Wigner function and Schrödinger equation in phase-space representation

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We discuss a family of quasidistributions (*s*-ordered Wigner functions of Agarwal and Wolf [Phys. Rev. D **2**, 2161 (1970); Phys. Rev. D **2**, 2187 (1970); Phys. Rev. D **2**, 2206 (1970)]) and its connection to the so-called phase space representation of the Schrödinger equation. It turns out that although Wigner functions satisfy the Schrödinger equation in phase space, they have a completely different interpretation.

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I. INTRODUCTION

Since the pioneering work of Wigner [1], generalized phase-space techniques have found useful applications in various branches of physics [2–5]. The main idea of this approach is to represent the density operator $\hat{\rho}$ as a function (quasidistribution) over the classical phase space (q,p). This function fully characterized the quantum state and enables one to express the quantum-mechanical expectations as averages of classical observables over the classical phase space. Moreover, it is the Wigner function that is directly related to the measurement. Then quantum tomographic methods [6,7] enable one to reconstruct the quantum state from the experimental data. Recently, the Wigner function was also applied to study quantum entanglement and related issues for continuous systems (see, e.g., [8–11]).

The Wigner function [1] is only one particular example of such a quasidistribution. An especially important role (e.g., in quantum optics) is played by the family of functions introduced by Cahill and Glauber [12] containing as the Wigner function, the Glauber-Sudarshan *P*-function [13,14], and the Husimi *Q*-function [15]. In this paper we analyze another lesser-known family introduced by Agarwal and Wolf [16]. Actually, all these quasidistributions correspond to the particular quantization procedure, that is, different ordering of $\hat{q}$ and $\hat{p}$ or, equivalently, different ordering of creation $\hat{a}^\dagger$ and $\hat{a}$ annihilation operators, respectively.

The procedure of representing quantum states by quasidistributions in phase-space is closely related to the phase space formulation of quantum mechanics based on the noncommutative product known as Moyal product [17,18] or more generally as a star product [19] (see [20,21] for the compact formulation of the standard quantum mechanics in terms of the Moyal product).

There is another phase-space representation of quantum mechanics based on the works of Torres-Vega and Frederic [22,23] (see also [24]). In this approach the (pure) quantum state is represented by the wave function $\psi(\Gamma)$, where Γ represents a point in phase space. It turns out that $\psi(\Gamma)$ satisfies so called Schrödinger equation in phase space. The quantity $|\psi(\Gamma)|^2$ is, therefore, treated as a probability distribution in phase space. This procedure was applied to study simple quantum systems [25–27]. In a recent paper Li *et al.* [28] found a general method of solving Schrödinger equation in phase space.

The aim of the present paper is to relate the standard phase-space approach based on quasidistributions functions to that of Torres-Vega and Frederic (TF). We show that one can easily produce the whole family of Schrödinger equation in phase space, which is closely related to the family of *s*-ordered Wigner functions W_s of Agarwal and Wolf [16]; that is, W_s are particular solutions of this family of equations. Now, according to the standard approach W_s defines the quasidistribution in phase space, whereas the TF approach implies that $|\psi_s|^2$, where $\psi_s = 2\pi\hbar\, W_s$, is a (true) probability distribution (i.e., $|\psi_s|^2 \geqslant 0$ and $\int |\psi_s|^2 d\Gamma = 1$). It should be stressed, that Schrödinger equation in phase space has an infinite number of solutions. It is a price one pays for using $\psi(\Gamma)$ instead of $\varphi(q)$. Each particular solution gives rise to the particular phase-space representation of ordinary wave function $\varphi(q)$ in position representation.

The paper is organized as follows. In Sec. II we present general approach to phase-space representation of the wave function and following [28] we discuss the general solution for Schrödinger equation in phase space. Section III introduces the whole *s* family of equations together with the family of solutions. Then, after recalling the formulas for star products in Sec. IV, we show that *s*-ordered Wigner functions do solve the family of Schrödinger equations. We end with some conclusions in Sec. VI.

