Path summation and von Neumann–like quantum measurements

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We demonstrate how a general von Neumann–like measurement can be analyzed in terms of histories (paths) constructed for the measured variable A. The Schrödinger state of a system in a Hilbert space of arbitrary dimensionality is decomposed into a set of substates, each of which corresponds to a particular detailed history of the system. The coherence between the substates may then be destroyed by meter(s) to a degree determined by the nature and the accuracy of the measurement(s) which may be of von Neumann, finite-time, or continuous type. The cases of a particle described by Feynman paths in the coordinate space and a qubit in a two-dimensional Hilbert space are studied in some detail.

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I. INTRODUCTION

Path integrals and, more generally, the path summation techniques [1–3] have found broad application in quantum mechanics. One advantage of such techniques is that they reduce the task of calculating quantum-mechanical amplitudes to straightforward summation over certain subsets of particle histories. It is tempting to use the path integral as a starting point for the analysis of quantum measurements. An analysis, based exclusively on devising a meter or a measurement model is usually incomplete, as it provides only a limited theoretical insight into the nature of the measured quantity [4,5]. Possessing the probability amplitudes for a complete set of the system's histories, allows one to construct the amplitude for a particular property simply by adding up amplitudes for those histories which share such a property. Yet imposing formal restrictions on quantum histories, such as Feynman paths [6,7], has the disadvantage of leaving open the question of how, if at all, the obtained probabilities relate to the observable quantities. It is advantageous, therefore, to analyze the case where the two approaches are complementary, and the probabilities assigned directly to the histories are automatically those one would measure with a suitable measuring device. One such example is measurements of von Neumann type. Historically, the path integral approach to measurements was pioneered in [8,9] and its later applications relevant to present work can be found in [8–18]. There are different types of quantum measurement to consider: instantaneous von Neumann measurements [19], most commonly used in applications such as quantum-information theory, finite-time measurements [20] studied in [13–17] in connection with the tunneling time problem, and continuous measurements [10–12], where a record of the particle's evolution is produced by a "measuring medium." In addition, measurements of the same type may differ in accuracy, depending on the strength of interaction between the system and a meter or an environment. Some peculiar properties of inaccurate "weak" measurements, proposed in [21], are discussed in [22].

The purpose of this paper is to establish general equivalence between the path summation formalism and the conventional meter-based analysis of von Neumann–like measurements in a Hilbert space of arbitrary dimension. Some work on the relation between the restricted Feynman path integral and von Neumann–like meters can be found in [13,14]. As we aim to justify the previous use of the path summation approach, as well as to expose its limitations, our development will be largely formal, and for detailed illustrations of the method the reader is referred to our earlier work [15,17,18]. We also note that alternative approaches to assigning quantum probabilities, such as the one based on consistent histories (CH) [23], are beyond the scope of the present analysis, although some discussion of the CH approach was given in [16]. The rest of the paper is organized as follows. In Sec. II we introduce a functional differential equation to generate a decomposition of the Schrödinger state of a system corresponding the most detailed set of histories (paths) for a particular variable A. In Sec. III we establish the link between the paths and the measurement amplitudes for a meter, or meters, employed to measure A. In Sec. IV we show how finite resolution of the meter can be incorporated into the path summation approach. In Sec. V we show that only the paths taking values among the eigenvalues of $\hat{A}$ contribute to the fine grained amplitude introduced in Sec. II, and VI extends the approach to all quantities that commute with $\hat{A}$. Section VII contains our conclusions.

II. DECOMPOSITION OF THE SCHRÖDINGER STATE

We will follow Ref. [16] in relating a general quantum measurement to a decomposition of the current state of the system $|\Psi(t)\rangle$ into a set of (generally, nonorthogonal) substates $|\Phi[n]\rangle$, where the index n refers to a particular history. This can be illustrated by a simple example, equivalent to the usual two-slit experiment. Let a wave packet $|\Psi_0\rangle$ be split (e.g., by means of a beam splitter) into two parts $|\Phi[n]\rangle$, $n=1, 2$, which thereafter travel along two different routes. At a later time t, the parts of the wave function are brought to-

