilde{I} - \tilde{Z})\tilde{X}\tilde{X}, (\tilde{I} - \tilde{Z})\tilde{Y}\tilde{Y}. \quad (30)$$

To find a set of operators which are mutually orthogonal and orthogonal to S_X , we use the following procedure. First we normalize the collective X error (with an overall factor of $1/\sqrt{6}$). We then require that the set of mutually orthogonal operators (30) be taken from the set of orthogonal basis elements, to another set of orthogonal elements. The appropriate mapping is an element of $SO(4)$ since it maps four orthonormal vectors to four orthonormal vectors. Therefore, form an $SO(4)$ matrix whose first column elements are all $1/\sqrt{6}$, and

whose remaining three columns provide the following coefficients (the set formed in this way is not unique):

$$S_{X_1} = \frac{1}{\sqrt{30}}(\tilde{I} + \tilde{Z})\tilde{I}\tilde{X} + \frac{1}{\sqrt{10}}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{X} + \frac{1}{\sqrt{30}}(\tilde{I} - \tilde{Z})\tilde{X}\tilde{X} - \sqrt{\frac{5}{6}}(\tilde{I} - \tilde{Z})\tilde{Y}\tilde{Y}, \quad (31)$$

$$S_{X_2} = -\frac{\sqrt{3}}{2}(\tilde{I} + \tilde{Z})\tilde{I}\tilde{X} + \frac{1}{2}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{X}, \quad (32)$$

$$S_{X_3} = -\frac{1}{2\sqrt{5}}(\tilde{I} + \tilde{Z})\tilde{I}\tilde{X} - \frac{1}{2}\sqrt{\frac{3}{5}}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{X} + \frac{2}{\sqrt{5}}(\tilde{I} - \tilde{Z})\tilde{X}\tilde{X}. \quad (33)$$

The result is three additional orthonormal basis elements which all act trivially on the DFS.

The same procedure can be used for the collective Y error [Eq. (26)] to obtain the following set of trivially acting errors:

$$S_{Y_1} = \frac{1}{\sqrt{30}}(\tilde{I} + \tilde{Z})\tilde{I}\tilde{Y} + \frac{1}{\sqrt{10}}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{Y} + \frac{1}{\sqrt{30}}(\tilde{I} - \tilde{Z})\tilde{X}\tilde{Y} - \sqrt{\frac{5}{6}}(\tilde{I} + \tilde{Z})\tilde{Y}\tilde{X}, \quad (34)$$

$$S_{Y_2} = -\frac{\sqrt{3}}{2}(\tilde{I} + \tilde{Z})\tilde{I}\tilde{Y} + \frac{1}{2}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{Y}, \quad (35)$$

$$S_{Y_3} = -\frac{1}{2\sqrt{5}}(\tilde{I} + \tilde{Z})\tilde{I}\tilde{Y} - \frac{1}{2}\sqrt{\frac{3}{5}}(\tilde{I} - \tilde{Z})\tilde{I}\tilde{Y} + \frac{2}{\sqrt{5}}(\tilde{I} - \tilde{Z})\tilde{X}\tilde{Y}. \quad (36)$$

Some of the lack of symmetry can be remedied by a different choice for the $SO(4)$ matrix. However, no significant simplification is obtained. The (diagonal) collective Z error may be completed by finding the remaining elements in the set of eight diagonal elements which span the subspace of diagonal elements of the algebra (the Cartan subalgebra). We now choose elements to complete the basis, including this set. We start with the following diagonal matrices

$$\tilde{Z}\tilde{I}\tilde{Z}, \quad \tilde{Z}\tilde{I}\tilde{I}, \quad (37)$$

which clearly also act as collective errors. Counting indicates there are 14 remaining basis elements. These can be taken to be

$$\begin{aligned} (\tilde{I} - \tilde{Z})\tilde{X}\tilde{Z}, \quad (\tilde{I} - \tilde{Z})\tilde{Y}\tilde{Z}, \quad (\tilde{I} - \tilde{Z})\tilde{Z}\tilde{X}, \\ (\tilde{I} - \tilde{Z})\tilde{Z}\tilde{Y}, \quad (\tilde{I} - \tilde{Z})\tilde{Z}\tilde{Z}, \end{aligned} \quad (38)$$

which act to annihilate the DFS; and

$$\begin{aligned} (\tilde{I} + \tilde{Z})\tilde{X}\tilde{X}, \quad (\tilde{I} + \tilde{Z})\tilde{X}\tilde{Y}, \quad (\tilde{I} + \tilde{Z})\tilde{X}\tilde{Z}, \\ (\tilde{I} + \tilde{Z})\tilde{Y}\tilde{X}, \quad (\tilde{I} + \tilde{Z})\tilde{Y}\tilde{Y}, \quad (\tilde{I} + \tilde{Z})\tilde{Y}\tilde{Z}, \end{aligned}$$

$$(\tilde{I} + \tilde{Z})\tilde{X}\tilde{X}, \quad (\tilde{I} + \tilde{Z})\tilde{Y}\tilde{Y}, \quad (\tilde{I} + \tilde{Z})\tilde{Z}\tilde{Z}, \quad (39)$$

which act nontrivially on the DFS in that they mix the μ factors of $|0_L\rangle, |1_L\rangle$.

One may verify that we have enumerated 64 basis elements, with the property that they are trace orthogonal and therefore span the space of three-qubit operators. These were chosen compatible with the appropriate subspaces of the three-qubit DFS. The advantage of explicitly listing a complete set of basis elements is that we may now classify the operators and their actions on the code space.

