Dirac quantization of the Pais-Uhlenbeck fourth order oscillator

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As a model, the Pais-Uhlenbeck fourth order oscillator with equation of motion: $(d^4q/dt^4)+(\omega_1^2+\omega_2^2)\times(d^2q/dt^2)+\omega_1^2\omega_2^2q=0$ is a quantum-mechanical prototype of a field theory containing both second and fourth order derivative terms. With its dynamical degrees of freedom obeying constraints due to the presence of higher order time derivatives, the model cannot be quantized canonically. We thus quantize it using the method of Dirac constraints to construct the correct quantum-mechanical Hamiltonian for the system, and find that the Hamiltonian diagonalizes in the positive and negative norm states that are characteristic of higher derivative field theories. However, we also find that the oscillator commutation relations become singular in the $\omega_1 \to \omega_2$ limit, a limit which corresponds to a prototype of a pure fourth order theory. Thus the particle content of the $\omega_1=\omega_2$ theory cannot be inferred from that of the $\omega_1 \neq \omega_2$ theory; and in fact in the $\omega_1 \to \omega_2$ limit we find that all of the $\omega_1 \neq \omega_2$ negative norm states move off shell, with the spectrum of asymptotic in and out states of the equal frequency theory being found to be completely devoid of states with either negative energy or negative norm. As a byproduct of our work we find a Pais-Uhlenbeck analog of the zero energy theorem of Boulware, Horowitz, and Strominger, and show how in the equal frequency Pais-Uhlenbeck theory the theorem can be transformed into a positive energy theorem instead.

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I. INTRODUCTION

While attention in physics has by and large concentrated on second order equations of motion, nonetheless, from time to time there has also been some interest in higher derivative theories, with a typical such higher order equation of motion being the scalar field equation of motion

$$(\partial_0^2 - \nabla^2)(\partial_0^2 - \nabla^2 + M^2)\phi(\bar{x},t) = 0, \qquad (1)$$

a wave equation which is based on both second and fourth order derivatives of the field. With the propagator associated with Eq. (1) being given by

$$D(k^2,M^2) = \frac{1}{k^2(k^2+M^2)} = \frac{1}{M^2}\left[\frac{1}{k^2} - \frac{1}{k^2+M^2}\right] \qquad (2)$$

in momentum space, the expectation of canonical reasoning (see e.g., Ref. [1] for a canonical study of theories of such second plus fourth order type) is that a quantization of the theory associated with Eq. (1) would, for $M^2 \neq 0$, lead to a $1/k^2 - 1/(k^2+M^2)$ dipole spectrum consisting of two species of particles, one possessing a positive signature and the other a negative or ghost signature. Indeed, part of the appeal of such propagators is that precisely because of this ghost signature, the propagator has much better behavior in the ultraviolet than a standard second order $1/k^2$ propagator, to thus enable this higher order theory to naturally address renormalization issues such as those associated with elementary particle self energies or with quantum gravitational fluctuations.

While a similar conclusion regarding the presence of ghosts might be anticipated to apply to pure fourth order theories as well [viz. the $M^2=0$ limit of Eq. (1)] [2], as we see from the form of Eq. (2), the $1/M^2$ prefactor multiplying the $1/k^2 - 1/(k^2+M^2)$ dipole term is singular, and thus is not reliable to infer the structure of the $M^2=0$ spectrum from that associated with that of $M^2 \neq 0$. In fact, it is not actually reliable to try to infer the spectrum associated with Eq. (1) via canonical reasoning at all, since even when $M^2 \neq 0$, the theory is constrained due to the presence of higher order time derivatives. Thus before drawing any conclusions at all, one must first quantize the theory in a way which fully takes these constraints into account, and only then identify the particle spectrum. To address this issue we shall thus effect a full Dirac constraint quantization [3] of a quantum-mechanical prototype of Eq. (1), something which has not previously been carried out in the literature. Specifically, we shall study a restricted version of Eq. (1) in which we specialize to field configurations of the form $\phi(\bar{x},t)=q(t)e^{i\bar{k}\cdot\bar{x}}$, configurations in which Eq. (1) then reduces to

$$\frac{d^4q}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2q}{dt^2} + \omega_1^2\omega_2^2 q = 0, \qquad (3)$$

where

$$\omega_1^2 + \omega_2^2 = 2\bar{k}^2 + M^2, \quad \omega_1^2\omega_2^2 = \bar{k}^4 + \bar{k}^2M^2, \qquad (4)$$

with Eq. (4) reducing to the equal frequency $\omega_1=\omega_2$ when $M=0$. As such, for general $\omega_1 \neq \omega_2$ Eq. (3) thus serves as a quantum-mechanical prototype of a field theory based on second order plus fourth order derivatives of the field, while becoming a prototype of a pure fourth order theory in the equal frequency limit. In addition to being a model which we

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can actually solve analytically, the quantum-mechanical prototype encapsulates many of the general issues associated with the quantization of the full field theory, since it is the time derivatives of the fields rather then their spatial derivatives which are of relevance for constructing conjugates of the fields. With the equation of motion given in Eq. (3) being derivable from the Pais-Uhlenbeck Lagrangian [4]

$$L = \frac{\gamma}{2}[\ddot{q}^2 - (\omega_1^2 + \omega_2^2)\dot{q}^2 + \omega_1^2\omega_2^2 q^2] \tag{5}$$

(γ is a constant), we shall thus quantize the theory starting from this Lagrangian [5]. This will allow us to establish that the properly constructed $\omega_1 \neq \omega_2$ Hamiltonian does indeed possess eigenstates of normal and ghost signature (states which themselves are then explicitly constructed), while also allowing us to monitor the $\omega_1 \to \omega_2$ limit, a limit, which like the above $M^2 \to 0$ limit, will be found to be highly singular [6].

II. CLASSICAL DIRAC HAMILTONIAN

For a quantization of the theory associated with the Lagrangian of Eq. (5) we would like to treat q and $\dot{q}$ as independent coordinates, but cannot immediately do so since if $\dot{q}$ is to be an independent coordinate, we could not use $\partial L/\partial\dot{q}$ as the canonical conjugate of q. To obtain a form which would be appropriate for quantization we therefore introduce a new variable $x(t)$ to replace $\dot{q}$, and compensate for doing so by additionally introducing a Lagrange multiplier $\lambda(t)$ and a substitute Lagrangian

$$L = \frac{\gamma}{2}[\dot{x}^2 - (\omega_1^2 + \omega_2^2)x^2 + \omega_1^2\omega_2^2 q^2] + \lambda(\dot{q} - x), \tag{6}$$

with the Dirac constraint method assuring us that the Hamiltonian, which is to ultimately emerge, will then be independent of the Lagrange multiplier. With the Lagrangian of Eq. (6) possessing three coordinate variables, q, x, and λ, we must introduce the three canonical momenta, p_x, p_q, and p_λ, conjugates which for the Lagrangian of Eq. (6) evaluate to

$$p_x = \frac{\partial L}{\partial \dot{x}} = \gamma\dot{x}, \quad p_q = \frac{\partial L}{\partial \dot{q}} = \lambda, \quad p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0. \tag{7}$$

To implement the Dirac method, as well as use these relations to construct the Legendre transform of the Lagrangian of Eq. (6)

$$H_L = p_x\dot{x} + p_q\dot{q} + p_\lambda\dot{\lambda} - L, \tag{8}$$

we also introduce two primary constraint functions for the canonical momenta which involve either the Lagrange multiplier or its conjugate

$$\phi_1 = p_q - \lambda, \quad \phi_2 = p_\lambda. \tag{9}$$

Then, rather than use H_L, the Dirac prescription is to instead use

$$H_1 = H_L + u_1\phi_1 + u_2\phi_2 \tag{10}$$

as the Hamiltonian, where H_1 is thus given by

$$H_1 = \frac{p_x^2}{2\gamma} + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 q^2 + \lambda x + u_1(p_q - \lambda) + u_2 p_\lambda. \tag{11}$$

