Semiclassical limit of the entanglement in closed pure systems

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We discuss the semiclassical limit of the entanglement for the class of closed pure systems. By means of analytical and numerical calculations, we obtain two main results: (i) the short-time entanglement does not depend on Planck's constant and (ii) the long-time entanglement increases as more semiclassical regimes are attained. On one hand, this result is in contrast with the idea that the entanglement should be destroyed when the macroscopic limit is reached. On the other hand, it emphasizes the role played by decoherence in the process of emergence of the classical world. We also found that, for Gaussian initial states, the entanglement dynamics may be described by an entirely classical entropy in the semiclassical limit.

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I. INTRODUCTION

Understanding the emergence of all aspects of the classical world from the quantum theory is far from being a trivial task. In fact, it is a recurrent question which has been asked since the early days of quantum mechanics. However, at the level of the expectation values of position and momentum, the classical limit is reasonably understood [1–4], according to the conceptual scheme we try to sketch in the next few lines.

Concerning the Newtonian trajectories, which are based on the assumption of complete knowledge of the initial conditions, their emergence from the quantum expectation values is predicted by the Ehrenfest theorem, valid for initially localized states. Within the Ehrenfest time scale, expectation values agree with the classical trajectories and, in this sense, the quantum mechanics recovers the determinism of the Newtonian theory. The classical limit ($\hbar \to 0$) in this case is in general asymptotic and does not require any further mechanism such as decoherence. In other words, there exists a formal classical limit for expectation values even in the context of closed systems. Although such a mathematical limit is not unique, one possible way to obtain it is particularly convenient: a two-degree-of-freedom classical Hamiltonian may be produced with the assumption of a separable dynamics of coherent states [5]. In this approach, the classical theory emerges as a consequence of the absence of the entanglement.

After the Ehrenfest time scale, the Newtonian mechanics is no longer able to mimic the quantum expectation values of observables and the single trajectory determinism is missed. This means that predictions can only be made by means of averages over ensembles. But, even in this case, classical equations of motion are applicable (within the Liouville formalism). This is ensured by a mechanism known as *decoherence* [6]. Environment, to which all systems are inevitably coupled, destroys quantum correlations and turns quantum predictions identical to those offered by the classical statistical formalism. Since the decoherence time is in general much smaller than the Ehrenfest time, the correspondence between the quantum and classical world will be complete for all instants of the dynamics.

Within the above described lines of thought, a natural step before the introduction of the action of the environment is to analyze the classical limit of an intrinsically quantum quantity—the *entanglement*—for the class of *closed pure systems*, for which the theoretical semiclassical limit ($\hbar \to 0$) is well established at the level of expectation values. Here, a word concerning the entanglement is in order, recalling that the entanglement is the greatest resource offered "exclusively" by the quantum world, which allows for the realization of the most challenging ideas, such as quantum information processing and quantum computation [7–9]. Since entanglement is regarded as an intrinsically quantum property with no classical analog, it seems to be an adequate modern tool for the study of the semiclassical behavior of quantum mechanics.

We then ask about the behavior of the entanglement in such situations in which an underlying classical dynamics is well known. This question motivates us to analyze the semiclassical limit of the entanglement dynamics for closed pure systems in which classicality is achieved by means of macroscopic coherent states, i.e., minimal uncertainty states $|\alpha\rangle$ with $|\alpha|^2 \gg 1$.

At first glance, one may be induced to expect a decrease in the entanglement in the semiclassical limit. However, we show analytically and also by means of numerical examples that this is not a precise idea: in fact, the short-time entanglement is $\hbar$-independent for closed systems (Sec. II).

This paper is organized as follows. In Sec. II, we discuss a short-time expansion for the entanglement in an arbitrary pure bipartite system. In Sec. III, a semiclassical analysis of the whole dynamics of the entanglement is numerically investigated for two different models, and an explanation in

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terms of mechanisms that are classically understandable as well is given. Section IV is reserved for some concluding remarks.

II. THE SHORT-TIME ENTANGLEMENT

For globally pure bipartite systems, there exist suitable entropic quantities obeying all requirements for an adequate entanglement measure [10,11]. Particularly useful is the reduced linear entropy, which offers in general the same information given by the von Neumann entropy but with a much less computational effort.