E. Classification of errors

In this subsection we classify the different types of operators in terms of how they affect the three-qubit DFS code. We will then discuss the effect of asymmetric exchange operations on this code in the next subsection.

As stated previously [63], we will not consider the errors of type (38) or (29) as elements of the algebra which would cause irrevocable loss of information (note, however, that one could in principle remove them by symmetrizing with respect to C^\perp). In addition, we have previously classified basis elements Eq. (24) as logical operations which give rise to logical errors when they are contained in the error algebra. The elements of the form Eq. (21) are a basis for the leakage type errors, while terms of the form Eqs. (24), (26), and (27), are collective operations and will not affect the DFS encoded states. The two diagonal elements (37) do not cause leakage and do not add anything beyond logical type errors. The remaining operations need to be interpreted.

After some consideration, it is clear that the S_{X_i}, S_{Y_i} operators are actually elements of the stabilizer subgroup. For a DFS this is defined as [11]

$$\hat{D}(\vec{v}) = \exp\left[\sum_{\alpha} v_{\alpha}(\hat{S}_{\alpha} - I \otimes M_{\alpha})\right], \quad (40)$$

where the elements of the vector \vec{v} , v_{α} are complex numbers, the \hat{S}_{α} come from the set which generates the collective error algebra \mathcal{A} (we can include H_S as S_0) and M_{α} is a matrix which only mixes noncode indices. Thus the linear combinations that are used in the sum that is in the argument of the exponential, will span the space of the primitives in the collective errors, and the S_{X_i}, S_{Y_i} are included in these linear combinations. Another way in which to see that these operators do not affect the code space is by acting with them on the code. The terms in S_{X_i} and S_{Y_i} of the form $(I + \tilde{Z})^{**}$ where the $*$ is any operator, are the only ones which act on the code space (with no effect), the others act on the orthogonal complement C^\perp . Thus these operations do not affect the code and are thus elements of the stabilizer.

The remaining errors (39) are products of elements of the logical operations and the collective operations. In this case the operator has the form $\mathcal{A}\mathcal{A}'$ which is a product of an operator which does nothing to the code (a stabilizer element) and a logical error.

Now that the complete set of basis elements spanning the three-qubit DFS has been explicitly represented, and we have

identified the action of these operators on the code, we may discuss their significance in a more practical setting. We will next decompose the errors that are seen as obstacles to building practical solid state quantum computing systems and discuss the effect of these errors in terms of the complete set of operations on the code space.

F. Errors in quantum-dot qubits

A dominant type of error in solid-state quantum dot quantum computing architectures arises from spin-orbit interactions [70]. Spin-orbit interactions couple charge degrees of freedom associated with the orbital wave functions to the spin degrees of freedom used to store and manipulate information. Since charge often interacts much more strongly with the environment [71], spin-orbit interactions give rise to decoherence in these devices.

There are several ways in which to treat the gating errors which arise due to anisotropic exchange interactions. One way is to treat them with a QECC, which, as stated in the Introduction, requires a substantial qubit overhead. A second way is to use shaped pulses [72]. A third way is to use dressed qubits [73]. Alternatively the spin-orbit interaction can also be used to construct a universal gate set [74,75]. Here we assume the form of spin-orbit interactions which give rise to errors which are of the same form as the asymmetric, or anisotropic exchange. However, we treat these as decohering (causing information loss), rather than unitary errors.

Consider a bilinear coupling in the physical basis of the form

$$H_{SB}^{(2)} = \sum_{i < j} \sum_{\alpha, \beta = \{x, y, z\}} g_{ij}^{\alpha\beta} \sigma_i^{\alpha} \sigma_j^{\beta} \otimes B_{ij}^{\alpha\beta}, \quad (41)$$

where $g_{ij}^{\alpha\beta}$ is a rank-2 tensor. The symmetrization procedure of [34], that prepares collective decoherence conditions, applies only to linear coupling, so will not work in this case. In this case we must consider the possibility of leakage. The bilinear term $g_{ij}^{\alpha\beta} \sigma_i^{\alpha} \sigma_j^{\beta}$ can be decomposed into (i) a scalar $g \vec{\sigma}_i \cdot \vec{\sigma}_j$, which is proportional to the Heisenberg exchange operator and thus has the effect of logical errors E ; (ii) a rank-1 tensor

$$\vec{\beta} \cdot (\vec{\sigma}_i \times \vec{\sigma}_j); \quad (42)$$

(iii) a mixed rank tensor

$$(\vec{\sigma}_i \cdot \vec{\gamma}_i)(\vec{\sigma}_j \cdot \vec{\gamma}_j) \quad (43)$$

which cannot couple the two $S=1/2$ states to each other, but can couple them to $S=3/2$ states causing leakage. Thus we see that the $S=3/2$ subspace acts as a source for leakage [from (ii) and (iii)], and that there is also the possibility of (noncollective) errors [from (ii)] which do not have the same effect on the $|0_L\rangle, |1_L\rangle$ states and therefore cause logical errors. Clearly, higher-order interactions $H_{SB}^{(n)}$ with $n > 2$ can cause similar leakage and logical errors. We also note that the scalar coupling can be included as a special case of the type (iii) errors. This is the way in which it will arise in our calculations.

We will now examine the errors (42) and (43) in the basis for encoded operations and thus identify their effect on the code space. To be specific, the errors in Eqs. (42) and (43) will be transformed into the tilde basis, which acts on the code space, by $U_{dfs} E_s U_{dfs}^\dagger$, where E_s is either of the form (42) or (43). We will then decompose these operations in the DFS (tilde) basis of Sec. IV D.