For this theory we define generalized Poisson brackets of the form

$$\{A,B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}\frac{\partial B}{\partial x} + \frac{\partial A}{\partial q}\frac{\partial B}{\partial p_q} - \frac{\partial A}{\partial p_q}\frac{\partial B}{\partial q} + \frac{\partial A}{\partial \lambda}\frac{\partial B}{\partial p_\lambda} - \frac{\partial A}{\partial p_\lambda}\frac{\partial B}{\partial \lambda}, \tag{12}$$

to obtain the canonical

$$\{x,p_x\} = \{q,p_q\} = \{\lambda,p_\lambda\} = 1. \tag{13}$$

Similarly, given the definition of Eq. (12), we find that the Poisson brackets of the constraint functions with the Hamiltonian H_1 are given by

$$\{\phi_1,H_1\} = \gamma\omega_1^2\omega_2^2 q - u_2 + \phi_1\{\phi_1,u_1\} + \phi_2\{\phi_1,u_2\},$$

$$\{\phi_2,H_1\} = -x + u_1 + \phi_1\{\phi_2,u_1\} + \phi_2\{\phi_2,u_2\}. \tag{14}$$

Consequently, both of the $\{\phi_1,H_1\}$ and $\{\phi_2,H_1\}$ Poisson brackets will vanish weakly (in the sense of Dirac) if we set

$$u_1 = x, \quad u_2 = \gamma\omega_1^2\omega_2^2 q. \tag{15}$$

On thus imposing these two conditions, H_1 is then replaced by a Hamiltonian

$$H_2 = \frac{p_x^2}{2\gamma} + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 q^2 + p_q x + \gamma\omega_1^2\omega_2^2 q p_\lambda. \tag{16}$$

With respect to this Hamiltonian the constraint functions obey Poisson bracket relations of the form

$$\{\phi_1,H_2\} = \gamma\omega_1^2\omega_2^2 p_\lambda, \quad \{\phi_2,H_2\} = \{p_\lambda,H_2\} = 0. \tag{17}$$

Hence, finally, if we now set $p_\lambda=0$ (to enforce $\{\phi_1,H_2\}=0$), the resulting algebra associated with the four-dimensional q,p_q,x,p_x sector of the theory will then (as befits a fourth order theory) be closed under commutation using Poisson brackets defined via

$$\{A,B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}\frac{\partial B}{\partial x} + \frac{\partial A}{\partial q}\frac{\partial B}{\partial p_q} - \frac{\partial A}{\partial p_q}\frac{\partial B}{\partial q}, \tag{18}$$

with the requisite classical Hamiltonian which we seek then being given by

$$H = \frac{p_x^2}{2\gamma} + p_q x + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 q^2, \tag{19}$$

and with the requisite Poisson bracket relations which define the classical theory being given by

$$\{x,p_x\} = 1, \quad \{q,p_q\} = 1,$$

$$\{x,H\} = \frac{p_x}{\gamma}, \quad \{q,H\} = x,$$

$$\{p_x,H\} = -p_q - \gamma(\omega_1^2+\omega_2^2)x, \quad \{p_q,H\} = \gamma\omega_1^2\omega_2^2 q. \quad (20)$$

With such Poisson bracket relations the canonical equations of motion thus take the form

$$\dot{x} = \frac{p_x}{\gamma}, \quad \dot{q} = x, \quad \dot{p}_x = -p_q - \gamma(\omega_1^2+\omega_2^2)x, \quad \dot{p}_q = \gamma\omega_1^2\omega_2^2 q, \quad (21)$$

to then enable us to both recover Eq. (3) and make the identification $x=\dot{q}$ in the solution. Additionally, in solutions that obey these equations of motion the Legendre transform $L=p_x\dot{x}+p_q\dot{q}-H$ is found to reduce to Eq. (5) just as it should. The Hamiltonian H of Eq. (19) is thus the correct one for the fourth order theory, and it thus is the one which is to be quantized.

III. CONNECTION WITH THE OSTROGRADSKI HAMILTONIAN

In considering the theory associated with Eq. (19), it is important to distinguish between the general Hamiltonian H as defined by Eq. (19) and the particular value H_{STAT} that it takes in the stationary path in which the equations of motion of Eq. (21) are imposed, with the great virtue of the Dirac procedure being that it allows us to define a classical Hamiltonian H which takes a meaning [as the canonical generator used in Eq. (20)] even for nonstationary field configurations [i.e., even for configurations for which $p_x\dot{x}+p_q\dot{q}-H$ does not reduce to the Lagrangian of Eq. (5)]. Thus, for instance, it is the Hamiltonian of Eq. (19), which defines the appropriate phase space for the problem, so that the path integral for the theory is then uniquely given by $\int[dq][dp_q][dx][dp_x]\exp[i\int dt(p_x\dot{x}+p_q\dot{q}-H)]$ as integrated over a complete set of classical paths associated with these four independent coordinates and momenta.

With regard to the stationary value that the classical H_{STAT} takes when the equations of motion are imposed,

$$H_{\mathrm{STAT}} = \frac{\gamma}{2}\ddot{q}^2 - \frac{\gamma}{2}(\omega_1^2+\omega_2^2)\dot{q}^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 q^2 - \gamma\dot{q}\frac{d^3q}{dt^3}, \quad (22)$$

we note that not only is this particular H_{STAT} time independent [see, e.g., Eqs. (25) and (26) below], it is also recognized as being the Ostrogradski [7] generalized higher derivative Hamiltonian associated with the Lagrangian of Eq. (5), viz. the Hamiltonian

$$H_{\mathrm{OST}} = \dot{q}\frac{\partial L}{\partial \dot{q}} + \ddot{q}\frac{\partial L}{\partial \ddot{q}} - \dot{q}\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) - L, \quad (23)$$

which Ostrogradski showed to be time independent in solutions that obey the generalized Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0, \quad (24)$$

while also being its associated time translation generator [8]. Moreover, not only is the classical H_{STAT} time independent in the unequal frequency solution $q(t)=a_1e^{-i\omega_1 t}+a_2e^{-i\omega_2 t}$ + c.c., where it evaluates to

$$H_{\mathrm{STAT}}(\omega_1 \neq \omega_2) = 2\gamma(\omega_1^2-\omega_2^2)(a_1^*a_1\omega_1^2 - a_2^*a_2\omega_2^2), \quad (25)$$

it is even time independent in the equal frequency $\omega_1=\omega_2=\omega$ temporal runaway solution $q(t)=c_1e^{-i\omega t}+c_2te^{-i\omega t}$ + c.c., where it evaluates to

$$H_{\mathrm{STAT}}(\omega_1=\omega_2) = 4\gamma\omega^2(2c_2^*c_2 + i\omega c_1^*c_2 - i\omega c_2^*c_1), \quad (26)$$

with a runaway in time not leading to a runaway in energy (i.e., there is an appropriately defined energy which is time independent in the temporal runaway solution).