The reduced linear entropy (RLE) for a bipartite system belonging to the Hilbert space $\mathcal{E}=\mathcal{E}_u\otimes\mathcal{E}_v$ is defined by

$$S(t) = 1 - \mathrm{Tr}_z[\rho_z^2(t)], \tag{1}$$

in which $\rho_{u,v}(t)=\mathrm{Tr}_{v,u}[\rho(t)]$ is the reduced density operator. The index z denotes the subsystems "u" and "v." The density operator ρ satisfies the von Neumann equation $\imath\hbar\dot{\rho}=[H,\rho]$, where H is the Hamiltonian operator. This is a dynamical measure of the purity of the subsystem z in the unitary bipartite dynamics of initially pure states. For this case, the Schmidt decomposition [12] guarantees that RLE is also a measure of the entanglement between the parts [13]. Furthermore, the Araki-Lieb inequality [14] guarantees that the RLE of the subsystems is identical for pure states.

In order to obtain a semiclassical expansion for the entanglement in the approximation of short times, we write Dyson's series for the density operator up to the order t^2,

$$\rho(t) \simeq \rho_0 + \frac{[H,\rho_0]}{\imath\hbar}t + \frac{[H,[H,\rho_0]]}{2(\imath\hbar)^2}t^2, \tag{2}$$

where $\rho_0=|u_0\rangle\langle u_0|\otimes|v_0\rangle\langle v_0|$ is an initially disentangled pure state composed of the product of bosonic coherent states corresponding to the subsystems u and v. Now we apply the partial trace over the subsystem u in the coherent state basis to get

$$\rho_v(t) \simeq \rho_v^{(0)} + \rho_v^{(1)}t + \frac{1}{2}\rho_v^{(2)}t^2, \tag{3a}$$

$$\rho_v^{(0)} = |v_0\rangle\langle v_0|, \tag{3b}$$

$$\rho_v^{(1)} = \int \frac{d^2u_1}{\pi}\frac{\langle u_1|[H,\rho_0]|u_1\rangle}{\imath\hbar}, \tag{3c}$$

$$\rho_v^{(2)} = \int \frac{d^2u_1}{\pi}\frac{\langle u_1|[H,[H,\rho_0]]|u_1\rangle}{(\imath\hbar)^2}. \tag{3d}$$

Then, the RLE defined by Eq. (1) may be written as

$$S(t) \simeq S^{(0)} + S^{(1)}t + S^{(2)}t^2, \tag{4a}$$

$$S^{(0)} = 1 - \mathrm{Tr}_v[\rho_v^{(0)}], \tag{4b}$$

$$S^{(1)} = - \mathrm{Tr}_v[\rho_v^{(0)}\rho_v^{(1)} + \rho_v^{(1)}\rho_v^{(0)}], \tag{4c}$$

$$S^{(2)} = - \mathrm{Tr}_v\left[\frac{\rho_v^{(0)}\rho_v^{(2)} + \rho_v^{(2)}\rho_v^{(0)}}{2} + \rho_v^{(1)}\rho_v^{(1)}\right], \tag{4d}$$

where $\mathrm{Tr}_v[\cdot]=\int(d^2v_1/\pi)\langle v_1|\cdot|v_1\rangle$.

Straightforward manipulations on these equations yield $S^{(0)}=S^{(1)}=0$. Consequently, $S(t)\simeq S^{(2)}t^2$. This result, which has already appeared in the literature [15], is a direct consequence of both the purity and the separability of the initial state.

Defining the dimensionless Hamiltonian

$$\mathbb{H} \equiv \frac{Ht}{\hbar}, \tag{5}$$

we put Eq. (4a) in the following compact form:

$$S(t) \simeq 2[\mathcal{C}_{00}(\mathbb{H}) + \mathcal{C}_{11}(\mathbb{H}) - \mathcal{C}_{10}(\mathbb{H}) - \mathcal{C}_{01}(\mathbb{H})], \tag{6}$$

in which he have defined the correlations

$$\mathcal{C}_{00} = (\langle\psi_0|\mathbb{H}|\psi_0\rangle)^2, \tag{7a}$$

$$\mathcal{C}_{01} = \langle\psi_0|\mathbb{H}\rho_1(0)\mathbb{H}|\psi_0\rangle, \tag{7b}$$

$$\mathcal{C}_{10} = \langle\psi_0|\mathbb{H}\rho_2(0)\mathbb{H}|\psi_0\rangle, \tag{7c}$$

$$\mathcal{C}_{11} = \langle\psi_0|\mathbb{H}^2|\psi_0\rangle, \tag{7d}$$

with $|\psi_0\rangle=|u_0\rangle\otimes|v_0\rangle$. Interestingly, by Eq. (6) we may put the conditions for the existence of short-time entanglement in terms of the inequality

$$2(\mathcal{C}_{00} + \mathcal{C}_{11} - \mathcal{C}_{10} - \mathcal{C}_{01}) > 0. \tag{8}$$