Let us first consider errors of the form (42). We will consider only the errors on physical qubits one and two and neglect leakage errors of the type (21). After omitting these two types of errors, the remaining errors are of the form

$$\frac{1}{\sqrt{3}}[\beta_{12}^x(I+\tilde{Z})\tilde{Y}\tilde{X} + \beta_{12}^y(I+\tilde{Z})\tilde{Y}\tilde{Y} + \beta_{12}^z(I+\tilde{Z})\tilde{Y}\tilde{Z}]. \quad (44)$$

Assuming that we can remove leakage errors using an appropriate LEO, our goal is to analyze the remaining errors to see how they affect the encoded qubits.

Note first that in Eq. (44) we are examining errors caused by interactions between qubits one and two. If only errors between qubits one and two are present, there exists an inherent asymmetry in the noise, thus we would not expect a DFS to protect against this type of error. However, it turns out that the basis elements (primitives) here are the same as those found in the interactions between qubits two and three and in qubits one and three. Cancellation or partial cancellation of this type of error will occur when the interactions between pairs of qubits combine to form only stabilizer operations, a situation which could be achieved through material/environment engineering or by dynamical decoupling symmetrization with respect to the code space. Here we will only consider the asymmetric case of a pair of qubits (qubits 1 and 2).

How should we interpret the errors in Eq. (44)? Consider the first term. Examining the logical errors, we see that the error present in Eq. (44) is proportional to $\bar{\sigma}_y S_X$ less those terms which are of the form $(I-\tilde{Z})^{**}$. Thus it is a logical Y multiplied by a collective X error less those errors which act only on C^\perp . The other terms are also proportional to a logical Y operation. This indicates that we may protect against all of the remaining errors with the concatenation of this DFS by a QECC which protects against logical Y errors. This can be accomplished by using a 3-to-1 QECC encoding. This may well be a significant advantage over a pure QECC given available resources in solid-state implementations of quantum computing.

Now we examine errors of the form (43). Here, as before, we will analyze only the effect of this type of error on physical qubits one and two. We will also again omit the leakage errors which are of the form (21). The remaining 10 terms of the DFS transformed errors (43) are

$$\begin{aligned} & \frac{\gamma_1^x \gamma_2^x - \gamma_1^y \gamma_2^y}{2\sqrt{3}}(I-\tilde{Z})\tilde{X}I + \frac{\gamma_1^y \gamma_2^x + \gamma_1^x \gamma_2^y}{2\sqrt{3}}(I-\tilde{Z})\tilde{Y}I \\ & + \frac{(-\gamma_1^z \gamma_2^y + \gamma_1^y \gamma_2^z)}{2\sqrt{3}}(I+\tilde{Z})\tilde{Y}\tilde{X} + \frac{\gamma_1^x \gamma_2^z - \gamma_1^z \gamma_2^x}{2\sqrt{3}}(I+\tilde{Z})\tilde{Y}\tilde{Y} \\ & + \frac{-\gamma_1^y \gamma_2^x + \gamma_1^x \gamma_2^y}{2\sqrt{3}}(I+\tilde{Z})\tilde{Y}\tilde{Z} + \frac{1}{3}(-\gamma_1^x \gamma_2^x - \gamma_1^y \gamma_2^y - \gamma_1^z \gamma_2^z) \end{aligned}$$

$$\begin{aligned} & \times (I+\tilde{Z})\tilde{Z}I + \frac{\gamma_1^z \gamma_2^x + \gamma_1^x \gamma_2^z}{2\sqrt{3}}(I-\tilde{Z})\tilde{Z}\tilde{X} + \frac{(\gamma_1^y \gamma_2^z + \gamma_1^z \gamma_2^y)}{2\sqrt{3}} \\ & \times (I-\tilde{Z})\tilde{Z}\tilde{Y} + \frac{1}{6}(-\gamma_1^x \gamma_2^x - \gamma_1^y \gamma_2^y + 2\gamma_1^z \gamma_2^z)(I-\tilde{Z})\tilde{Z}\tilde{Z} \\ & + \frac{1}{3}(-\gamma_1^x \gamma_2^x - \gamma_1^y \gamma_2^y - \gamma_1^z \gamma_2^z)\tilde{Z}II. \end{aligned} \quad (45)$$

Terms 1,2,7,8,9 act on C^\perp and thus have no effect on the code. Term 10 is a combination of exchange operations which does not affect the code subspace. Terms 3,4,5,6 act on the code space. They act either as a logical error (term 6) or as a logical error composed with a collective error. The logical errors must be treated using other methods. In this case, we see that all errors other than term 6 give rise to logical Y errors. To remove all errors in the system, we have different choices which could be good alternatives depending upon the physical system. First, we could eliminate the errors using more decoupling pulses. Second, we could choose a system where the term 6 is negligible. This would enable complete elimination of the errors with one other decoupling pulse. Third, if term 6 is negligible, we could treat the Y errors with the concatenation of the DFS with a QECC. The required QECC would use only three qubits to encode one.