gether in the same spatial region, so that the state of the system is a sum of two overlapping components ($n=1,2$)

$$|\Psi(t)\rangle = \sum_n |\Phi(t|[n])\rangle, \tag{1}$$

each corresponding to a particular history. Two cases must then be considered separately. For an isolated system in a pure state $|\Psi(t)\rangle$, the two routes are interfering alternatives, no probabilities can be assigned, and one can only use Eq. (1) to see how the components interfere to produce $|\Psi(t)\rangle$. If, on the other hand, the two alternatives have been made exclusive, e.g., by reversing the direction of the particle's spin when traveling along one of the routes, one finds the system (after tracing out the spin variable) in a mixed state and is able to assign probabilities for taking a particular path. In either case it is Eq. (1) that contains all available information about the particle's past.

The same reasoning applies to a more general case. Let $\hat{A}$ be the operator representing some variable A of a quantum system, which at the time $t=T$ is in a state $|\Psi(T)\rangle$. We ask next what, if anything, can be said about the value $\varphi(t')$ of the variable A within an interval $0\leqslant t'\leqslant T$. It is sufficient to assume that A may take a real value at each t', i.e., that the sequence of values $\varphi(t')$ is a continuous but not necessarily differentiable real function of time. We will refer to $\varphi(t')$ as a path, or history, which, in the following, will replace the discrete index n in Eq. (1). Just as in the above example, the path taken by an isolated system cannot be known, but there must be a substate $|\Phi(T|[\varphi])\rangle$ (square brackets are used to indicate the functional dependence) representing the (virtual) contribution the path makes to the Schrödinger state $|\Psi(T)\rangle$ so that the functional

$$\eta[\varphi] = \langle q|\Phi(T|[\varphi])\rangle \tag{2}$$

yields the conditional probability amplitude to find the system in a state $|q\rangle$ at $t=T$ provided A had the values $\varphi(t')$ at $0\leqslant t'\leqslant T$. As in Eq. (1), a coherent sum over all available paths must restore $|\Psi(T)\rangle$,

$$|\Psi(T)\rangle = \sum_{paths} |\Phi(T|[\varphi])\rangle \equiv \int D\varphi |\Phi(T|[\varphi])\rangle, \tag{3}$$

where the symbol $\int D\varphi$ incorporates integrations over all $\varphi(t')$'s, including the end values $\varphi(0)$ and $\varphi(T)$ (see Appendix A). The substates $|\Phi(t|[\varphi])\rangle$ still need to be defined and, for reasons that will become clear later, we do so with the help of the functional differential equation [24]

$$i\partial_t|\Phi(t|[\varphi])\rangle = \left\{\hat{H} - i\hat{A}\frac{\delta}{\delta\varphi(t_-)}\right\}|\Phi(t|[\varphi])\rangle, \tag{4}$$

where $\hat{H}$ is the Hamiltonian of the isolated system and $|\Phi(t|[\varphi])\rangle$ is subjected to the initial condition

$$|\Phi(t=0|[\varphi])\rangle = |\Psi_0\rangle\delta[\varphi]. \tag{5}$$

In Eqs. (4) and (5), $|\Psi_0\rangle \equiv |\Psi(t=0)\rangle$ is the initial state of the system at $t=0$, the subscript "$-$" (to be omitted in the following) indicates that the variational derivative is taken at the time just preceding the current time t, and $\delta[\varphi]$ is the δ functional, such that for any functional $F[\varphi]$, the integral $\int D\varphi\, F[\varphi]\delta[\varphi] = F[\varphi\equiv 0]$ (see Appendix A). Summing Eqs. (4) and (5) and over all paths φ and using the identity $\int D\varphi\, \delta F[\varphi]/\delta\varphi(t) = 0$ [see Eq. (A5) of Appendix A] shows that at any t, the substates $|\Phi(t|[\varphi])\rangle$ add up to $|\Psi(t)\rangle$, as prescribed by Eq. (3). One should note that the index $[\varphi]$ belongs to the fixed set $\{\varphi\}$ of all continuous real functions taking arbitrary values in the interval $[0,T]$, whereas the time t in Eq. (4) may be less than, equal to, or greater than T. Equation (4) describes, therefore, "spreading" of a system, initially localized at the zero path $\varphi(t')\equiv 0$, across the functional space $\{\varphi\}$, and, when integrated to $t=T$, yields the required set $|\Phi(T|[\varphi])\rangle$. Keeping $\varphi(t')$ explicitly independent of t helps simplify the use of Eq. (4) in subsequent derivations. Equation (4) formally resembles that for a von Neumann meter [19] (with the meter's momentum replaced by the functional derivative over φ) and we shall explore this similarity further. The explicit form of $|\Phi(t|[\varphi])\rangle$ can be obtained by writing it as a Fourier integral,