Now we look for a semiclassical expansion for the short-time entanglement given by Eq. (6) in the basis of coherent states. We start by defining a general two-degree-of-freedom classical Hamiltonian,

$$\mathcal{H}(q_u,p_u,q_v,p_v) = \sum_{n,m,l,k} c_{nmlk}q_u^n p_u^m q_v^l p_v^k. \tag{9}$$

This is an explicit $\hbar$-independent function. Following the ordered quantization procedure [4], we obtain the corresponding Hamiltonian operator

$$H = \mathcal{S}_u\mathcal{S}_v \sum_{n,m,l,k} c_{nmlk}Q_u^n P_u^m Q_v^l P_v^k, \tag{10}$$

with $[Q_z,P_z]=\imath\hbar$. $\mathcal{S}_z=\exp[-(\imath\hbar/2)\partial_{Q_z}\partial_{P_z}]$ is the symmetric ordering operator [4]. This is a suitable quantization method for our purposes since it factorizes the dependences in $\hbar$. There are other sources of $\hbar$, namely the operator $\mathbb{H}$ and the parametrizations of the coherent states for the phase space.

The calculations of all terms in Eq. (6) are made by inserting Eq. (10) in Eq. (7). Then, using the unities of the coherent states basis, namely $\mathbf{1}_u=\int(d^2u_1/\pi)|u_1\rangle\langle u_1|$ and $\mathbf{1}_v=\int(d^2v_1/\pi)|v_1\rangle\langle v_1|$, we are led to

$$\frac{\hbar^2}{2t^2}S(t) = (\langle u_0v_0|H|u_0v_0\rangle)^2 + \int \frac{d^2u_1}{\pi}\frac{d^2v_1}{\pi}\langle u_0v_0|H|u_1v_1\rangle\langle u_1v_1|H|u_0v_0\rangle - \int \frac{d^2u_1}{\pi}\langle u_0v_0|H|u_1v_0\rangle\langle u_1v_0|H|u_0v_0\rangle - \int \frac{d^2v_1}{\pi}\pi\langle u_0v_0|H|u_0v_1\rangle\langle u_0v_1|H|u_0v_0\rangle. \quad (11)$$

The function $\langle u_0v_0|H|u_1v_1\rangle$ is the most general kernel we have to manipulate in the calculation of the correlations. By Eq. (10) we obtain

$$\langle u_0v_0|H|u_1v_1\rangle = \sum_{n,m,k,l} c_{nmkl}\langle u_0|\mathcal{S}_uQ_u^nP_u^m|u_1\rangle \times \langle v_0|\mathcal{S}_vQ_v^kP_v^l|v_1\rangle. \quad (12)$$

The properties of the symmetric ordering operator [4] allow us to write

$$\frac{\langle u_0|\mathcal{S}_uQ_u^nP_u^m|u_1\rangle}{\langle u_0|u_1\rangle} = e^{(1/2)\partial_{u_0^*}\partial_{u_1}}[(q_u^{(01)})^n(p_u^{(01)})^m], \quad (13a)$$

$$q_u^{(ij)} \equiv \frac{\langle u_i|Q_u|u_j\rangle}{\langle u_i|u_j\rangle}, \quad p_u^{(ij)} \equiv \frac{\langle u_i|P_u|u_j\rangle}{\langle u_i|u_j\rangle}, \quad (13b)$$

with similar expressions for v. Notice that q^{ij} and p^{ij} correspond to the usual canonical phase-space pair when $i=j$. Otherwise, when $i\neq j$, they are complex numbers composed of combinations of distinct canonical pairs.

Given the usual parametrization of the coherent state label u (or v) for the phase space [Eq. (18)], we may write $\partial_u = \sqrt{(\hbar/2)}(\partial_{q_u} - \imath\partial_{p_u})$. Considering smooth wave packets at $t=0$, we may regard this partial derivative as a small parameter and the exponential operator in Eq. (13a) may be expanded in few terms. After performing such adequate expansions in Eq. (13a) we return to Eq. (12) to calculate the semiclassical expansion of the kernel.