Let us assume that we can introduce another decoupling pulse to eliminate the logical Z error (term 6) so that we only have logical Y errors present in our computation. This could be accomplished, for example, by implementing an extra $\pi/2$ -logical- X rotation in half the LEO time. To eliminate the remaining Y errors, we concatenate the DFS with a QECC [28] by encoding the DFS-protected logical qubits into a QECC which is protected against Y errors. We may describe this briefly as follows. We use the state consisting of three logical (DFS) qubit states to construct the following nine-qubit code (nine physical qubits)

$$|\psi_E\rangle = a|0_L 0_L 0_L\rangle + b|1_L 1_L 1_L\rangle.$$

Suppose that a logical Y error occurs on one of the logical qubits, say the second. Then, in terms of the encoded DFS states, the logical qubit undergoes the following transformation:

$$|\psi_E\rangle = a|0_L 0_L 0_L\rangle + b|1_L 1_L 1_L\rangle \rightarrow -ia|0_L 1_L 0_L\rangle + ib|1_L 0_L 1_L\rangle. \quad (46)$$

Detecting errors on this logical state may be accomplished by detecting differences between any two pair of DFS encoded qubits according to the stabilizer formalism [4]. Alternatively, the original Shor code [1] which protects against X errors (or equivalently Z errors) on physical qubits may be modified for our purposes by simply changing local bases on each logical qubit to describe the formalism in terms of the Y errors. (Every operator and state would be rotated by $\pi/2$ about the Z axis in the logical basis.) The advantage of such a concatenation is described in [28]. The fidelity is effectively decoupled from the error rate on logical qubits and depends only on the strength of the perturbation which causes the asymmetries in the system-bath interaction and disturbs the DFS encoding. We should note that, whereas our

formalism to this point is valid for any Heisenberg exchange based quantum computing scheme, this final readout and correction of the error on the QECC encoded qubit will depend upon the details of the system.

Physical implementations will dictate which combination of decoupling pulses, materials engineering, or a QECC is a valid option for the elimination of remaining errors after leakage has been removed.

V. LEOs AND THE 4-QUBIT DFS

The four qubit DFS encodes one logical qubit using a subspace of four physical qubits. It protects a single logical qubit from collective errors of any type (bit-flip, phase-flip, or both). It has been shown that the exchange interaction is sufficient for implementing a universal set of gating operations on this system while preserving the DFS [11,40]. In this section we will answer the following questions: (1) Is there a LEO available in solid-state implementations using the 4-qubit DFS, which relies on the exchange interaction? (2) Is there a canonical set of such gates? (3) After leakage is removed, can we completely remove all errors in asymmetric exchange, and if so, how?

A. The four-qubit DFS

The four-qubit DFS contains two singlet states for the representative qubit, three triplets and a spin 2, or quintuplet. The singlet states, which represent the logical zero and one of the DFS encoded qubit are given by

$$S^0 = |0_L\rangle = \frac{1}{2}(|0101\rangle + |1010\rangle - |0110\rangle - |1001\rangle) \quad (47)$$

and

$$S^1 = |1_L\rangle = \frac{1}{\sqrt{12}}(2|0011\rangle + 2|1100\rangle - |0110\rangle - |1001\rangle - |0101\rangle - |1010\rangle), \quad (48)$$

where as before, e.g., $|0101\rangle = |1/2\rangle \otimes |-1/2\rangle \otimes |1/2\rangle \otimes |-1/2\rangle$ in the standard angular momentum basis. S^0 and S^1 comprise the code space \mathcal{C} for this DFS and the remaining states are the states in \mathcal{C}^\perp .

The triplet states are given by

$$T_{11}^1 = |11\rangle = \frac{1}{2}(|0100\rangle + |1000\rangle - |0001\rangle - |0010\rangle),$$

$$T_{10}^1 = |10\rangle = \frac{1}{\sqrt{2}}(|1100\rangle - |0011\rangle),$$

$$T_{1-1}^1 = |1-1\rangle = \frac{1}{2}(|1110\rangle + |1101\rangle - |1011\rangle - |0111\rangle),$$

$$T_{11}^2 = |11\rangle = \frac{1}{\sqrt{2}}(|0001\rangle - |0010\rangle),$$

$$T_{10}^2 = |10\rangle = \frac{1}{2}(|1001\rangle + |0101\rangle - |1010\rangle - |0110\rangle),$$

$$T_{1-1}^2 = |1-1\rangle = \frac{1}{\sqrt{2}}(|1101\rangle - |1110\rangle),$$

$$T_{11}^3 = |11\rangle = \frac{1}{\sqrt{2}}(|0100\rangle - |1000\rangle),$$

$$T_{10}^3 = |10\rangle = \frac{1}{2}(|0110\rangle + |0101\rangle - |1010\rangle - |1001\rangle),$$

$$T_{1-1}^3 = |1-1\rangle = \frac{1}{\sqrt{2}}(|0111\rangle - |1011\rangle). \quad (49)$$

The spin-2 representation is given by the following set of states (a quintuplet)

$$Q_{22} = |22\rangle = |0000\rangle,$$

$$Q_{21} = |10\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle),$$

$$Q_{20} = |1-1\rangle = \frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle),$$

$$Q_{2-1} = |2-1\rangle = \frac{1}{2}(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle),$$

$$Q_{2-2} = |2-2\rangle = |1111\rangle. \quad (50)$$

We will refer to this set of states as the set of DFS states with the logical elements comprising \mathcal{C} and the other states (T and Q states) comprising \mathcal{C}^\perp . In the latter part of this section we will again use a DFS basis for the operators and a transformation U_{dfs} to change from the computational basis set of states and operators to the DFS sets. Let us now discuss logical operations on the DFS.