The value we have obtained for $H_{\mathrm{STAT}}(\omega_1=\omega_2)$ in Eq. (26) reminds us of the zero energy theorem derived by Boulware *et al.* [9] for fourth order conformal gravity, another typical fourth order theory. Specifically, Boulware *et al.* constructed a classical energy for the fourth order gravity theory, and found that it would vanish identically if the gravitational field solutions were required to be asymptotically flat. The analog condition for the equal frequency Pais-Uhlenbeck theory (a prototype of a pure fourth order field theory) would be to demand the absence of runaway solutions in time, and thus to set $c_2=0$, a condition which would then yield for $H_{\mathrm{STAT}}(\omega_1=\omega_2)$ none other than the value zero. As we thus see, it is the restriction to an analog of asymptotic flatness which leads to a zero value for the energy in the equal frequency Pais-Uhlenbeck case, with there being no $c_1^*c_1$ type term present in $H_{\mathrm{STAT}}(\omega_1=\omega_2)$ even though $H_{\mathrm{STAT}}(\omega_1\neq\omega_2)$ contains both $a_1^*a_1$ and $a_2^*a_2$ type terms. Since the c_1 mode can only make a contribution to $H_{\mathrm{STAT}}(\omega_1=\omega_2)$ when c_2 is nonzero, we can anticipate that in the quantization of the theory to be presented below the c_1 mode will not give rise to a propagating on-shell state, with the eigenspectrum of the equal frequency quantum Hamiltonian possessing not the number of energy eigenstates states associated with a two-dimensional harmonic oscillator [viz. the dipole structure exhibited in Eq. (2)], but rather possessing only the number of energy eigenstates associated with a one-dimensional one. A further interesting feature of the form we have obtained for $H_{\mathrm{STAT}}(\omega_1=\omega_2)$ in Eq. (26) is that there is a choice of sign for the parameter γ for which the pure c_2 contribution to the energy, viz. $8\gamma\omega^2c_2^*c_2$, is then positive definite. On its own then, the runaway mode does not give rise to any disease such as a negative energy, with it yielding a positive one instead. Therefore, there is no need to require the absence of runaway solutions in the equal frequency Pais-Uhlenbeck theory (by demanding an analog of asymptotic flatness [10]) since in and of themselves they give rise to perfectly acceptable energies, and do not need to be avoided. Since the runaway solution classical energies are well behaved, we can thus anticipate that the quantum energy eigenspectrum associated with the c_2 mode sector will not possess any energy eigenstates with either negative energy or negative norm, even while the structure found for the unequal frequency $H_{\mathrm{STAT}}(\omega_1\neq\omega_2)$ in Eq. (25) indicates that there will be energy eigenstates of either negative energy or negative norm in the quantization of the unequal frequency case.

IV. CONSTRUCTING THE UNEQUAL FREQUENCY FOCK SPACE

With the Hamiltonian of Eq. (19) being defined for both stationary and nonstationary classical paths, a canonical quantization of the theory can be readily obtained by replacing (i times) the Poisson brackets of Eq. (20) by canonical equal time commutators. However, without reference to the explicit structure of the Hamiltonian itself, we note first that the identification (as suggested but not required by the equations of motion [11])

$$q(t) = a_1 e^{-i\omega_1 t} + a_2 e^{-i\omega_2 t} + \text{H.c.},$$

$$p_q(t) = i\gamma\omega_1\omega_2^2 a_1 e^{-i\omega_1 t} + i\gamma\omega_1^2\omega_2 a_2 e^{-i\omega_2 t} + \text{H.c.},$$

$$x(t) = -i\omega_1 a_1 e^{-i\omega_1 t} - i\omega_2 a_2 e^{-i\omega_2 t} + \text{H.c.},$$

$$p_x(t) = -\gamma\omega_1^2 a_1 e^{-i\omega_1 t} - \gamma\omega_2^2 a_2 e^{-i\omega_2 t} + \text{H.c.} \qquad (27)$$

then furnishes us with a Fock space representation of the quantum-mechanical commutation relations

$$[x,p_x] = [q,p_q] = i, \quad [x,q] = [x,p_q] = [q,p_x] = [p_x,p_q] = 0 \qquad (28)$$

at all times provided that

$$[a_1,a_1^\dagger] = \frac{1}{2\gamma\omega_1(\omega_1^2 - \omega_2^2)},$$

$$[a_2,a_2^\dagger] = \frac{1}{2\gamma\omega_2\ (\omega_2^2 - \omega_1^2)]}, \quad [a_1,a_2^\dagger] = 0, \quad [a_1,a_2] = 0. \qquad (29)$$

(Here and throughout both ω_1 and ω_2 are taken to be positive.) Then in this convenient Fock representation the quantum-mechanical Hamiltonian is found to take the form

$$H = 2\gamma(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + \tfrac{1}{2}(\omega_1 + \omega_2) \qquad (30)$$

with its associated commutators as inferred from Eq. (20) then automatically being satisfied. With the quantity $\gamma(\omega_1^2 - \omega_2^2)$ being taken to be positive for definitiveness, we see that the $[a_2, a_2^\dagger]$ commutator is negative definite, and with H being diagonal in the a_1, a_2 occupation number basis, we see that the state defined by

$$a_1|\Omega\rangle = 0, \quad a_2|\Omega\rangle = 0 \qquad (31)$$

is its ground state [12], that the states

$$|+1\rangle = [2\gamma\omega_1(\omega_1^2 - \omega_2^2)]^{1/2} a_1^\dagger|\Omega\rangle,$$

$$|-1\rangle = [2\gamma\omega_2(\omega_1^2 - \omega_2^2)]^{1/2} a_2^\dagger|\Omega\rangle \qquad (32)$$

are both positive energy eigenstates with respective energies ω_1 and ω_2 above the ground state, that the state $|+1\rangle$ has a norm equal to plus one, but that the state $|-1\rangle$ has norm minus one, a ghost state. Thus, as anticipated, the correct Hamiltonian for the unequal frequency theory can be diagonalized in a basis of positive and negative norm states, with the relevant negative norm state wave functions being explicitly constructed via Eq. (32) and its multiparticle generalizations. With the eigenstates of H labeling the asymptotic states associated with scattering in the presence of any interaction Lagrangian L_I, which might be added on to the original Lagrangian L of the theory, the effect of L_I would be expected to induce transitions between asymptotic in and out states of opposite norm, with the unequal frequency theory then being nonunitary [13]. However, even though the Hamiltonian of the unequal frequency theory does have ghost eigenstates, since the commutation relations given in Eq. (29) become singular in the equal frequency limit while both the Hamiltonian of Eq. (30) and the normalized $|+1\rangle$ and $|-1\rangle$ states develop zeros, we will have to exercise some caution in trying to discover exactly what happens to the ghost states when this limit is taken.

V. CONSTRUCTING THE EQUAL FREQUENCY FOCK SPACE

To explore the equal frequency limit it is convenient to introduce new Fock space variables according to

$$a_1 = \frac{1}{2}\left(a - b + \frac{2b\omega}{\epsilon}\right), \quad a_2 = \frac{1}{2}\left(a - b - \frac{2b\omega}{\epsilon}\right),$$

$$a = a_1\left(1 + \frac{\epsilon}{2\omega}\right) + a_2\left(1 - \frac{\epsilon}{2\omega}\right), \quad b = \frac{\epsilon}{2\omega}(a_1 - a_2), \qquad (33)$$

where

$$\omega = \frac{(\omega_1 + \omega_2)}{2}, \quad \epsilon = \frac{(\omega_1 - \omega_2)}{2}. \qquad (34)$$

These variables are found to obey commutation relations of the form

$$[a,a^\dagger] = \lambda, \quad [a,b^\dagger] = \mu, \quad [b,a^\dagger] = \mu, \quad [b,b^\dagger] = \nu, \quad [a,b] = 0, \qquad (35)$$

where

$$\lambda = \nu = -\frac{\epsilon^2}{16\gamma(\omega^2 - \epsilon^2)\omega^3}, \quad \mu = \frac{(2\omega^2 - \epsilon^2)}{16\gamma(\omega^2 - \epsilon^2)\omega^3}. \qquad (36)$$

As such, the introduction of the operators a, $a^\dagger$, b, $b^\dagger$ is initially nothing more than a rewriting of the original a_1, $a_1^\dagger$, a_2, $a_2^\dagger$ operators. Their utility derives from the fact that when expressed in terms of them the coordinate $q(t)$ of Eq. (27) is rewritten as

$$q(t) = e^{-i\omega t}\left[(a - b)\cos\epsilon t - \frac{2ib\omega}{\epsilon}\sin\epsilon t\right] + \text{H.c.}, \qquad (37)$$

and thus has a well defined $\epsilon \to 0$ limit, viz.