The next step is to calculate the integrals in Eq. (11), which is done by means of the following useful formula [4]:

$$\int \frac{d^2u}{\pi}|\langle u_0|u\rangle|^2 f(u_0,u) = [e^{(\hbar/4)\nabla_u^2}f(u_0,u)]_{u=u_0}, \quad (14)$$

with $\nabla_u^2 = \partial_{q_u}^2 + \partial_{p_u}^2$. This relation emerges from the fact that $|\langle u_0|u\rangle|^2$ is a Gaussian weight function. Once again we may get a semiclassical expression by expanding the exponential operator in the first orders in $\hbar$. In fact, the procedure described up to here must be performed in a consistent way such that the short-time RLE may be written as

$$S(t) \simeq t^2[O(\hbar^{-2}) + O(\hbar^{-1}) + O(\hbar^0) + \cdots]. \quad (15)$$

Notice that the classical limit $\hbar \to 0$ seems to yield a divergence of the short-time entanglement, which would be a quite unexpected result. However, an exhaustive calculation shows that $O(\hbar^{-2}) = O(\hbar^{-1}) = 0$ and $O(\hbar^0) \neq 0$, in general. This derivation is really lengthy and tedious, but does not require any further recipe beyond that presented above and it will be omitted here.

Concluding this calculation, in a semiclassical regime, where $\hbar$ is finite, but arbitrarily smaller than a typical action of the system, the RLE is given by

$$S(t) \simeq t^2 O(\hbar^0), \quad (16)$$

showing that the short-time entanglement *does not depend* on $\hbar$ in a semiclassical regime.

The constant $O(\hbar^0)$ is constituted by a sum of terms involving first-order derivatives (in phase-space variables) of the Hamiltonian function (9). Note that the possibility of expressing the first contributions to the entanglement dynamics in terms of classical quantities is a direct consequence of the special basis used in the calculation.

Result (16), which is general within the class of bipartite bosonic systems in pure states, attests to the fact that the entanglement does not tend asymptotically to the classically expected limit $S(t)=0$, as does those well-behaved quantities such as expectation values of canonical operators Q and P.

Also it has to be remarked that the quadratic dependence in time found for the short-time entanglement is independent of the integrability of the underlying classical system, i.e., it is always algebraic no matter whether the classical system is chaotic or regular. In an approach based on the Loschmidt echo dynamics [16], different behavior has been predicted for the entanglement in chaotic and regular regimes. However, this has not been done for the time scale we are considering, and furthermore, in that case the entanglement generation is obtained by means of two distinct unitary evolutions (echo operator). In this sense, there is no conflict with our results. In fact, the quadratic dependence of the short-time linear entropy has already been found in [15]. It is just a consequence of the initial separability of the quantum state. Recent studies also confirm our results concerning the algebraic dependence [17] and the $\hbar$ independence [18] of the short-time entanglement.

Next we present two numerical examples confirming the above result and we also discuss the long-time behavior of the entanglement.

III. NUMERICAL ANALYSIS

A. Classical states

Since we are interested in the quantum-to-classical aspect of the entanglement, we avoid initial states that are initially entangled, mixed, or delocalized. The natural choice is the minimum uncertainty coherent states defined by

$$|v\rangle = D(v)|r\rangle, \quad (17a)$$

$$D(v) = \begin{cases} \exp[v\hat{a}^+ - v^*\hat{a}], \\ \exp\left[\frac{\arctan|v|}{|v|}(v\hat{J}_+ - v^*\hat{J}_-)\right]. \end{cases} \quad (17b)$$

D was defined, respectively, for the harmonic oscillator and for the angular momentum (spins). The reference state $|r\rangle$ stands for the ground states $|0\rangle$, in the Fock basis, and $|J,-J\rangle$, in the angular momentum basis.

The coherent state label, v, has the following common parametrization in terms of classical phase-space variables:

$$v = \begin{cases} \dfrac{q+\imath p}{\sqrt{2\hbar}}, \\ \dfrac{q+\imath p}{\sqrt{\hbar J-(q^2+p^2)}}. \end{cases} \tag{18}$$

For bosonic coherent states, the classicality is attained by taking $|v|^2$ sufficiently macroscopic to guarantee that the average energy $\langle v|H|v\rangle$ will be much greater than a typical spectral distance $E_{n+1}-E_n$. In this case, taking $|v|^2 \to \infty$ is mathematically equivalent to requiring $\hbar \to 0$. On the other hand, for spin coherent states, the classicality emerges by means of a more sophisticated limit [19]: $J\to\infty$, $\hbar\to 0$, and $\hbar\sqrt{J(J+1)}\approx \hbar J = 1$.

We consider Hamiltonian operators like

$$H = H_1 \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes H_2 + H_{12}, \tag{19}$$

the last term being a nonlinear coupling which is able to entangle initially separable coherent states.