B. Gate operations and LEOs

Physical gates were given in Ref. [11] and shown to be compatible with the DFS. These are given by the exchange interaction between pairs of physical qubits. The logical “ X ” operation is given by

$$\bar{X} = \frac{1}{\sqrt{3}}(E_{23} - E_{13}), \quad (51)$$

where, again, E_{ij} is the exchange operation between qubits i and j , and I_2 is the 2×2 identity matrix. The logical “ Z ” operation is given by

$$\bar{Z} = -E_{12} \quad (52)$$

and \bar{Y} can be obtained from these two. However, there is a distinct difference between this set of logical operations and the analogous set in Sec. IV. The difference is that naturally occurring logical operations on the three-qubit DFS have ei-

genvalue zero on the states in \mathcal{C}^\perp . This is not the case for the four-qubit DFS. For example, all states in \mathcal{C}^\perp have eigenvalues of $+1$ for states in \mathcal{C}^\perp , with the exception of T^3 whose states have eigenvalue -1 when acted on by \bar{Z} . These gates are not “canonical” in our sense and so the projection onto the code subspace is not automatic. One must either use another set of gating operations which are projective or use the generalized LEO of Sec. III B. We explore each of these two possibilities in the next two subsections.

C. A LEO from exchange

As noted in [37] there exist physically available operations which are “canonical” in many circumstances, meaning that they are projective onto the code subspace. The definition of \bar{Z} given in [11] (and the previous section) is not canonical. We therefore seek to construct a physically available LEO. We do this using the definition of the generalized LEO given in Sec. III B.

Let the (square of the) total spin angular momentum operator be denoted \vec{S}^2 with eigenvalue $S(S+1)$. Then

$$4\vec{S}^2 = \left(\sum_i \vec{\sigma}_i \right)^2,$$

where $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)$ are the Pauli matrices acting on the i th qubit. Therefore

$$\vec{S}^2/2 = \frac{12 + 2 \sum_{i < j} \vec{\sigma}_i \cdot \vec{\sigma}_j}{8}$$

gives an appropriate LEO of the form given in Eq. (9). This can be seen as follows. On the $S=0$ (singlet) subspaces the operator gives zero. On the $S=1$ subspaces the operator gives 1 and on the $S=3$ subspace it gives 3. Therefore the appropriate LEO is given by

$$R_L = \exp\{-i\pi S^2/2\},$$

which reproduces the LEO of Eq. (9).

Now consider a modified set of logical operations:

$$\bar{X} \rightarrow \bar{X}' = \bar{X} + S^2/2, \quad (53)$$

$$\bar{Y} \rightarrow \bar{Y}' = \bar{Y} + S^2/2, \quad (54)$$

$$\bar{Z} \rightarrow \bar{Z}' = \bar{Z} + S^2/2. \quad (55)$$

This set may be used to obtain an appropriate LEO by exponentiation, e.g.,

$$R_L = \exp\{-i\pi(\bar{Z}')\}.$$

Similarly we can construct a LEO using \bar{X}' or \bar{Y}' . Since the operator \vec{S}^2 is composed of exchange interactions, it is also experimentally available.

D. A canonical LEO from exchange

One may ask the question: what set of gates would be canonical if we use only exchange operations? One way to

answer this question is to do the following calculation. Start with operations which act as \bar{X} , \bar{Y} and \bar{Z} and have eigenvalue zero on the states in \mathcal{C}^\perp . These are operators in the DFS (i.e., the tilde) basis. Then use the DFS transformation to transform from the DFS basis back to the physical basis to find the set of physical interactions necessary to perform canonical gating operations. We now use this procedure to find such a LEO.

The DFS basis for this system has logical operations which transform between the two one-dimensional spin-0 representations. These may be represented by ordinary Pauli matrices which act only on the 2×2 block. The operations which perform these logical operations are labelled using the spin-0 index and a 0, 1 degeneracy index, $O_{\lambda_1, \lambda_2}^{0,0}$, which in this case has only a single entry for each pair λ_1, λ_2 . According to the definition, canonical logical operations would have the following form in the DFS basis:

$$\bar{\sigma}_i = \begin{pmatrix} \sigma_i & 0_{14 \times 2} \\ 0_{2 \times 14} & 0_{14 \times 14} \end{pmatrix}, \quad (56)$$

where $0_{m \times n}$ is an $m \times n$ matrix of zeros. This operation acts simultaneously as a projector onto the code subspace and a Pauli operator on the encoded state. Now, let the DFS transformation be given by U_{dfs} , the computational basis states be given by $|\psi_c\rangle$, and the DFS states be given by $|\psi_{dfs}\rangle$:

$$U_{dfs}|\psi_c\rangle = |\psi_{dfs}\rangle. \quad (57)$$

Then the logical operations are related to the operations in the computational basis by

$$\bar{\sigma}_i|\psi_{dfs}\rangle = U_{dfs}\bar{\sigma}_i^c|\psi_c\rangle, \quad (58)$$

where $\bar{\sigma}_i$ is the canonical logical operation in the logical basis and $\bar{\sigma}_i^c$ is the logical operation in the computational basis. This physical realization of the canonical operations will be found using

$$U_{dfs}^{-1}\bar{\sigma}_i U_{dfs} = \bar{\sigma}_i^c. \quad (59)$$

Using only the exchange operations between qubits i and j , $E_{ij} = I + \vec{\sigma}_i \cdot \vec{\sigma}_j$, the canonical operations are given by

$$\bar{\sigma}_x^c = (2I - E_{13})(2I - E_{24}) - (2I - E_{23})(2I - E_{14}) \quad (60)$$

and

$$\begin{aligned} \bar{\sigma}_z^c &= 2(2I - E_{34})(2I - E_{12}) - (2I - E_{13})(2I - E_{24}) - (2I - E_{23}) \\ &\quad \times (2I - E_{14}) \end{aligned} \quad (61)$$

and the commutator of these two gives the third logical element. It is clear from this form that 4-body interactions are required to construct canonical logical operations using only exchange operations. Although there are methods for constructing these from more fundamental interactions [76,77] so that they may be useful for some quantum computing purposes (e.g., simulations) they are likely impractical for BB controls due to time constraints. Alternatively, it has been shown that these four-body interactions naturally arise in spin-coupled quantum dots with significant effect, and it has been suggested that one might take advantage of these effects when designing gating operations [78].