$$q(t, \epsilon = 0) = e^{-i\omega t}(a - b - 2ib\omega t) + \text{H.c.} \qquad (38)$$

Similarly, the Hamiltonian of Eq. (30) which takes the form

$$H = 8\gamma\omega^2\epsilon^2(a^\dagger a - b^\dagger b) + 8\gamma\omega^4(2b^\dagger b + a^\dagger b + b^\dagger a) + \omega \qquad (39)$$

in the new variables, then limits to

$$H(\epsilon=0) = 8\gamma\omega^4(2b^\dagger b + a^\dagger b + b^\dagger a) + \omega, \tag{40}$$

while the following commutators of interest have limiting form:

$$[H(\epsilon=0),a^\dagger] = \omega(a^\dagger + 2b^\dagger), \quad [H(\epsilon=0),a] = -\omega(a+2b),$$

$$[H(\epsilon=0),b^\dagger] = \omega b^\dagger, \quad [H(\epsilon=0),b] = -\omega b,$$

$$[a+b,a^\dagger+b^\dagger] = 2\hat{\mu}, \quad [a-b,a^\dagger-b^\dagger] = -2\hat{\mu},$$
$$[a+b,a^\dagger-b^\dagger] = 0, \tag{41}$$

where now $\mu(\epsilon=0)=\hat{\mu}=1/(8\gamma\omega^3)$. Thus even while the relation between the (a,b) and (a_1,a_2) sets of operators is explicitly singular in the limit, the limiting prescription obtained by using the a and b operators of Eq. (33) nonetheless leads to the $\epsilon=0$ operator algebra given in Eqs. (40) and (41) in which there are no singular terms at all. However, this is not the only way to take the $\epsilon\to 0$ limit, as we could instead take the limit not of the operators of the unequal frequency theory but rather of the states. There are thus two possible ways to construct the equal frequency theory—we can either start with the $\epsilon=0$ operator algebra and construct a new Fock space for it from scratch or we can construct the equal frequency states as the $\epsilon\to 0$ limit of the unequal frequency Fock space states. With these two prescriptions not being equivalent, we shall explore both of them, and discuss first the limit of the operator algebra.

A. Taking the limit of the operator algebra

For the operator algebra limit, we use the $\epsilon\to 0$ limit of the a, $a^\dagger$, b, $b^\dagger$ operators as the dynamical variables, with use of these operators then enabling us to construct an $\epsilon=0$ theory which is well defined. For this theory the $\epsilon=0$ Fock space is then built on the Fock vacuum $|\Omega\rangle$ defined by $a|\Omega\rangle=b|\Omega\rangle=0$, [we use this basis since, according to Eq. (29), states built on the unequal frequency Fock vacuum in which $a_1|\Omega\rangle=0$, $a_2|\Omega\rangle=0$ become undefined in the $\epsilon=0$ limit], with the state $|\Omega\rangle$ being an eigenstate of H with energy ω. Moreover, in this $\epsilon\to 0$ limit we see that the $[a,a^\dagger]$ and $[b,b^\dagger]$ commutators both vanish, with, as we shall see below, there actually being two ways rather than one in which the Fock space states can implement this vanishing, depending on how the $\epsilon\to 0$ limit is actually taken. However, before constructing either of these two ways, we note that because the effect of the Hamiltonian is to shift $a^\dagger$ by $2b^\dagger$ in Eq. (41), and because (unlike the unequal frequency case) neither of the operators $(a^\dagger \pm b^\dagger)$ which diagonalize the Fock space basis acts as a ladder operator for the Hamiltonian [14], we can anticipate that the eigenspectrum of $H(\epsilon=0)$ will be quite different from the unequal frequency eigenspectrum.

Because $b^\dagger$ does act as a ladder operator for $H(\epsilon=0)$, it is possible to construct a one particle energy eigenstate, viz. the state $b^\dagger|\Omega\rangle$ with energy 2ω, but if we try to normalize it (our first way to realize the $\epsilon=0$ Fock space) the vanishing of the $[b,b^\dagger]$ commutator would entail that this state would have to have zero norm. On its own a zero norm state (unlike a negative norm state) does not lead to loss of probability, but its very existence entails the existence of negative norm states elsewhere in the theory [15]. Thus, with the state $a^\dagger|\Omega\rangle$ also having zero norm, we immediately construct the states

$$|\pm\rangle = \frac{(a^\dagger \pm b^\dagger)}{(2\hat{\mu})^{1/2}}|\Omega\rangle, \tag{42}$$

states which obey

$$\langle +|+\rangle = 1, \quad \langle -|-\rangle = -1, \quad \langle +|-\rangle = 0. \tag{43}$$

However, unlike the orthogonal positive and negative norm states $|\pm 1\rangle$ of Eq. (32) which are eigenstates of the unequal frequency Hamiltonian, this time we find that neither of the $|\pm\rangle$ states is an eigenstate of the $\epsilon=0$ Hamiltonian since

$$H(\epsilon=0)|\pm\rangle = 2\omega|\pm\rangle + 4\omega b^\dagger|\Omega\rangle. \tag{44}$$

Thus even while the $\epsilon=0$ Fock space possesses negative norm ghost states, this time they can only exist off shell (where they can still regulate Feynman diagrams) but cannot materialize as on shell asymptotic in and out states. With a similar situation being found in the two particle sector [16] where only the state $(b^\dagger)^2|\Omega\rangle$ with energy 3ω is an energy eigenstate, we see that the only states which are eigenstates of $H(\epsilon=0)$ are those of the form $(b^\dagger)^n|\Omega\rangle$, with all such states having zero norm and positive energy [17].

Now at first it is quite perplexing that the $\epsilon=0$ theory possesses far fewer energy eigenstates than the $\epsilon\neq 0$ theory, possessing only the number of eigenstates associated with a one-dimensional harmonic oscillator rather than a two-dimensional one. The reason for such an outcome derives from the fact that while the normal situation for square matrices is that the number of independent eigenvectors of a square matrix is the same as the dimensionality of the matrix, there are certain matrices, known as defective matrices, for which this is not in fact the case. A typical example of such a defective matrix is the non-Hermitian, Jordan block form, two-dimensional matrix

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{45}$$

Specifically, while this matrix has two eigenvalues both of which are real (despite the lack of hermiticity) and equal to 1 (the trace of M is equal to 2 and its determinant is equal to 1), solving the equation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p+q \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{46}$$

leads only to $q=0$, with there thus only being one eigenvector despite the twofold degeneracy of the eigenvalue, with the space on which the matrix M acts not being complete [18].