II. SCHRÖDINGER EQUATION IN PHASE SPACE

There is no a unique way to represent a quantum state as a wave function $\psi = \psi(\Gamma)$, where Γ represents a point in a classical phase (q,p). In the standard approach one usually uses a coordinate $\varphi(q)$ or momentum $\tilde{\varphi}(p)$ representations, respectively. To pass from $\varphi(q)$ to $\psi(\Gamma)$ one has to invent an integral transformation

$$\psi(\Gamma) = \int K(\Gamma;q')\varphi(q')dq', \tag{1}$$

where $K(\Gamma;q')$ denotes the integral kernel. Functions $\psi(\Gamma)$ defined by the above formula form a proper subspace $\mathcal{H}_K$ of the Hilbert space $L^2(\mathbb{R}^2)$. The unitarity of transformation $L^2(\mathbb{R}) \ni \varphi(q) \rightarrow \psi(\Gamma) \in \mathcal{H}_K$ requires

$$\int \overline{K}(\Gamma;q')K(\Gamma;q'')d\Gamma = \delta(q'-q''), \tag{2}$$

where $d\Gamma$ denotes a measure on the phase space. Clearly, there is a huge freedom in choosing K. Performing the following "gauge transformation"

$$K(\Gamma;q) \to e^{if(\Gamma)}K(\Gamma;q), \tag{3}$$

with $f(\Gamma)$ being an arbitrary real function, one obtains a new kernel still satisfying (2).

In the literature there are several well-known examples of such a transform. Perhaps the most famous is the Bargmann (or Bargmann-Segal) transform defined by [29,30]

$$K_{\mathrm{B}}(z;q) = \pi^{-1/4}\exp\left\{-\tfrac{1}{2}(z^2+q^2)+\sqrt{2}zq\right\}, \tag{4}$$

where z is a complex number, i.e., one uses $\mathbb{R}^2 \cong \mathbb{C}$. The corresponding space $\mathcal{H}_{\mathrm{B}}$ of entire functions $\psi(z)$ equipped with the following inner product

$$\langle\psi_1|\psi_2\rangle_{\mathrm{B}} = \int \psi_1^*(z)\psi_2(z)d\mu(z), \tag{5}$$

where $d\mu(z)=\pi^{-1}e^{-|z|^2}d^2z$, is known as the Bargmann-Segal representation of the Hilbert space.

A closely related kernel is connected to the coherent states representation [31]

$$K_{\mathrm{CS}}(q,p;q') = \langle\Gamma|q'\rangle, \tag{6}$$

where

$$\langle\Gamma|q'\rangle = (\lambda^2\pi)^{-1/4}\exp\left\{-\frac{(q'-q)^2}{2\lambda^2} - \frac{ip}{\hbar}(q'-q)\right\}, \tag{7}$$

with $|\Gamma\rangle$ denoting the standard Glauber coherent state corresponding to

$$\Gamma = (\lambda^{-1}q + i\lambda\hbar^{-1}p)/\sqrt{2}. \tag{8}$$

The parameter λ is a natural length scale defined by the mass and frequency of the oscillator, i.e., $\lambda=\sqrt{\hbar/\mu\omega}$. The corresponding Hilbert space $\mathcal{H}_{\mathrm{CS}}$ carries the following inner product:

$$\langle\psi_1|\psi_2\rangle_{\mathrm{CS}} = \int \psi_1^*(\Gamma)\psi_2(\Gamma)d\Gamma, \tag{9}$$

with

$$d\Gamma = \frac{dqdp}{2\pi\hbar}. \tag{10}$$

From now on we shall use the dimensionless convention (10) for $d\Gamma$. Note that in this convention $\psi(\Gamma)$ is dimensionless, whereas the kernel $K(\Gamma:q')$ has the same dimension as the wave function in the position representation $\varphi(q)$.

Actually, Eq. (6) was a starting point in constructing phase-space representation of quantum mechanics of Torres-Vega and Frederick [22,23]. They showed that if $\varphi(q)$ satisfies the standard Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\varphi(q,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\varphi(q,t), \tag{11}$$

then $\psi(\Gamma)$ obtained from $\varphi(q)$ via $K_{\mathrm{CS}}(\Gamma;q)$ satisfies the following Schrödinger equation in phase space

$$i\hbar\frac{\partial}{\partial t}\psi(\Gamma,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V\left(q+i\hbar\frac{\partial}{\partial p}\right)\right]\psi(\Gamma,t). \tag{12}$$