$$|\Phi(t|[\varphi])\rangle = \int D\lambda \exp\left(i\int_0^T \lambda(t')\varphi(t')dt'\right)|\Phi(t|[\lambda])\rangle. \tag{6}$$

Inserting Eq. (6) into Eq. (4) shows that the functional Fourier transform $|\Phi(t|[\lambda])\rangle$ satisfies the Schrödinger equation in which a time-dependent term $\lambda(t)\hat{A}$ has been added to the original Hamiltonian $\hat{H}$,

$$i\partial_t|\Phi(t|[\lambda])\rangle = \{\hat{H} + \lambda(t)\hat{A}\}|\Phi(t|[\lambda])\rangle, \tag{7}$$

$$|\Phi(t=0|[\lambda])\rangle = \tilde{\delta}|\Psi_0\rangle \tag{8}$$

where $\tilde{\delta}$ is the constant defined in Eq. (A8). It can, therefore, be formally written as $\exp[-i\int_0^t(\hat{H}+\lambda(t')\hat{A})dt']|\Psi_0\rangle\tilde{\delta}$, so that from Eq. (6) we have

$$|\Phi(t|[\varphi])\rangle \sim \int D\lambda \exp\left(i\int_0^T \lambda(t)\varphi(t)dt\right) \times\exp\left(-i\int_0^T\left[\hat{H}+\lambda(t')\hat{A}\right]\theta_t(t')dt'\right)|\Psi_0\rangle, \tag{9}$$

where $\theta_t(z)\equiv 1$ for $z<t$ and 0 otherwise. It is readily seen that by a time $t<T$ the operator term only affects $t'\leqslant t$ so that only the paths such that $\varphi(t')\equiv 0$, $t<t'<T$ may have nonzero substates $|\Phi(t|[\varphi])\rangle$ assigned to them [Fig. 1(a)]. This suggests the following tentative interpretation for Eq. (4) and the initial condition (5). Consider a continuous array of meters with pointer positions $\varphi(t')$, $0<t'<T$, such that the meter with the position $\varphi(t)$ "fires" at the time t. Initially, all the pointers are set to zero. At $t<T$ some of the meters have fired, "recording" a history $\varphi(t')$, $0<t'<t$, while those with $\varphi(t')$, $t<t'<T$, have not yet been enacted. Once the elapsed time exceeds T, the amplitudes for all paths are fixed and no longer subject to change with t. The analogy with a

classical data recorder monitoring the value of some variable A suggests labeling (4) the "quantum recorder" equation. Note, however, that while in the classical case a unique record is produced as the time progresses, a "quantum recorder" (4) employs the complete set $\{\varphi\}$ of all virtual records and assigns a time-dependent (possibly zero) substate $|\Phi(t|[\varphi])\rangle$ to each path. Equation (4), which, to our knowledge, has not appeared in the literature previously, provides the most detailed information about the possible past values of A and will be the basis of our analysis. Next we show that the probability amplitudes for different measurements of A can be obtained by coherent summation of the fine grained substates $|\Phi(t|[\varphi])\rangle$, subject to appropriate restrictions.