E. Decomposition of the algebra

The detailed analysis of Sec. IV becomes much more complicated for the four-qubit DFS. The set of operators for the Hilbert space of four qubits contains $(2^4)^2 = 2^8 = 256$ basis elements. This set obviously becomes prohibitively large for such a detailed analysis as the number of qubits grows. However, there are several key observations which summarize our three-qubit analysis and which are quite general. (These can also be seen from the algebraic decomposition of Sec. II B.) The operations in the DFS basis may be classified as follows: Logical operations, operations on \mathcal{C}^\perp , collective operations, leakage operations, sums and/or products of one or more of these types. Thus identifying these types will enable the identification of general operations in terms of these. As in the treatment of the 3-qubit DFS, we will assume that the LEO can be implemented so that leakage errors are irrelevant. In this case, we need only remove logical operations (this will also remove operations which are products of logical operations and another type). Therefore our goal is to identify logical operations and then extract those operators in the spin-orbit coupling terms which give rise to logical type errors, whether or not they are combined with collective errors.

Logical errors are those generated by logical operations given in Sec. V B. In order to discern their effect on the DFS, we may transform these to the DFS basis by $U_{dfs} O U_{dfs}^\dagger = \tilde{O}$, where the transformation U_{dfs} is defined by the transformation from the computational basis states to the logical (DFS) basis states of Sec. V A.

The extraction of logical operations and operations which are products of logical operations with another are readily performed with MATHEMATICA. We first note the form of logical operations when we choose an ordered logical, or DFS, basis. The first two entries for this ordered basis are $|0_L\rangle$ and $|1_L\rangle$. Logical basis elements for the algebra, when restricted to the code subspace, have the form

$$(I + \tilde{Z})(I + \tilde{Z})(I + \tilde{Z})\tilde{O}_i, \quad (62)$$

where $\tilde{O}_i \in \{\tilde{X}, \tilde{Y}, \tilde{Z}\}$. These observations imply that, from Sec. III, B is a 2×2 matrix, C is a 14×14 matrix, E is 2×14 and E^\perp is 14×2 .

It is clear from Eq. (62) that on this two-dimensional subspace, these operators span the space of logical operations. As noted above, we could include products of these with other operations and/or include operations on \mathcal{C}^\perp , by a direct sum. However, these have no effect on the code once leakage errors have been removed, and eliminating logical elements eliminates the other errors as well.

The collective errors, which also have no effect on the code annihilate the code elements (they are singlet states). Therefore, in this case the collective errors are trivial.

Having established the correspondence between the relevant parts of the argument for the three-qubit DFS and the requirements for noise-free evolution for the four-qubit DFS, we may now analyze the errors which occur on this code in solid-state qubits.

F. Errors in quantum-dot qubits

As was done for the three-qubit DFS, we will examine errors corresponding to Eqs. (42) and (43). We will again suppose that the errors of the type (i) are included as type (iii), and that an appropriate LEO is experimentally available, so that we may neglect leakage errors. We therefore decompose the four-qubit DFS transformed terms, Eqs. (42) and (43), into the DFS basis elements. In this case the representations used for the DFS states are not completely symmetric in the four qubits. We must therefore consider two pairs of interactions, the interactions between qubits one and two, and between qubits two and three.

The error arising from Eq. (42) between physical qubits 1 and 2, and neglecting leakage operations and operations on \mathcal{C}^\perp , consists of just one term:

$$-\frac{1}{3\sqrt{2}}(\gamma_1^x \gamma_2^x + \gamma_1^y \gamma_2^y + \gamma_1^z \gamma_2^z)(I + \tilde{Z})(I + \tilde{Z})(I + \tilde{Z})\tilde{Z}. \quad (63)$$

There are two different terms for the Eq. (42) when acting between qubits 2 and 3:

$$\frac{1}{6\sqrt{2}}(\gamma_2^x \gamma_3^x + \gamma_2^y \gamma_3^y + \gamma_2^z \gamma_3^z)(I + \tilde{Z})(I + \tilde{Z})(I + \tilde{Z})\tilde{Z}, \quad (64)$$

$$\frac{1}{2\sqrt{6}}(\gamma_2^x \gamma_3^x + \gamma_2^y \gamma_3^y + \gamma_2^z \gamma_3^z)(I + \tilde{Z})(I + \tilde{Z})(I + \tilde{Z})\tilde{X}. \quad (65)$$

A rather stark difference between the three-qubit DFS and the four-qubit DFS is the fact that no errors of the form Eq. (43) contribute to logical errors on this code. We therefore need not correct this type of error. These types of errors occur in zinc-blende type semiconductor structures which have a broken inversion symmetry. Knowing that removing all leakage errors can render this type of error insignificant is an important advantage of this code.

Another quite striking difference between the two codes is that the scalar term, errors of the form (i), are the only types present. Thus if this type of error could be eliminated by materials engineering, no errors would be present on the encoded qubits.

To correct all errors of the forms in Eqs. (42) and (43), without the ability to eliminate the scalar coupling, for the four-qubit DFS code, we may apply the same methods as in the three-qubit case. (1) More decoupling pulses could be applied to eliminate the logical X and Z errors. (2) Another decoupling pulse could be used to eliminate the Z type errors and a three-logical-qubit QECC could be used to eliminate the X type error.