Actually, performing the gauge transformation $\psi'(q,p) = e^{-ipq/2\hbar}\psi(q,p)$ one finds a more symmetric formula

$$i\hbar\frac{\partial}{\partial t}\psi'(\Gamma,t) = \left[\frac{1}{2m}\hat{P}^2 + V(\hat{Q})\right]\psi'(\Gamma,t), \tag{13}$$

where

$$\hat{Q} = \frac{q}{2} + i\hbar\frac{\partial}{\partial p}, \tag{14}$$

$$\hat{P} = \frac{p}{2} - i\hbar\frac{\partial}{\partial q},$$

satisfy $[\hat{Q},\hat{P}]=i\hbar$ and, therefore, they define phase-space representation of position and momentum. This particular representation corresponds to the gauge-transformed coherent states kernel $e^{-ipq/2\hbar}K_{\mathrm{CS}}(q,p;q')$. Note that Eq. (14) give a highly reducible representation of the commutation relation. To compensate this reducibility, admissible wave functions $\psi(\Gamma)$ have to satisfy certain supplementary condition. Following Harriman [24], this condition is given by

$$\left(\frac{1}{\lambda}\Delta\hat{Q} - \frac{i\lambda}{\hbar}\Delta\hat{P}\right)\psi(\Gamma) = 0, \tag{15}$$

where $\Delta\hat{Q}=\hat{Q}-q$, $\Delta\hat{P}=\hat{P}-p$. Let us note that (15) may be regarded as a defining condition for the Hilbert space $\mathcal{H}_K$. Clearly, there are square-integrable functions of two real variables that do not satisfy this condition; that is, $\mathcal{H}_K$ is a proper subspace of $L^2(\mathbb{R}^2)$. In the case of Bargmann kernel (4), such a condition restricts the space of square-integrable functions to the class of entire functions on the complex plane.

Recently, the following stationary Schrödinger equation

$$\left[\frac{1}{2m}\left(\frac{p}{2} - i\hbar\frac{\partial}{\partial q}\right)^2 + V\left(\frac{q}{2} + i\hbar\frac{\partial}{\partial p}\right)\right]\psi(\Gamma) = E\psi(\Gamma), \tag{16}$$

was postulated in [28]. Now, $\psi(\Gamma)$ denotes an arbitrary phase-space representation; that is, the integral kernel $K(\Gamma;q)$ in (1) is not specified. The general solution of (16) reads as follows [Eq. (11) in [28]):

$$\psi(\Gamma) = e^{-iqp/2\hbar}\int g(y)\varphi(q+y)e^{-(i/\hbar)py}dy, \tag{17}$$

where $g(y)$ is an arbitrary nonzero function and $\varphi(q)$ is the eigenfunction of the Schrödinger equation in coordinate representation corresponding to the eigenvalue E. Note that the function $g(y)$ uniquely defines an integral kernel K_g by

$$K_g(q,p;q') = e^{-ip(q'-q/2)/\hbar} g(q'-q). \tag{18}$$

Note that

$$\int \overline{K}(\Gamma;q')K(\Gamma;q'')d\Gamma = \delta(q'-q'') \int g^*(q'-q)g(q''-q)dq, \tag{19}$$

and hence unitarity condition (2) implies

$$\int |g(y)|^2 dy = 1. \tag{20}$$

In particular, the following Gaussian:

$$g(y) = (\pi\lambda^2)^{-1/4} e^{-y^2/2\lambda^2} \tag{21}$$

does satisfy (20) and one finds for the corresponding $K_g(q,p;q')=e^{-ipq/2\hbar}K_{\mathrm{CS}}(q,p;q')$.

III. A FAMILY OF THE SCHRÖDINGER EQUATION IN PHASE SPACE

Let us observe that the representation (14) may be generalized to the whole family of representations. It is convenient to scale phase-space variables $(q,p)\to(2q,2p)$ and to introduce

$$\hat{Q}_s = q + (1-s)\frac{i\hbar}{2}\frac{\partial}{\partial p}, \tag{22}$$

$$\hat{P}_s = p - (1+s)\frac{i\hbar}{2}\frac{\partial}{\partial q},$$

with $s \in \mathbb{R}$. One can easily verify that $[\hat{Q}_s,\hat{P}_s]=i\hbar$.