B. Dicke model

First, we investigate the entanglement in one of the most important models in quantum optics (QO) and more recently in quantum computation (QC), namely the N-atom Jaynes-Cummings model (or Tavis-Cummings model) [20]. In QO it is mostly known in the so-called rotating-wave approximation (RWA), and describes the coupling of a monochromatic field of frequency ω with N noninteracting two-level atoms with level separation ϵ. Recently, in connection with QC, the model received renewed attention from the solid-state community working on "phonon cavity quantum dynamics" [21,22], Josephson junctions and quantum dots [23], quantum chaos [24], and quantum phase transitions [25]. These works are concerned with the model in the original form conceived by Dicke [26] without the RWA. The Hamiltonian in the last case is composed of the following terms:

$$H_1 = \epsilon J_z, \quad H_2 = \hbar\omega a^\dagger a,$$

$$H_{12} = \frac{G}{\sqrt{2J}}(aJ_+ + a^\dagger J_-) + \frac{G'}{\sqrt{2J}}(a^\dagger J_+ + aJ_-), \tag{20}$$

in which the operators a and $a^\dagger$ are the usual harmonic-oscillator creation and destruction operators associated with the field, and $\hat{J}_+$,$\hat{J}_-$, $\hat{J}_z$ are angular momentum ladder operators. Here, the spin algebra is associated with the atoms, with the total spin given by $J=N/2$. G and G' are real coupling constants, and generally could be taken unequal.

This model possesses a classical counterpart $\mathcal{H}_{N\text{-JCM}}(J)$ obtained in Ref. [27] which presents several interesting properties, e.g., chaotic behavior. Another peculiar feature is the scaling property

$$\frac{\mathcal{H}_{N\text{-JCM}}(J)}{J} = \frac{\mathcal{H}_{N\text{-JCM}}(J')}{J'}. \tag{21}$$

If this relation is satisfied for the pairs $[E(J),J]$ and $[E(J'),J']$, then the associated dynamics will be totally equivalent.

The initial state $|\psi_0\rangle = |v_a\rangle \otimes |v_f\rangle$ is thus constructed by means of the following process. We use the formula $\mathbf{r}_J = \mathbf{r}_1\sqrt{J}$ in order to rescale the vector of initial conditions $\mathbf{r}_J = (q_a,p_a,q_f,p_f)$, keeping the dynamics for $J=1$ as a reference. This guarantees automatically that we obtain a dynamics $[E(J),J]$ equivalent to the unitary $[E(1),1]$. Then, we construct the coherent initial state by putting $\mathbf{r}_J$ in Eqs. (18) and (17). Resuming, we chose initial quantum states with different J, but producing always the same classical dynamics. The semiclassical conditions in this case are satisfied by increasing J and keeping $\hbar$ constant (as it indeed occurs in nature). It may be shown that this is totally equivalent to the mathematical condition $\hbar = 1/J$, with $J\to\infty$ and fixed initial conditions.

In Fig. 1 we show numerical results for the entanglement as a function of time for several values of J. Actually, for long times, the quantity plotted is a kind of mean entanglement of the states "associated" to a given classical trajectory. We calculated it as follows: given a certain classical trajectory, we choose a set of M initial states $|\psi_i(0)\rangle$ centered at points along the trajectory, and calculate the respective RLE's, $S_i(t)$. Then, the averaged quantity is given by

$$S_m(t) \equiv \frac{1}{M}\sum_{i=1}^{M} S_i(t). \tag{22}$$

Such a calculation allows us to observe a smooth mean behavior for entanglement without the characteristic oscillations associated with the border effect [28]. The mean long-time entanglement is then suitably described by a fitting expression [29] given by

$$S_m(t) = A_0(1-e^{-A_1 t}), \tag{23}$$

where A_0 and A_1 are fitting parameters.

At least two aspects are remarkable in this numerical result: (i) the short-time behavior, proportional to t^2, is J-independent and (ii) the plateau value increases with J.

The aspect (i) has already been predicted in Sec. II, but for a different class of Hamiltonians. The results shown in Fig. 1 are, therefore, a numerical verification of Eq. (16) in a more general context, in which the Hamiltonian operator H is not obtained from a classical function by means of a quantization based on bosonic coherent (Gaussian) states. Such an $\hbar$ independence induces us to believe that the rising in the entanglement is determined just by classical sources, as the local departure of neighboring phase-space points. This is indeed suggested by previous studies on quantum chaos [30]. But the main point is that entanglement does not diminish as more macroscopic regimes are attained.

The second aspect may seem even more curious at first glance, since it means that the entanglement increases as we tend to a more semiclassical limit. We will return to this analysis after our second example.