However, the important difference between the two codes is that the antisymmetric term, also known as a Dzyaloshinski-Moriya term, is not present after the implementation of an appropriate LEO. Therefore, if this term is a dominant source of errors in a particular implementation of quantum dot quantum computing, then the four-qubit DFS has a distinct advantage over the three-qubit DFS.

VI. CONCLUSIONS

We have given methods for the implementation of leakage elimination operators (LEOs) which are, in many circumstances, physically available in experiments. These LEOs eliminate a large and important class of errors; those that would serve to destroy a subspace encoding. The methods for producing these have shown promise in many experiments and here we have generalized the method for producing such LEOs.

Symmetrization by dynamical decoupling can be performed by decoupling the code from its orthogonal subspace or the orthogonal subspace from the code. In the first case, we can, in principle, remove all errors from the code space. While this is not true when the decoupling is performed with respect to the orthogonal subspace, the advantage of this method is that it does not introduce further errors if imperfect decoupling controls are used.

For the three- and four-qubit DFSs, we have given a method for producing LEOs using the exchange interaction

which is physically available in solid-state qubit implementations of quantum computing. We have shown that after leakage errors have been removed, the remaining bilinear couplings which could give rise to errors may be treated with either a QECC or a cycle involving extra decoupling pulses. For the four-qubit DFS there are similarities in our ability to experimentally treat errors. However, it has two advantages over the three-qubit DFS. First, only scalar type bilinear interactions affect the code. Second, eliminating leakage errors from the four-qubit DFS removes the Dzyaloshinski-Moriya interaction errors. These are present in many semiconductors which have a broken inversion symmetry and are the main part of the anisotropic exchange interaction.

ACKNOWLEDGMENTS

D.A.L. acknowledges financial support from AFOSR (F49620-01-1-0468), the Sloan Foundation, NSERC, and the Connaught Fund.

-
- [1] P. W. Shor, Phys. Rev. A **52**, R2493 (1995).
 - [2] A. R. Calderbank and P. W. Shor, Phys. Rev. A **54**, 1098 (1996).
 - [3] A. Steane, Rep. Prog. Phys. **61**, 117 (1998).
 - [4] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).
 - [5] D. Gottesman, Ph.D. thesis, California Institute of Technology, Pasadena, CA, 1997, quant-ph/9705052.
 - [6] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
 - [7] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
 - [8] L.-M. Duan and G.-C. Guo, Phys. Rev. A **57**, 737 (1998).
 - [9] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
 - [10] E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. **84**, 2525 (2000).
 - [11] J. Kempe, D. Bacon, D. A. Lidar, and K. B. Whaley, Phys. Rev. A **63**, 042307 (2001).
 - [12] D. A. Lidar, D. Bacon, J. Kempe, and K. B. Whaley, Phys. Rev. A **63**, 022306 (2001).
 - [13] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
 - [14] M. Ban, J. Mod. Opt. **45**, 2315 (1998).
 - [15] L.-M. Duan and G. Guo, Phys. Lett. A **261**, 139 (1999).
 - [16] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).
 - [17] P. Zanardi, Phys. Lett. A **258**, 77 (1999).
 - [18] D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999).
 - [19] M. S. Byrd and D. A. Lidar, Phys. Rev. A **67**, 012324 (2003).
 - [20] L. Viola, S. Lloyd, and E. Knill, Phys. Rev. Lett. **83**, 4888 (1999).
 - [21] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **85**, 3520 (2000).
 - [22] D. Vitali and P. Tombesi, Phys. Rev. A **65**, 012305 (2002).
 - [23] M. S. Byrd and D. A. Lidar, Quantum Inf. Process. **1**, 19 (2001).
 - [24] D. G. Cory, R. Laflamme, E. Knill, L. Viola, T. F. Havel, N. Boulant, G. Boutis, E. Fortunato, S. Lloyd, R. Martinez, C. Negrevergne, M. Pravia, Y. Sharf, G. Teklemariam, Y. S. Weinstein, and W. H. Zurek, Fortschr. Phys. **48**, 875 (2000).
 - [25] G. S. Agarwal, M. O. Scully, and H. Walther, Phys. Rev. Lett. **86**, 4271 (2001).
 - [26] C. Uchiyama and M. Aihara, Phys. Rev. A **66**, 032313 (2002).
 - [27] C. Uchiyama and M. Aihara, Phys. Rev. A **68**, 052302 (2003).
 - [28] D. A. Lidar, D. Bacon, and K. B. Whaley, Phys. Rev. Lett. **82**, 4556 (1999).
 - [29] P. Zanardi, Phys. Rev. A **60**, R729 (1999).
 - [30] L. Viola, Phys. Rev. A **66**, 012307 (2002).
 - [31] M. S. Byrd and D. A. Lidar, Phys. Rev. Lett. **89**, 047901 (2002).
 - [32] M. S. Byrd and D. A. Lidar, J. Mod. Opt. **50**, 1285 (2002).
 - [33] G. Alber, Th. Beth, Ch. Charney, A. Delgado, M. Grassl, and M. Mussinger, Phys. Rev. Lett. **86**, 4402 (2001).
 - [34] L.-A. Wu and D. A. Lidar, Phys. Rev. Lett. **88**, 207902 (2002).
 - [35] K. Khodjasteh and D. A. Lidar, Phys. Rev. Lett. **89**, 197904 (2002).
 - [36] D. A. Lidar and L.-A. Wu, Phys. Rev. A **67**, 032313 (2003).
 - [37] L.-A. Wu, M. S. Byrd, and D. A. Lidar, Phys. Rev. Lett. **89**, 127901 (2002).
 - [38] Y. Zhang, Z.-W. Zhou, B. Yu, and G.-C. Guo, Phys. Rev. A **69**, 042315 (2004).
 - [39] M. S. Byrd, L.-A. Wu, and D. A. Lidar, J. Mod. Opt. **51**, 2449 (2004).
 - [40] D. Bacon, J. Kempe, D. A. Lidar, and K. B. Whaley, Phys. Rev. Lett. **85**, 1758 (2000).
 - [41] D. Bacon, J. Kempe, D. A. Lidar, K. B. Whaley, and D. P. DiVincenzo, in *Proceedings of the 1st International Conference on Experimental Implementations of Quantum Computation*, edited by R. Clark (Rinton, Princeton, NJ, 2001), p. 257, quant-ph/0102140.
 - [42] D. P. DiVincenzo, D. Bacon, J. Kempe, G. Burkard, and K. B. Whaley, Nature (London) **408**, 339 (2000).
 - [43] J. Levy, Phys. Rev. Lett. **89** 147902 (2002).
 - [44] S. C. Benjamin, Phys. Rev. A **64**, 054303 (2001).
 - [45] L.-A. Wu and D. A. Lidar, Phys. Rev. A **65** 042318 (2002).