In analogy to (16) let us postulate the following family of Schrödinger equations:

$$\left[\frac{1}{2m}\left(p-(1+s)\frac{i\hbar}{2}\frac{\partial}{\partial q}\right)^2 + V\left(q+(1-s)\frac{i\hbar}{2}\frac{\partial}{\partial p}\right)\right]\times\ \psi_s(\Gamma) = E\psi_s(\Gamma). \tag{23}$$

To solve this equation assume that for $s\neq -1$

$$\psi_s(q,p) = \exp\left\{\frac{-2i}{\hbar}\frac{pq}{1+s}\right\}\phi_s(q,p). \tag{24}$$

One obtains the following equation for ϕ_s:

$$\left[-\frac{\hbar}{2m}\frac{(1+s)^2}{4}\frac{\partial^2}{\partial q^2} + V\left(\frac{2q}{s+1}+(1-s)\frac{i\hbar}{2}\frac{\partial}{\partial p}\right)\right]\times\phi_s(q,p) = E\phi_s(q,p). \tag{25}$$

Now we expand the potential V as a Taylor's series about $[i\hbar(1-s)/2]\partial/\partial p$ for given q and use the partial Fourier transform

$$\phi_s(q,p) = \int \chi_s(q,y)e^{-ipy/\hbar}. \tag{26}$$

Further more, multiplying both sides of (25) by $\exp\{-ipy'/\hbar\}$ and integrating over p, one obtains

$$\left[-\frac{\hbar^2}{2m}\frac{(1+s)^2}{4}\frac{\partial^2}{\partial q^2} + V\left(\frac{2}{s+1}q+\frac{1-s}{2}y\right)\right]\chi_s(q,y) = E\chi_s(q,y), \tag{27}$$

which defines the standard Schrödinger equation in the ξ representation

$$\xi = \frac{2}{s+1}q + \frac{1-s}{2}y. \tag{28}$$

Therefore, the general solution of (27) reads as follows:

$$\chi_s(q,y) = g_s(y)\varphi(\xi), \tag{29}$$

where $g_s=g_s(y)$ is an arbitrary nonzero function and $\varphi(\xi)$ satisfies

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi^2} + V(\xi)\right]\varphi(\xi) = E\varphi(\xi). \tag{30}$$

Finally, the general solution of (24) has the following form:

$$\psi_s(q,p) = \exp\left\{-\frac{2i}{\hbar}\frac{pq}{s+1}\right\}\int dy\, e^{-ipy/\hbar}\times g_s(y)\varphi\left(\frac{2}{s+1}q + \frac{1-s}{2}y\right). \tag{31}$$

Clearly, for each $g_s(y)$ it defines a family of kernels K_g^s

$$\psi_s(q,p) = \int K_g^s(q,p;q')\varphi(q')dq', \tag{32}$$

given by

$$K_g^s(q,p;q') = \frac{2}{1-s}g_s\left(\frac{2q'}{1-s} - \frac{4q}{1-s^2}\right)\times\exp\left(-\frac{2i}{1-s}\frac{p(q'-q)}{\hbar}\right). \tag{33}$$

Again, the requirement of unitarity (2) implies the following condition for the function $g_s(s\neq -1)$:

$$\int |g_s(y)|^2 dy = \frac{2}{|1+s|}. \tag{34}$$

Note that for $s=1$ the formula for the kernel considerably simplifies

$$K_g^{s=1}(q,p;q') = e^{-ipq/\hbar}\tilde{g}(p)\delta(q-q'), \tag{35}$$

where $\tilde{g}$ stands for the Fourier transform of g. In this case the ave function $\varphi(q)$ has the following phase-space representation

$$\psi(q,p) = e^{-ipq/\hbar}\tilde{g}(p)\varphi(q). \tag{36}$$

IV. STAR PRODUCT

Now we show that the family of Eqs. (23) is closely related to the family of quasidistributions function in phase space.