For the case of interest to us here, the action of the unequal frequency Hamiltonian of Eq. (39) on the one-particle states $a^\dagger|\Omega\rangle$, $b^\dagger|\Omega\rangle$ yields

$$Ha^\dagger|\Omega\rangle = \frac{1}{2\omega}[(4\omega^2+\epsilon^2)a^\dagger|\Omega\rangle + (4\omega^2-\epsilon^2)b^\dagger|\Omega\rangle],$$

$$Hb^{\dagger}|\Omega\rangle = \frac{1}{2\omega}[\epsilon^2 a^{\dagger}|\Omega\rangle + (4\omega^2 - \epsilon^2)b^{\dagger}|\Omega\rangle]. \quad (47)$$

For this sector we can define a matrix

$$M(\epsilon) = \frac{1}{2\omega}\begin{pmatrix} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{pmatrix} \quad (48)$$

whose eigenvalues are given as $2\omega+\epsilon$ and $2\omega-\epsilon$ [the trace of $M(\epsilon)$ is 4ω and its determinant is $4\omega^2-\epsilon^2$]. For such eigenvalues, energy eigenvectors which obey

$$H|2\omega \pm \epsilon\rangle = (2\omega \pm \epsilon)|2\omega \pm \epsilon\rangle \quad (49)$$

are then readily constructed as

$$|2\omega \pm \epsilon\rangle = [\pm \epsilon a^{\dagger} + (2\omega \mp \epsilon)b^{\dagger}]|\Omega\rangle. \quad (50)$$

As we see, as long as $\epsilon \neq 0$, the two one-particle sector eigenvectors of the Hamiltonian H are distinct. However when we let ϵ go to zero, the two eigenvectors in Eq. (50) collapse onto a single eigenvector, viz. the vector $b^{\dagger}|\Omega\rangle$, the two eigenvalues collapse onto a common eigenvalue, viz. 2ω, and the matrix $M(\epsilon=0)$ of Eq. (48) becomes (2ω times) the defective one given in Eq. (45). Consequently, while the dimensionality of the a, $a^{\dagger}$, b, $b^{\dagger}$ based Fock space does not change in the $\epsilon \to 0$ limit (i.e., the dimensionality of the Fock space remains that of a two-dimensional harmonic oscillator), nonetheless, in the limit the Hamiltonian becomes defective, with $H(\epsilon=0)$ thus possessing far fewer eigenvectors than $H(\epsilon \neq 0)$, and with the dimensionality of the space of energy eigenvectors of $H(\epsilon=0)$ actually being the same as that of a one-dimensional rather than a two-dimensional harmonic oscillator [19]. As we thus see, the $\epsilon \to 0$ limit is indeed highly singular. Finally, with Eq. (47) reducing to

$$H(\epsilon = 0)a^{\dagger}|\Omega\rangle = 2\omega[a^{\dagger}|\Omega\rangle + b^{\dagger}|\Omega\rangle],$$

$$H(\epsilon = 0)b^{\dagger}|\Omega\rangle = 2\omega b^{\dagger}|\Omega\rangle \quad (51)$$

when $\epsilon=0$, we confirm again that $H(\epsilon=0)$ only has one eigenvector, the zero norm $b^{\dagger}|\Omega\rangle$ with eigenvalue 2ω.

In addition to these zero norm states, the $\epsilon=0$ Fock space also possesses harmonic oscillator coherent states as well. To construct these particular states it is convenient to introduce new operators

$$\alpha = a + b, \quad \beta = a - b \quad (52)$$

which obey

$$[\alpha,\alpha^{\dagger}] = 2\hat{\mu}, \quad [\beta,\beta^{\dagger}] = -2\hat{\mu}, \quad [\alpha,\beta^{\dagger}] = 0, \quad [\beta,\alpha^{\dagger}] = 0,$$
$$[\alpha,\beta] = 0. \quad (53)$$

In terms of these operators the Hamiltonian of Eq. (40) may be rewritten as

$$H(\epsilon = 0) = \frac{\omega}{2\hat{\mu}}[2\alpha^{\dagger}\alpha - \alpha^{\dagger}\beta - \beta^{\dagger}\alpha] + \omega, \quad (54)$$

with $\alpha^{\dagger}$ and $\beta^{\dagger}$ then being found to obey the relations

$$[H(\epsilon = 0),\alpha^{\dagger}] = \omega(2\alpha^{\dagger} - \beta^{\dagger}), \quad [H(\epsilon = 0),\beta^{\dagger}] = \omega\alpha^{\dagger},$$

$$[H(\epsilon = 0),\alpha^{\dagger} - \beta^{\dagger}] = \omega(\alpha^{\dagger} - \beta^{\dagger}),$$

$$[H(\epsilon = 0),\alpha - \beta] = -\omega(\alpha - \beta). \quad (55)$$

In terms of these operators we construct the coherent state

$$|g\rangle = e^{\alpha^{\dagger}\beta^{\dagger}}|\Omega\rangle, \quad (56)$$

where the vacuum is again the one which a and b, and thus α and β, annihilate. Using the well-known relation $e^{A+B} = e^A e^B e^{-[A,B]/2} = e^B e^A e^{-[B,A]/2}$ which holds when $[A,[A,B]] = [B,[A,B]] = 0$, it can readily be shown that $|g\rangle$ has positive norm, viz. $\langle g|g\rangle = \langle\Omega|e^{\alpha^{\dagger}\beta^{\dagger}}e^{\alpha\beta}e^{(\beta^{\dagger}\beta-\alpha^{\dagger}\alpha-1)}|\Omega\rangle = 1/e$. Similarly, using the well-known relation $e^A B e^{-A} = B + [A,B] + [A,[A,B]]/2 + [A,[A,[A,B]]]/6 + \ldots$, it can also be shown that application of the Hamiltonian to this state yields

$$H(\epsilon = 0)|g\rangle = \omega(\alpha^{\dagger 2} - \beta^{\dagger 2} + 2\alpha^{\dagger}\beta^{\dagger})|g\rangle, \quad (57)$$

with the coherent state thus not being an energy eigenstate. This same analysis generalizes to any other state of the form of an exponential of a string of creation operators acting on the vacuum, since the commutator of $H(\epsilon=0)$ with the string yields another string also consisting purely of creation operators, a string which thus commutes with the original string of creation operators. Thus whether we use states with a definite number of particles (such as $a^{\dagger m}b^{\dagger n}|\Omega\rangle$) or states with an indefinite number of particles (such as $|g\rangle$), we find that only the $b^{\dagger n}|\Omega\rangle$ states are energy eigenstates, with all the $\epsilon \neq 0$ negative norm states having moved off shell in the limit. Consequently, in the $\epsilon=0$ theory there are no energy eigenstates with negative norm or negative energy.

Another interesting aspect of the $\epsilon=0$ theory is that even while it is a fourth order theory, the $\epsilon=0$ theory only possesses the number of observable states that would be found in an ordinary second order theory. Consequently, in a field-theoretic generalization of the equal frequency Pais-Uhlenbeck fourth order oscillator, we would anticipate that there would only be one observable particle in the theory rather than the two that would be associated with the field-theoretic analog of the unequal frequency case as exhibited schematically in Eq. (2). Replacing a pure second order field theory by a pure fourth order one would thus not be expected to bring about any change in the number of observable states.

Since the above equal frequency theory possesses zero norm states, as a quantum-mechanical theory it is thus somewhat unconventional. Even though the physical viability of the theory is not in question since the theory has been shown to possess no energy eigenstates with negative norm, nonetheless, it is still of interest to try to recast the theory in an alternate form in which the theory has a more conventional look to it. Thus we seek to reinterpret the zero norm states as positive norm ones by using an unconventional scalar product with which to define the norm. To this end, we note that in constructing the eigenstates of a Hamiltonian, all that is needed is the introduction of a Hilbert space on which the Hamiltonian is to act to the right on a set of ket vectors $|\psi\rangle$, with there being no need to introduce the dual vector bra states of those ket vectors for this purpose, with the structure of the energy eigenspectrum thus being in no way sensitive to the structure of the dual space vectors [20]. The choice of definition of a scalar product is thus independent of the structure of the energy eigenspectrum, and while one ordinarily defines the dual bra vector $\langle\psi|$ simply as the conjugate of $|\psi\rangle$, other choices are possible, and all of them provide legitimate formulations of quantum mechanics [21].