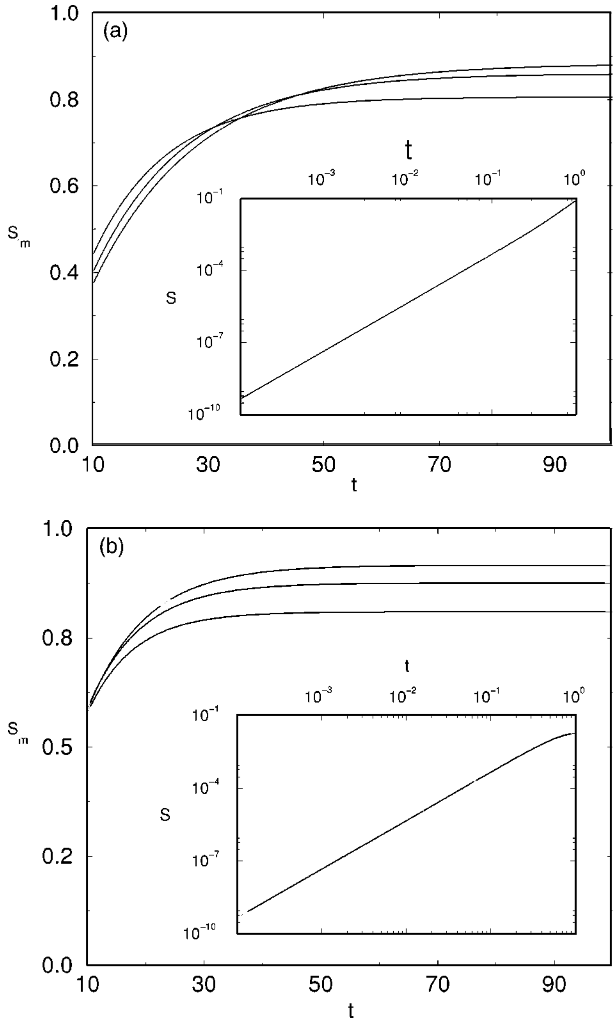

FIG. 1. Mean long-time entanglement for initial coherent states centered in (a) a periodic orbit and in (b) a chaotic trajectory. The insets show the result for the short-time entanglement, which behaves like $S=0.05t^2$, for all values of J. Upper curves correspond to larger values of J, which assumes the values 3.5, 6.5, and 10.5. In these calculations we used $\epsilon=\omega=1.0$, $G=G'=0.35$, and $\hbar=1$, in arbitrary units.

C. Coupled nonlinear oscillators

We thus proceed to show analytical results of the entanglement dynamics in a second model, for which it is possible to understand the behavior of the entanglement in the presence of an effect which has no classical analog, namely the quantum interference. This is done for a model of two resonant oscillators coupled nonlinearly which in certain regimes describe a coupled Bose-Einstein condensate (BEC). The terms of the Hamiltonian (19) are

$$H_k = \hbar\omega\left(a_k^\dagger a_k + \tfrac{1}{2}\right) \quad (k=1,2),$$

$$H_{12} = \hbar\lambda(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1\hat{a}_2^\dagger) + \hbar^2 g(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1)^2. \quad (24)$$

Such a Hamiltonian has been applied to describe an ideal two-mode BEC in the equal scattering length situation [31] with internal Josephson coupling [32]. Recently, the entanglement dynamics between the modes for initially disentangled coherent states was exactly calculated [33]. We reproduce here the analytical result for the RLE,

$$S(t) = 1 - e^{-2|\beta_1(t)|^2}\sum_{n,m}\frac{|\beta_1(t)|^{2n}}{n!}\frac{|\beta_1(t)|^{2m}}{m!} \times e^{-4|\beta_2(t)|^2\sin^2[\hbar g t(n-m)]}, \quad (25)$$

in which $\beta_{1,2}(t)=(\alpha_{1,2}e^{-\imath\omega t}\cos\lambda t-\imath\alpha_{2,1}e^{-\imath\omega t}\sin\lambda t)$ and $\alpha_i=(q_i+\imath p_i)/\sqrt{2\hbar}$. Result (25) predicts that the system recovers totally its purity (disentangled states) at

$$t=\frac{\pi}{g\hbar}, \quad (26)$$

after passing by an intricate interference process (see [33] for details).