To describe all quantum phenomena in phase space, we have to determine a relationship between operators and functions on the classical phase space. This correspondence is of course not unique. The most famous is based on the Wigner–Weyl transform $\mathcal{F}_{\rm WW}$: if $f(q,p)$ is a phase-space function then one defines the corresponding operator $\hat{F}$

$$\hat{F} = \mathcal{F}_{\rm WW}(f),$$

by

$$\hat{F} = \int d\sigma \int d\tau \tilde{f}(\sigma,\tau) e^{i(\sigma\hat{q}+\tau\hat{p})}, \tag{37}$$

where $\tilde{f}$ denotes the Fourier transform of f

$$\tilde{f}(\tau,\sigma) = \frac{1}{2\pi}\int d\sigma \int d\tau\, f(q,p) e^{-i(\sigma q+\tau p)}. \tag{38}$$

The inverse transform, i.e., $\hat{F}\rightarrow f$, is defined as follows:

$$f(q,p) = \int dy \left\langle q - \frac{1}{2}y|\hat{F}|q + \frac{1}{2}y\right\rangle e^{ipy/\hbar}. \tag{39}$$

If $\hat{F}$ corresponds to a density operator $\hat{\rho}$ then its inverse Wigner-Weyl transform recovers celebrated Wigner function $W=\mathcal{F}_{\rm WW}^{-1}(\hat{\rho})$

$$W(q,p) = \frac{1}{\pi\hbar}\int dy \langle q-y|\hat{\rho}|q+y\rangle e^{2ipy/\hbar}. \tag{40}$$

Now, the noncommutative multiplication of operators introduces the following noncommutative multiplication of functions:

$$\hat{F}_1 \cdot \hat{F}_2 = \mathcal{F}_{\rm WW}(f_1 \star f_2). \tag{41}$$

The formula for the star product $\star$ was derived long ago by Groenewold [17] and Moyal [18]

$$f_1 \star f_2 = f_1 \exp\left\{\frac{i\hbar}{2}(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q)\right\} f_2, \tag{42}$$

where $\overleftarrow{\partial}(\overrightarrow{\partial})$ act on the left (right) side. The Moyal product is associative

$$(f_1 \star f_2)\star f_3 = f_1 \star (f_2 \star f_3),$$

but it is noncommutative

$$f_1 \star f_2 \neq f_2 \star f_1.$$

Recall, that the operator $e^{a\partial_x}$ acts as a generator of translations: $e^{a\partial_x}f(x)=f(x+a)$. Therefore, the defining formula (42) may be rewritten in the following form:

$$f_1(q,p)\star f_2(q,p) = f_1\left(q + \frac{i\hbar}{2}\partial_p, p - \frac{i\hbar}{2}\partial_q\right) f_2(q,p). \tag{43}$$

Now we can make crucial observation: Equation (23) for $s=0$ has the following form:

$$H \star \psi_{s=0} = E\psi_{s=0}, \tag{44}$$

where $H(q,p)$ is the classical Hamilton function.

It turns out that the similar structure may be established also for $s\neq 0$. Let us introduce the following family of Wigner-Weyl transforms:

$$\hat{F}_s = \mathcal{F}^s_{\rm WW}(f)$$

by

$$\hat{F}_s = \int d\sigma \int d\tau \tilde{f}(\sigma,\tau) e^{i(\sigma\hat{q}+\tau\hat{p})} e^{-is\sigma\tau/2}. \tag{45}$$

Clearly, for $s=0$ one recovers (37). This formula enables one to introduce the family of star products$\star_s$

$$\hat{A}_s \cdot \hat{B}_s = \mathcal{F}^s_{\rm WW}(a\star_s b), \tag{46}$$

which reduces to (41) for $s=0$. One easily finds

$$a\star_s b = a \exp\left\{\frac{i\hbar}{2}((1-s)\overleftarrow{\partial}_q\overrightarrow{\partial}_p - (1+s)\overleftarrow{\partial}_p\overrightarrow{\partial}_q)\right\} b, \tag{47}$$

which is equivalent to

$$a(q,p)\star_s b(q,p) = a\left(q + (1-s)\frac{i\hbar}{2}\partial_p, p - (1+s)\frac{i\hbar}{2}\partial_q\right) b(q,p). \tag{48}$$

Therefore, the family of Schrödinger equations in phase space (23) may be rewritten as follows:

$$H\star_s \psi_s = E\psi_s. \tag{49}$$

This shows that the family of Eq. (23), which is a direct generalization of equations used in [22,28] is closely related to the noncommutative structure induced by the family of star products.