Thus for our purposes here, we note that with both the equal frequency theory $a^\dagger|\Omega\rangle$ and $b^\dagger|\Omega\rangle$ states being zero norm states, if their respective conjugates are canonically defined as $\langle\Omega|a$, $\langle\Omega|b$, in terms of the operators α and β which diagonalize the Fock algebra, we instead define the dual of the state $\beta^\dagger|\Omega\rangle$ not to be the standard $\langle\Omega|\beta$ but to be minus one times it (viz. $-\langle\Omega|\beta$) instead, while at the same time we continue to keep the dual of $\alpha^\dagger|\Omega\rangle$ to be the standard $\langle\Omega|\alpha$. With such a dual state definition, the dual of $b^\dagger|\Omega\rangle=(1/2)(\alpha^\dagger-\beta^\dagger)|\Omega\rangle$ is given by $(1/2)\langle\Omega|(\alpha+\beta)$, viz. by $\langle\Omega|a$, with the overlap of $b^\dagger|\Omega\rangle$ with its own dual then being given by $(1/4)\langle\Omega|(\alpha+\beta)(\alpha^\dagger-\beta^\dagger)|\Omega\rangle=\hat{\mu}=1/8\gamma\omega^3$, a norm which, for positive γ, is then positive rather than zero. Similarly, the dual of $a^\dagger|\Omega\rangle=(1/2)(\alpha^\dagger+\beta^\dagger)|\Omega\rangle$ is given by $(1/2)\langle\Omega|(\alpha-\beta)$, viz. by $\langle\Omega|b$, with the overlap of $a^\dagger|\Omega\rangle$ with its own dual then being given by $(1/4)\langle\Omega|(\alpha-\beta)(\alpha^\dagger+\beta^\dagger)|\Omega\rangle=\hat{\mu}=1/8\gamma\omega^3$, a norm which is then also positive for the same choice of sign of γ. Moreover, the overlap of the dual of $b^\dagger|\Omega\rangle$ with $a^\dagger|\Omega\rangle$ is given by $(1/4)\langle\Omega|(\alpha+\beta)(\alpha^\dagger+\beta^\dagger)|\Omega\rangle$, an overlap which is zero, while the overlap of the dual of $a^\dagger|\Omega\rangle$ with $b^\dagger|\Omega\rangle$ is given by $(1/4)\langle\Omega|(\alpha-\beta)(\alpha^\dagger-\beta^\dagger)|\Omega\rangle$, an overlap which is also zero. With respect to this definition of scalar product then, the states $a^\dagger|\Omega\rangle$ and $b^\dagger|\Omega\rangle$ form an orthonormal basis in which all states have a positive norm. Thus because we do not need to specify the dual vectors in order to construct energy eigenkets, we have flexibility in defining what we mean by a norm and a scalar product, and can thus make the equal frequency theory zero norm states become orthonormal positive norm states instead, and can do so without in any way altering the fact that $b^\dagger|\Omega\rangle$ is an energy eigenket while $a^\dagger|\Omega\rangle$ is not.

In addition to the above formal construction of dual vectors, there is also an explicit operational way to implement it. Specifically, we note that before we seek to define a new scalar product, the standard scalar products are given by $\langle\Omega|aa^\dagger|\Omega\rangle=0$, $\langle\Omega|ab^\dagger|\Omega\rangle=\hat{\mu}$, $\langle\Omega|ba^\dagger|\Omega\rangle=\hat{\mu}$, and $\langle\Omega|bb^\dagger|\Omega\rangle=0$. We now introduce an operator C whose action on $\alpha^\dagger|\Omega\rangle$ and $\beta^\dagger|\Omega\rangle$ is of the form $C\alpha^\dagger|\Omega\rangle=\alpha^\dagger|\Omega\rangle$, $C\beta^\dagger|\Omega\rangle=-\beta^\dagger|\Omega\rangle$. In consequence, its action on $a^\dagger|\Omega\rangle$ and $b^\dagger|\Omega\rangle$ is of the form $Ca^\dagger|\Omega\rangle=b^\dagger|\Omega\rangle$, $Cb^\dagger|\Omega\rangle=a^\dagger|\Omega\rangle$, an action which thus interchanges $a^\dagger|\Omega\rangle$ and $b^\dagger|\Omega\rangle$ with each other. With respect to this operator C we thus define scalar products which evaluate to $\langle\Omega|aCa^\dagger|\Omega\rangle=\hat{\mu}$, $\langle\Omega|aCb^\dagger|\Omega\rangle=0$, $\langle\Omega|bCa^\dagger|\Omega\rangle=0$, and $\langle\Omega|bCb^\dagger|\Omega\rangle=\hat{\mu}$. Thus with $\langle\Omega|aC$ being defined as the dual of $a^\dagger|\Omega\rangle$, and with $\langle\Omega|bC$ being defined as the dual of $b^\dagger|\Omega\rangle$, we thus generate an orthonormal basis. Now we saw above that the dual of $a^\dagger|\Omega\rangle$ is given as $\langle\Omega|b$ while the dual of $b^\dagger|\Omega\rangle$ is given as $\langle\Omega|a$. In this construction then we thus identify the dual of $a^\dagger|\Omega\rangle$ as $\langle\Omega|aC=\langle\Omega|b$, and the dual of $b^\dagger|\Omega\rangle$ as $\langle\Omega|bC=\langle\Omega|a$. Since the role of C is to interchange $a^\dagger|\Omega\rangle$ and $b^\dagger|\Omega\rangle$, we can thus conveniently describe the $a^\dagger|\Omega\rangle$, $b^\dagger|\Omega\rangle$ 2 space in the language of Pauli spinors. Thus on defining $a^\dagger|\Omega\rangle=|\uparrow\rangle$, $b^\dagger|\Omega\rangle=|\downarrow\rangle$, we see that the action of C on $a^\dagger|\Omega\rangle$ and $b^\dagger|\Omega\rangle$ can be written as $C|\uparrow\rangle=|\downarrow\rangle$, $C|\downarrow\rangle=|\uparrow\rangle$. Similarly, its action on $\alpha^\dagger|\Omega\rangle=(|\uparrow\rangle+|\downarrow\rangle)$ and $\beta^\dagger|\Omega\rangle=(|\uparrow\rangle-|\downarrow\rangle)$ can be written as $C(|\uparrow\rangle+|\downarrow\rangle)=(|\uparrow\rangle+|\downarrow\rangle)$, $C(|\uparrow\rangle-|\downarrow\rangle)=-(|\uparrow\rangle-|\downarrow\rangle)$. In the $a^\dagger|\Omega\rangle$, $b^\dagger|\Omega\rangle$ 2 space then the operator C can be represented by none other than the Pauli matrix σ_1, to thus establish that the needed C does indeed exist. Then given this definition of conjugate states, if we define wave functions via $\psi_a(x)=\langle x|a^\dagger|\Omega\rangle$, $\psi_b(x)=\langle x|b^\dagger|\Omega\rangle$ with respective conjugates $\psi_a^*(x)=\langle\Omega|aC|x\rangle$, $\psi_b^*(x)=\langle\Omega|bC|x\rangle$, we are able to obtain the conventional orthonormality relations for wave functions, viz. $\int dx\psi_a^*(x)\psi_a(x)=\int dx\langle\Omega|aC|x\rangle\langle x|a^\dagger|\Omega\rangle=\langle\Omega|aCa^\dagger|\Omega\rangle=\hat{\mu}$, $\int dx\psi_a^*(x)\psi_b(x)=0$, $\int dx\psi_b^*(x)\psi_a(x)=0$, $\int dx\psi_b^*(x)\psi_b(x)=\hat{\mu}$.

With regard to the Hamiltonian of the theory, we note that with respect to our definition of conjugate states, its matrix elements evaluate to $\langle\Omega|aCHa^\dagger|\Omega\rangle=1/4\gamma\omega^2$, $\langle\Omega|aCHb^\dagger|\Omega\rangle=0$, $\langle\Omega|bCHa^\dagger|\Omega\rangle=1/4\gamma\omega^2$, and $\langle\Omega|bCHb^\dagger|\Omega\rangle=1/4\gamma\omega^2$. As we see, with respect to this definition of conjugates, the Hamiltonian remains defective, with its diagonal elements still being real. With there thus still only being one energy eigenstate, viz. the original $b^\dagger|\Omega\rangle$, with which to characterize asymptotic in and out S-matrix states, the theory remains unitary not because H has became Hermitian with respect to the new definition of conjugate states, but because it has remained defective. While we have thus shown that one could equivalently either work with an equal frequency theory with zero norm states or recast the theory as one with positive norm states instead, to conclude this paper we now present an entirely different approach to the equal frequency theory, one that will lead to positive norm states without any need to introduce noncanonical dual vector states at all.