Following the reasoning presented in Sec. I, our argumentation will be supported by a classical entropy, which is defined within the classical theory of ensembles. It describes the classical correlation dynamics of initially separable probabilities. This quantity has been shown to evolve very closely to its quantum counterpart, the reduced linear entropy, in cases that are initially separable and in which intrinsic quantum effects are absent [13]. Similar approaches may be found in [34]. We then introduce the *classical reduced linear entropy* (CRLE), defined in Ref. [13] as follows:

$$S_{cl}(t) = 1 - \frac{\mathrm{tr}_k[P_k(t)]}{\mathrm{tr}_k[P_k(0)]}, \quad (27)$$

in which the partial classical trace is given by $\mathrm{tr}_k=\int dq_k dp_k$ for $k=1,2$. $P_{1,2}$ denotes the reduced probability function obtained by $\mathrm{tr}_{2,1}[P(t)]$, in which the joint probability $P(t)$ is a function satisfying the Liouville equation,

$$\frac{\partial P}{\partial t} = \{\mathcal{H},P\}. \quad (28)$$

$\mathcal{H}$ is the classical Hamiltonian function. The normalization in Eq. (27) is necessary to guarantee both an adequate dimensional unity and $S_{cl}(0)=0$. The connection between quantum and classical worlds at $t=0$ is established by taking

$$P(0) = \frac{\langle v_1 v_2|\rho_0|v_2 v_1\rangle}{2\pi\hbar}, \quad (29)$$

where ρ_0 is the pure density operator at $t=0$. The normalization is chosen in such a way that $\mathrm{tr}P=1$. The initial distribution (29) is the unique source of $\hbar$ in the CRLE dynamics. Thus, the semiclassical regime in the classical Liouvillian formalism is associated to strongly localized initial distributions.

The numerical results for both quantum and classical reduced linear entropies are shown in Fig. 2. The dependence of CRLE in time reflects the fact that classical probabilities also become correlated, i.e., initially independent probability distributions are transformed into conditional ones as the dynamical evolution takes place. In some cases, similarities between these quantum and classical quantities indicate that entanglement has indeed a strong statistical character [13].

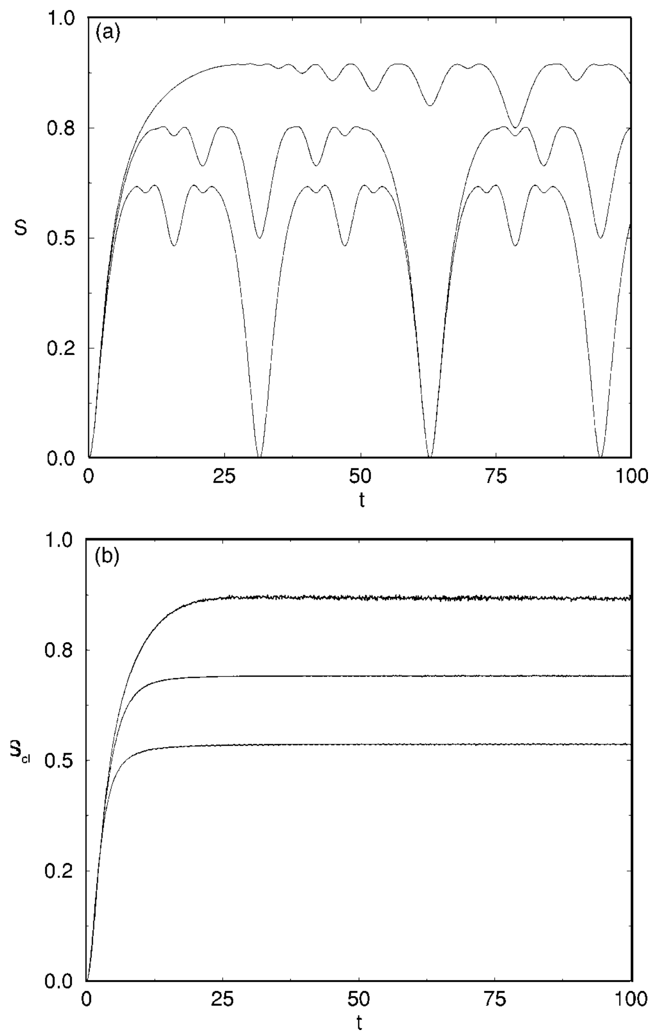

FIG. 2. (a) Entanglement between two Bose-Einstein condensates and (b) the classical reduced linear entropy for the classical counterpart as a function of time. Higher plateaus were attained for smaller values of $\hbar$. The parameter values used were $\omega=1$, $g=0.1$, $\lambda=0.2$, and $q_i=p_i=1$, and $\hbar$ assumed the values 0.1, 0.5, and 1.0 (arbitrary units).