V. WIGNER FUNCTION VERSUS PHASE-SPACE WAVE FUNCTION

Both Wigner function $W(q,p)$ and the wave function $\psi(\Gamma)$ are objects defined on the classical phase space. $\psi(\Gamma)$ satisfies (44)

$$H \star \psi = E\psi. \tag{50}$$

What about W? It turns out that the stationary Wigner function is uniquely determined by the following two equations [20]:

$$H \star W = W \star H = EW; \tag{51}$$

that is, W satisfies the same equations as ψ and, additionally, it fulfills $W\star H=EW$, which is equivalent to

$$\left[\frac{1}{2m}\left(p + \frac{i\hbar}{2}\partial_q\right)^2 + V\left(q - \frac{i\hbar}{2}\partial_p\right)\right] W(q,p) = EW(q,p). \tag{52}$$

We stressed, that there are infinite solutions of (50) and is only one solution of (51). Clearly, the Wigner function does belong to the solutions of (50). Indeed, taking

$$g(y) = \varphi^*(-y/2), \tag{53}$$

Eq. (17) implies

$$\psi(\Gamma) = \int dy\; e^{-ipy/\hbar}\varphi^*\left(q-\frac{y}{2}\right)\varphi\left(q+\frac{y}{2}\right); \tag{54}$$

that is, $\psi(\Gamma)=2\pi\hbar W$, where $W(q,p)$ is a Wigner function corresponding to φ. We stress that the choice for $g(y)$ in (53) is state dependent and not universal, i.e., it reproduces the Wigner function for φ only. Other wave functions are mapped to $\psi(\Gamma)$, which are not related to their Wigner functions.

As an example let us compare the solutions of (50) and (51) for the harmonic oscillator. Taking $g(y)$ as in (21) one obtains from (17) [22,24,27,28] the following formulas corresponding to nth energy eigenstate $\varphi_n(q)$:

$$\psi_n(\Gamma) = \frac{1}{\sqrt{n!}}\Gamma^{*n}\exp\left(-\frac{H}{2\hbar\omega}\right), \tag{55}$$

with Γ given by (8), whereas

$$W_n(q,p) = \frac{(-1)^n}{\pi\hbar}L_n\left(\frac{4H}{\hbar\omega}\right)\exp\left(-\frac{2H}{\hbar\omega}\right), \tag{56}$$

where L_n denotes nth Laguerre polynomial. Because of $|\Gamma|^2 = H/\hbar\omega$, one has for the probability distribution of transition from φ_n to the coherent state $|\Gamma\rangle$

$$|\psi_n(\Gamma)|^2 = \frac{1}{n!}\left(\frac{H}{\hbar\omega}\right)^n \exp-\left(\frac{H}{\hbar\omega}\right). \tag{57}$$

Clearly, both $|\psi_n|^2$ and W_n depends only on the oscillator energy H and both are normalized according to

$$\int |\psi_n|^2 d\Gamma = \int W_n dq dp = 1. \tag{58}$$

Moreover, it its easy to show

$$\int W_n^2 dq dp = \frac{1}{2\pi\hbar}. \tag{59}$$

Now, the family of Wigner-Weyl transforms $\mathcal{F}^s_{\mathrm{WW}}$ enables one to introduce the following family of Wigner functions: $W_s=(\mathcal{F}^s_{\mathrm{WW}})^{-1}(\hat{\rho})$, where $\hat{\rho}$ stands for the density operator. One finds

$$W_s(q,p) = \frac{1}{2\pi\hbar}\int dy\; e^{ipy/\hbar} \times \left\langle q-(1-s)\frac{y}{2}\middle|\hat{\rho}\middle|q+(1+s)\frac{y}{2}\right\rangle, \tag{60}$$

which reduces to $W(q,p)$ for $s=0$. The family W_s was introduced by Agarwal and Wolf [16]. It satisfies two basic properties: it is normalized

$$\int W_s(q,p) dq dp = 1,$$

and for any quantum observable $\hat{F}$

$$\mathrm{Tr}(\hat{F}\hat{\rho}) = \int W_s(q,p) f_{-s}(q,p) dq dp,$$

where $f_{-s}=(\mathcal{F}^{-s}_{\mathrm{WW}})^{-1}(\hat{F})$. For $s=0$ the last formula reproduces well known property of the Wigner function