- [46] D. A. Lidar and L.-A. Wu, Phys. Rev. Lett. **88** 017905 (2002).
- [47] J. Kempe, D. Bacon, D. P. DiVincenzo, and K. B. Whaley, Quantum Inf. Comput. **1**, 33 (2002).
- [48] J. Kempe and K. B. Whaley, Phys. Rev. A **65**, 052330 (2002).
- [49] D. A. Lidar, L.-A. Wu, A. Blais, Quantum Inf. Process. **1**, 155 (2002).
- [50] M. Mohseni and D. A. Lidar, Phys. Rev. Lett. **94**, 040507 (2005).
- [51] L. Viola, J. Mod. Opt. **51**, 2357 (2004).
- [52] P. Facchi, S. Tasaki, S. Pascazio, H. Nakazato, A. Tokuse, and D. A. Lidar, Phys. Rev. A **71**, 022302 (2004); P. Facchi, D. A. Lidar, and S. Pascazio, Phys. Rev. A **69**, 022314 (2004).
- [53] L. Viola and E. Knill, Phys. Rev. Lett. **90**, 037901 (2003).
- [54] K. Shiokawa and D. A. Lidar, Phys. Rev. A **69**, 030302(R) (2004).
- [55] L. Faoro and L. Viola, Phys. Rev. Lett. **92**, 117905 (2004).
- [56] P. Zanardi, Phys. Rev. A **63**, 012301 (2002).
- [57] P. Zanardi and S. Lloyd, Phys. Rev. Lett. **90** 067902 (2003).
- [58] L. Tian and S. Lloyd, Phys. Rev. A **62**, 050301 (2000).
- [59] J. Preskill, in *Introduction to Quantum Computation and Information*, edited by H. K. Lo, S. Popescu, and T. P. Spiller (World Scientific, Singapore, 1999).
- [60] J. P. Palao and R. Kosloff, Phys. Rev. A **68**, 062308 (2003).
- [61] J. P. Palao and R. Kosloff, Phys. Rev. A **69**, 059901 (2004).
- [62] S. E. Sklarz and D. J. Tannor, quant-ph/0404081.
- [63] When we are discussing elements of the error algebra in the Kraus representation which are elements of \mathcal{A} , these actually correspond to projections onto the space orthogonal to projections onto \mathcal{C}^\perp . However, we do not want to consider these here. We will consider their exponentials.
- [64] We should emphasize that although we are discussing a bang-bang pulse sequence, our arguments hold for any suitable set of decoupling pulses.
- [65] J. Preskill, Proc. R. Soc. London, Ser. A **454**, 385 (1998).
- [66] E. Knill, R. Laflamme, and W. Zurek, Science **279**, 342 (1998).
- [67] A. Bohm, *Quantum Mechanics: Foundations and Applications*, 3rd ed. (Springer-Verlag, New York, 1993), Chap. 5.
- [68] We note that the corresponding Eq. (74) in Ref. [11] contains incorrect numerical factors, which we have fixed here.
- [69] We are referring to trace orthogonality, i.e., using the Hilbert-Schmidt inner product.
- [70] K. V. Kavokin, Phys. Rev. B **64**, 075305 (2001).
- [71] G. Burkard, D. Loss, and D. P. DiVincenzo, Phys. Rev. B **59**, 2070 (1999).
- [72] N. E. Bonesteel, D. Stepanenko, and D. P. DiVincenzo, Phys. Rev. Lett. **87**, 207901 (2004); G. Burkard and D. Loss, Phys. Rev. Lett. **88**, 047903 (2002).
- [73] L.-A. Wu and D. A. Lidar, Phys. Rev. Lett. **91**, 097904 (2003).
- [74] L.-A. Wu and D. A. Lidar, Phys. Rev. A **66**, 062314 (2002).
- [75] D. Stepanenko and N. E. Bonesteel, Phys. Rev. Lett. **93**, 140501 (2004).
- [76] X. Wang and P. Zanardi, Phys. Rev. A **65**, 032327 (2002).
- [77] L.-A. Wu, M. S. Byrd, and D. A. Lidar, Phys. Rev. Lett. **89**, 057904 (2002).
- [78] A. Mizel and D. A. Lidar, Phys. Rev. Lett. **92**, 077903 (2004); Phys. Rev. B **70**, 115310 (2004).