Here, however, remarkable differences emphasize the role played by an intrinsic quantum effect, as is the interference. In fact, it has been shown that interference is the major responsible for such flagrant differences, since there is no analog effect in the classical formalism [33].

Concerning the semiclassical limit, we see that quantum recurrences, and, consequently, quantum interferences, are postponed by the dynamics as $\hbar$ is made small as compared with the classical action $|\beta_1|^2+|\beta_2|^2$. This is indeed predicted analytically by Eq. (26). Accordingly, the resemblance between quantum and classical entropies tends to increase in semiclassical regimes. In this sense, these results allow us to consider the CRLE as the very semiclassical limit of the entanglement. This is an indication that in a semiclassical regime, the occurrence of the entanglement is due solely to spreading effects of the wave function, i.e., statistical effects present also in the classical theory of ensembles.

The entanglement dynamics of the system (24) presents the same qualitative general features of our precedent example, namely short-time behavior independent of $\hbar$ (see [35] for the analytical demonstration) and higher plateaus for more classical regimes. Actually, this scenario has been shown to be quite general for a wide class of nonlinear systems [36].

These results lead us to separate the dynamics of the entanglement in two main time scales. The first one, namely the short-time scale, may be regarded as a classical scale in which entanglement is determined essentially by the vicinity of the initial condition in classical phase space. Accordingly, it has been shown that this scale is connected with the Ehrenfest time for the system (24) [33]. Then, this is a situation in which, although the subsystems are entangled, the quantum state still allows the correspondence between quantum expectation values and classical trajectories.

The second scale, associated to long times (for which the Ehrenfest theorem does not apply), contains all the allowed quantum effects, like interferences, and also indicates the fact that the entropy is extensive with the number of pure states accessible to the dynamics. These remarkably different time scales and the extensivity of the entropy have already been mentioned in slightly different contexts in [33,37], respectively. Also, other time scales have been established in [35].

All results shown here point to the same important conclusion: in closed pure bipartite systems, entanglement exists even in arbitrarily semiclassical regimes. On the other hand, in a strictly formal classical limit ($\hbar=0$), entanglement is indeed expected to disappear completely, since classical points have no statistics associated. In fact, this is a crucial assumption for obtaining the Newtonian trajectories from the quantum formalism [5]. But this is a peculiar mathematical limit which will never be accessible in the physical world. In this sense, our results also indicate that the formal classical limit of entanglement is rather singular for closed systems.

The analysis presented above concerns a theoretical framework for the classical limit of quantum mechanics in closed pure systems. It is a mathematical limit which ensures that quantum mechanics is a universal theory which recovers the classical results in a particular regime of parameters. In closed systems, the limit is asymptotic for expectation values but seems to be rather singular for the entanglement. However, this conceptual scenario does not match the real world of quantum open systems. In this sense, our results must be regarded as the starting point for a more fundamental analysis that takes into account the effects of the environment, to which every real system is coupled. As is well known, decoherence tends to wash out quantum superpositions and the entanglement is expected to disappear under such situation. However, a rigorous test of such assertions requires an adequate entanglement measure for mixed states of continuous variable systems, which is still a very controversial issue.

IV. CONCLUDING REMARKS

We have shown by two numerical examples and by analytical calculations that entanglement does not decrease as the semiclassical regime is attained asymptotically in closed pure systems. In fact, the results point to an increasing amount of entanglement as the equilibrium is attained.

This behavior may be understood by noticing that the semiclassical regime avoids quantum interferences, but it is not able to eliminate the spreading of the wave packet. Even extremely localized coherent states perceive the local spreading caused by the classical flux of trajectories associated. Furthermore, since we note that the entropy is extensive with the number of pure quantum states of the density operator, the increase in the long-time entanglement in the semiclassical limit becomes just a natural fact.

Recently, it was shown that entanglement can always arise in the interaction of an arbitrarily large system in any mixed state with a single qubit in a pure state [38]. This result stresses the role played by the subsystem purity as an enforcer of entanglement, even in a thermodynamic limit of high temperatures. Here, we are concerned with a different kind of classicality, namely that attained by $\hbar \to 0$. Furthermore, we have studied strictly the case where the subsystems are initially in pure states. The common point in these investigations is that the regarded systems do not interact with the environment. In this sense, *unitarity* seems to be a key word in such apparent contradicting semiclassical limits. Then, there is a remaining question to be answered: Is decoherence indeed able to provide the "expected" semiclassical limit for the entanglement?

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