B. Taking the limit of the Hilbert space

While we have seen that it is possible to define an unconventional scalar product with respect to which all of the $\epsilon=0$ theory zero norm Fock states then have positive norm instead, it is also of interest to determine whether we could construct an alternate $\epsilon\to 0$ limiting theory in which all energy eigenstates would have a positive norm even when the conventional definition of conjugate states is used. To this end we note that since we obtained zero norm states in the above by first going to the $\epsilon=0$ limit of the operator algebra before constructing the states, to avoid such zero norm states an alternate procedure would be to first construct a set of states with nonzero norm, and then take the $\epsilon\to 0$ limit while holding the norms of these states fixed. With the algebra of the a and b operators already being defined in Eq. (35) even prior to taking the $\epsilon\to 0$ limit, we can thus construct a basis for the $\epsilon\neq 0$ Fock space via states built on a vacuum which obeys $a|\Omega\rangle=b|\Omega\rangle=0$ rather than on one which obeys $a_1|\Omega\rangle=0$, $a_2|\Omega\rangle=0$, and then explore the $\epsilon\to 0$ limit of those particular states. We thus build normalized $\epsilon\neq 0$ states such as (the choice $\gamma<0$ assures the positivity of λ and ν) the one particle

$$|1,0\rangle=\frac{a^\dagger}{\lambda^{1/2}}|\Omega\rangle,\quad |0,1\rangle=\frac{b^\dagger}{\nu^{1/2}}|\Omega\rangle, \tag{58}$$

the two particle

$$|2,0\rangle = \frac{(a^\dagger)^2}{2^{1/2}\lambda}|\Omega\rangle, \quad |1,1\rangle = \frac{a^\dagger b^\dagger}{(\mu^2+\lambda\nu)^{1/2}}|\Omega\rangle,$$
$$|0,2\rangle = \frac{(b^\dagger)^2}{2^{1/2}\nu}|\Omega\rangle, \tag{59}$$

and so on.

Even while this particular basis is not a basis of eigenstates of the $\epsilon \neq 0$ Hamiltonian, for the $\epsilon \neq 0$ theory this particular basis is just as complete a basis as the one built out of the $\epsilon \neq 0$ eigenstates. However, unlike the $\epsilon \neq 0$ energy eigenstate basis, remarkably, and crucially for our purposes here as it will turn out, every single state constructed via the $\epsilon \neq 0$ $a^\dagger$ and $b^\dagger$ operators is found to have a positive norm. It is not however an orthogonal basis since overlaps such as $\langle 1,0|0,1\rangle = \mu/(\lambda\nu)^{1/2}$ are nonzero, overlaps which actually become singular in the $\epsilon \to 0$ limit. Since this basis is complete we can use it to define a particular limiting procedure in which the normalization of each of the particle states in Eqs. (58) and (59) and their multiparticle generalization is held fixed while ϵ is allowed to go to zero. Then in such a limit we find from Eq. (58) that $a^\dagger|\Omega\rangle$ and $b^\dagger|\Omega\rangle$ both become null vectors. However, despite this, we cannot conclude that the creation operators annihilate the vacuum identically in this limit since matrix elements such as $\langle\Omega|[a,b^\dagger]|\Omega\rangle = \mu$ do not vanish in the limit $[\mu(\epsilon=0)=\hat{\mu}\neq 0]$, with product operator actions such as a acting on $b^\dagger|\Omega\rangle$ being singular. Instead, the creation operators must be thought of as annihilating the vacuum weakly (i.e., in some but not all matrix elements), but not strongly as an operator identity. Now since neither of the states $a^\dagger|\Omega\rangle$ or $b^\dagger|\Omega\rangle$ survives in the limit, $H(\epsilon=0)$ now has no one particle eigenstates at all. With the two particle states $(a^\dagger)^2|\Omega\rangle$ and $(b^\dagger)^2|\Omega\rangle$ also becoming null in the limit, the only two particle state which is found to survive in this limit is the positive norm state $a^\dagger b^\dagger|\Omega\rangle/\hat{\mu}$ (since the $[a,b^\dagger]$ commutator does not vanish), and, quite remarkably, in this limit the state is also found to actually become an energy eigenstate, viz.

$$H(\epsilon=0)\frac{a^\dagger b^\dagger}{\hat{\mu}}|\Omega\rangle = \frac{3\omega}{\hat{\mu}}a^\dagger b^\dagger|\Omega\rangle + \frac{2\omega}{\hat{\mu}}(b^\dagger)^2|\Omega\rangle \equiv \frac{3\omega}{\hat{\mu}}a^\dagger b^\dagger|\Omega\rangle, \tag{60}$$

with the energy of this two-particle state being equal to 3ω, which is nicely positive. With this analysis immediately generalizing to the higher multiparticle states as well, we see that the only states which then survive in the limit are states of the form $(a^\dagger b^\dagger)^n|\Omega\rangle$, positive norm states which also become positive energy eigenstates in the limit, with the observable sector of the theory thus being completely acceptable. We thus recognize two limiting procedures, first taking the limit of the algebra and then constructing the Fock space, or first constructing the Fock space and then taking the limit of its states, with this latter limiting procedure being an extremely delicate one in which the only states that survive as observable ones are composite [22]. Thus, to conclude, we see that in the equal frequency limit the quantum theory based on the fourth order Lagrangian of Eq. (5) is, in fact, completely acceptable (in fact, technically, what we have shown is that the presence of energy eigenstates with negative norm in the unequal frequency theory is simply not a reliable indicator as to their possible presence in the equal frequency case), and it would thus be of interest to see the degree to which our results here might carry over to full fourth order field theories [23].

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[1] K. S. Stelle, Phys. Rev. D **16**, 953 (1977); Gen. Relativ. Gravit. **9**, 353 (1978).

[2] With pure second order theories not possessing ghosts, and with second plus fourth order theories possessing them, it might be presumed that the ghosts arise from the pure fourth order theory itself rather than through an interplay between the second and fourth order theories. However, whether pure fourth order theories are to possess energy eigenstates with negative norm is something which has to be investigated in and of itself, and we are not aware of any demonstration in the literature that pure fourth order theories actually do possess negative norm energy eigenstates.

[3] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University Press, New York, 1964).

[4] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[5] The need to properly take constraints into account was first noted by us in an earlier version of this paper [P. D. Mannheim and A. Davidson, e-print hep-th/0001115 (2000)] where the Dirac quantization was first presented, and also by Hawking and Hertog [S. W. Hawking and T. Hertog, Phys. Rev. D **65**, 103515 (2002)] who quantized the Pais-Uhlenbeck oscillator by Feynman path integration techniques.

[6] Indication of the singular nature of this limit is already seen at the classical level, with the equal frequency solution $q(t) = c_1 e^{-i\omega t} + c_2 t e^{-i\omega t} + \text{c.c.}$ to Eq. (3) (where $\omega_1 = \omega_2 = \omega$) having a time behavior quite different from that of the unequal frequency solution where $q(t) = a_1 e^{-i\omega_1 t} + a_2 e^{-i\omega_2 t} + \text{c.c.}$

[7] M. Ostrogradski, "Memoires sur les equations differentielles relatives au probleme des isoperimetres," Mem. Ac. St. Petersbourg, VI **4**, 385 (1850).

[8] D. Bak, D. Cangemi, and R. Jackiw, Phys. Rev. D **49**, 5173 (1994).

[9] D. G. Boulware, G. T. Horowitz, and A. Strominger, Phys. Rev. Lett. **50**, 1726 (1983).