$$\mathrm{Tr}(\hat{F}\hat{\rho}) = \int W(q,p) f(q,p) dq dp.$$

Moreover, W_s provides correct quantum marginals

$$\int dq\; W_s(q,p) = \langle p|\hat{\rho}|p\rangle \tag{61}$$

$$\int dp\; W_s(q,p) = \langle q|\hat{\rho}|q\rangle.$$

It turns out that stationary s-Wigner functions W_s are uniquely determined by

$$H\star_s W_s = W_s\star_s H = E W_s. \tag{62}$$

Equation (49) for ψ_s has infinite number of solutions, whereas the set of two Eqs. (62) has only one solution

$$\psi_s(\Gamma) = \int dy\; e^{-ipy/\hbar} \times \varphi^*\left(q-\frac{s+1}{2}y\right)\varphi\left(q+\frac{1-s}{2}y\right), \tag{63}$$

i.e., $\psi_s(\Gamma)=2\pi\hbar W_s(q,p)$. Therefore, W_s is only one particular solution of (49). It is easy to see that taking

$$g_s(y) = \varphi^*\left(-\frac{s+1}{2}y\right) \tag{64}$$

in (31) one obtains $\psi_s(\Gamma)$ given by (63). In particular for $s=1$ one obtains so-called Kirkwood-Rihaczek function $K(q,p)=W_{s=1}(q,p)$, which, in the case of pure state φ, reduces to

$$K(q,p) = e^{ipq/\hbar}\tilde{\varphi}^*(p)\varphi(q). \tag{65}$$

It was introduced by Kirkwood [32] as an alternative for the Wigner function. Then, in 1968, the same formula was rediscovered by Rihaczek [33] in the context of signal time-frequency distributions (see [34] for a useful review). Recently, this function was analyzed and applied in various contexts in [35–40].

VI. DISCUSSION

Both the phase-space wave function $\psi(\Gamma)$ and s-ordered Wigner function W_s encode the entire information about the quantum state $\varphi(q)$. Because of the basic property

$$\int |\psi(\Gamma)|^2 d\Gamma = 1, \tag{66}$$

some authors call $|\psi(\Gamma)|^2$ a probability distribution in phase space. Clearly, quantum mechanics does not allow for a genuine probability distribution in q and p! To interpret $|\psi(\Gamma)|^2$ correctly note that Eq. (1) may be rewritten as the following inner product

$$\psi(\Gamma) = \langle\varphi_\Gamma|\varphi\rangle, \tag{67}$$

where

$$\varphi_\Gamma(q') = K^*(\Gamma;q'). \tag{68}$$

Let us consider kernels defined by (18). Then, due to (20), $\varphi_\Gamma(q')$ is a normalized wave function in the position representation. Therefore, $|\psi(\Gamma)|^2$ is the probability density of transition from the state φ to state φ_Γ. In particular for the coherent state kernel K_{CS} one has $|\psi(\Gamma)|^2 = |\langle\Gamma|\varphi\rangle|^2$ which defines the Husimi function for the state φ.

Now, the Wigner function defined quasi-distribution such that $\int W\, dqdp = 1$. Since W is a special solution of the Schrödinger equation in phase space one has

$$W_\varphi(q,p) = \frac{1}{2\pi\hbar}\langle\varphi_\Gamma|\varphi\rangle, \tag{69}$$

where W_φ is the Wigner function corresponding to φ. Equations (33) and (53) imply

$$\varphi_\Gamma(q') = 2\varphi^*(2q - q')e^{-2ip(q'-q)/\hbar}. \tag{70}$$

It should be stressed that the phase formulation based on the wave function $\psi(\Gamma)$ is restricted to pure states only, whereas the approach based on Wigner function works perfectly for general mixed states ρ. Therefore, this approach is much more general. Note that for mixed states one has

$$\int W^2 d\Gamma \leqslant \frac{1}{(2\pi\hbar)^2}, \tag{71}$$

and the equality holds for pure states only. Therefore, for general mixed states $(2\pi\hbar)^2W^2$ cannot be interpreted as a probability distribution.

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