[10] While asymptotic flatness is natural to second order theories where the exterior solution to the second order Poisson equation $\nabla^2\phi(r) = g(r)$ is given by the falling potential $\phi(r>R) = -\alpha/r$ where $\alpha = \int_0^R dr' g(r') r'^2$, for the fourth order $\nabla^4\phi(r) = g(r)$ the solution is instead given by [P. D. Mannheim and

D. Kazanas, Gen. Relativ. Gravit. **26**, 337 (1994)] the rising potential $\phi(r>R)=-\beta/r+\gamma r$, where $\beta=\int_0^R dr' g(r')r'^4/6$, $\gamma=-\int_0^R dr' g(r')r'^2/2$. Since there is no reason to expect $\int_0^R dr' g(r')r'^2$ to vanish for an arbitrary source, even in the static sector asymptotic flatness would not appear to be an appropriate restriction to place on fourth order theories.

[11] In general, equal time commutators such as $[q(t),p_q(t)]=i$ can be satisfied at all times by an $[a,a^\dagger]=1$ commutator defined via $q(t)=ae^{if(t)}/2^{1/2}+\text{H.c.}$, $p_q(t)=ia^\dagger e^{-if(t)}/2^{1/2}+\text{H.c.}$ with the function $f(t)$ being arbitrary. The occupation number Fock space can thus be defined independent of the structure of H, and even independent of any interaction terms that might also be added on to the free particle H. Thus in general the dimensionality of the Fock space basis need not be the same as that of the eigenspectrum of H, and their basis states need not be in one to one correspondence.

[12] Identifying $a_2^\dagger$ as the annihilator of the Fock vacuum would eliminate negative norm states but would leave H without any lower bound on the ground state energy.

[13] In their study of the Pais-Uhlenbeck oscillator, Hawking and Hertog actually argue that interactions do not in fact lead to any such transitions, thus making the $\omega_1\neq\omega_2$ theory viable.

[14] It is $b^\dagger$ itself which acts as a ladder operator, though $a^\dagger$ does not.

[15] The unequal frequency theory states $|+1\rangle\pm|-1\rangle$ both have zero norm.

[16] The zero norm states $(b^\dagger)^2|\Omega\rangle$ and $(a^\dagger)^2|\Omega\rangle$ can be combined into positive and negative norm states while the state $a^\dagger b^\dagger|\Omega\rangle/\hat{\mu}$ has norm plus one.

[17] A further unusual property of these states is that the one particle matrix element $\langle\Omega|q(t)b^\dagger|\Omega\rangle$ is given by $\hat{\mu}e^{-i\omega t}$ rather than by the coefficient of the b field operator in Eq. (38), with there thus being a mismatch between the second quantized states and the first quantized wave functions.

[18] While we ordinarily use Hermitian matrices in quantum mechanics since a Hermitian matrix necessarily has real eigenvalues, there is, nonetheless, no converse theorem that would require the eigenvalues of a non-Hermitian matrix to be complex. Rather, hermiticity is only sufficient to secure real eigenvalues but not necessary, as the matrix M of Eq. (45) for instance directly demonstrates. Indeed, if we take a diagonal matrix M_1 and add on to it a second matrix M_2 whose diagonal entries are all zero and whose only nonzero entries are all located on only one of the two sides of the leading diagonal, then no matter what specific values these nonzero entries actually take, the secular equations for M_1 and $M_3=M_1+M_2$ will be absolutely identical, with M_2 not contributing to $|M_3-\lambda I|=|M_1-\lambda I|$ at all. As such, we can think of matrices such as M_2 as being divisors of zero, so that just as the addition of zero to an ordinary number does not change its value, the addition of a divisor of zero such as M_2 to a diagonal matrix does not change its eigenvalues. While there is no change in the eigenvalues, there can nonetheless still be a change in the eigenvectors, with there being a possible reduction in the number of eigenvectors in certain cases. It is thus possible to have a non-Hermitian N-dimensional Hamiltonian with N real eigenvalues but less than N eigenvectors. Apart from our interest here in the application of this phenomenon to the equal frequency fourth order oscillator, in passing we note that if applied to neutrinos, it could result in the nonobservability of any right-handed or sterile neutrinos which might accompany the ordinary left-handed ones.

[19] This reduction in the number of energy eigenstates in the $\epsilon\to 0$ limit is a quantum-mechanical reflection of the fact that the equal frequency classical energy $H_{\text{STAT}}(\omega_1=\omega_2)$ given in Eq. (26) becomes completely independent of the coefficient c_1 of the classical solution $q(t)=c_1e^{-i\omega t}+c_2te^{-i\omega t}+\text{c.c.}$ when the c_2 coefficient is set equal to zero, with c_1 thus not representing a dynamical degree of freedom whose quantum analog could materialize as an on shell state.

[20] The authors are indebted to Dr. A. Smilga for a very helpful comment in this regard and also would like to thank him for informing them of his own recent work on ghost issues in fourth order theories—A. V. Smilga, e-print hep-th/0407231 (2004).

[21] For further discussion of such quantum-mechanical issues, see C. M. Bender, D. C. Brody, and H. F. Jones, Am. J. Phys. **71**, 1095 (2003).

[22] A model in which a field has a positive frequency part which annihilates the vacuum strongly and a negative frequency part which annihilates the vacuum weakly in a way such that only certain multiparticle states survive would appear to be a possible candidate mechanism for quark confinement, with quarks themselves then only existing off shell.

[23] Within such pure fourth order theories one of particular interest is the fourth order conformal gravity theory, a theory which has been found capable [P. D. Mannheim, Astrophys. J. **561**, 1 (2001)] of readily resolving the dark matter and dark energy problems which currently challenge the standard second order Newton-Einstein gravitational theory. The conformal theory is based on the imposition of the local Weyl conformal invariance $g_{\mu\nu}(x)\to e^{2\alpha(x)}g_{\mu\nu}(x)$, and thus has a unique gravitational action of the form $I_W=-\alpha_g\int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}$ where $C^{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor and α_g is a dimensionless coupling constant. With the general conformal gravity rank two gravitational tensor $-(-g)^{-1/2}\delta I_W/\delta g_{\mu\nu}=2\alpha_g W^{\mu\nu}$ reducing to $W^{\mu\nu}=\Pi^{\mu\rho}\Pi^{\nu\sigma}K_{\rho\sigma}/2-\Pi^{\mu\nu}\Pi^{\rho\sigma}K_{\rho\sigma}/6$ in a $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ linearization around flat spacetime (here $K^{\mu\nu}=h^{\mu\nu}-\eta^{\mu\nu}h^{\alpha}{}_{\alpha}/4$ and $\Pi^{\mu\nu}=\eta^{\mu\nu}\partial^\alpha\partial_\alpha-\partial^\mu\partial^\nu$), we find in the conformal gauge $\partial_\nu g^{\mu\nu}-g^{\mu\sigma}g_{\nu\rho}\partial_\sigma g^{\nu\rho}/4=0$ [viz. a gauge condition which is left invariant under $g_{\mu\nu}(x)\to e^{2\alpha(x)}g_{\mu\nu}(x)$] that the source-free region gravitational fluctuation wave equation then reduces to $\alpha_g(\partial_0^2-\nabla^2)^2K^{\mu\nu}=0$. With this equation of motion being decoupled in its tensor indices, we see that each tensor component precisely obeys none other than the $M^2=0$ limit of Eq. (1). Since the conformal gravity theory is power counting renormalizable (due to the dimensionlessness of α_g), it would thus be of interest to see if the structure we have found for the equal frequency Pais-Uhlenbeck theory might carry over to the conformal gravity theory, since it might then be possible to construct a fully renormalizable, fully unitary gravitational theory in four spacetime dimensions, one which despite its fourth order equation of motion, would nonetheless only possess one on-shell graviton and not two.