III. RESTRICTED PATH SUMS AND METERS

In the previous section we have produced a decomposition of the state of a quantum system into substates labeled by the record of a variable A. It remains to demonstrate that such decomposition is meaningful, i.e., that Eq. (2) and similar expressions relate to valid physical probabilities. Such a justification is usually obtained by showing either that the probabilities satisfy some formal criterion for a system in isolation [6,7], or that they describe the outcomes of a measurement should the system be coupled to a suitable meter [16]. We will follow the latter route and start by demonstrating that for $t \leqslant T$ the weighted sum of the substates,

$$|\Phi(t|\vec{\lambda})\rangle_{\vec{\beta}} \equiv \int D\varphi |\Phi(t|[\varphi])\rangle \exp\left(i\sum_{j=1}^{M} \lambda_j \int_0^T \beta_j(t')\varphi(t')dt' \right) \tag{10}$$

where $\beta_j(t)$, $j=1,2,\ldots,M$, are some known functions of time, satisfies a Schrödinger-like equation (we will omit the subscript $\vec{\beta}$)

$$i\partial_t |\Phi(t|\vec{\lambda})\rangle = \left\{ \hat{H} + \sum_{i=1}^{M} \lambda_j \beta_j(t) \hat{A} \right\} |\Phi(t|\vec{\lambda})\rangle \tag{11}$$

with the initial condition

$$|\Phi(t=0|\vec{\lambda})\rangle = |\Psi_0\rangle. \tag{12}$$

Equation (11) is readily obtained if Eq. (4) is multiplied by $\exp[i\Sigma_{i=1}^{M}\lambda_j \int_0^T \beta_j \varphi\, dt']$, integrated over $\int D\varphi$, and the term containing the variational derivative $\delta/\delta\varphi(t)$ is integrated by parts. Equation (12) then follows upon inserting Eq. (5) into Eq. (10). Taking a further Fourier transform with respect to $\vec{\lambda}(\vec{\lambda}\vec{f} \equiv \Sigma_{j=1}^{M}\lambda_j f_j)$,

$$|\Phi(t|\vec{f})\rangle \equiv (2\pi)^{-M} \int_{-\infty}^{\infty} \exp(i\vec{\lambda}\vec{f}) |\Phi(t|\vec{\lambda})\rangle d\vec{\lambda}, \tag{13}$$

yields the equation of motion for $|\Phi(t|\vec{f})\rangle$,

$$i\partial_t |\Phi(t|\vec{f})\rangle = \left\{ \hat{H} - i\sum_{j=1}^{M} \partial_{f_j} \beta_j(t) \hat{A} \right\} |\Phi(t|\vec{f})\rangle, \tag{14}$$

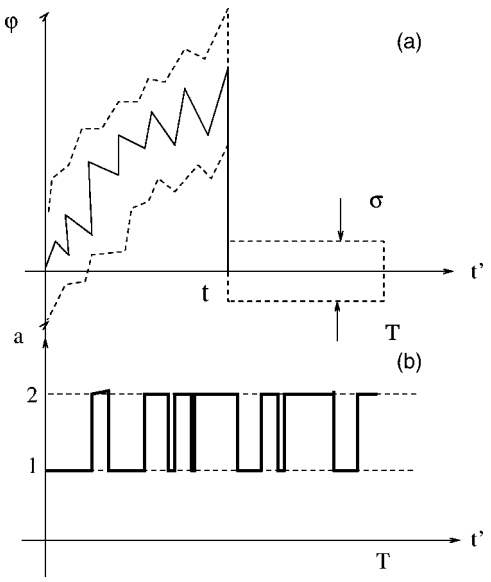

FIG. 1. (a) A schematic diagram of a path $\varphi(t')$ which contributes to $|\Phi(t|[\phi])\rangle$ at $t<T$ (solid). Also shown by a dashed line is the tube which contains the paths, contributing to $|\Psi(t|[\varphi])\rangle$, obtained by Gaussian coarse graining with the width σ. (b) An eigenpath which contributes to $|\Phi(T|[\varphi])\rangle$ for a two-level system ($a_1=1, a_2=2$).

$$|\Phi(t=0|\vec{f})\rangle = |\Psi_0\rangle \prod_{j=1}^{M} \delta(f_j). \tag{15}$$

It is readily seen that Eq. (14) describes a system governed by the Hamiltonian $\hat{H}$ which, in addition, interacts with M external degrees of freedom playing the role of meters with f_j, $j=1,2,\ldots,M$. Each coupling of the von Neumann type contains the product of the measured quantity $\hat{A}$ and the meter's momentum $-i\partial_{f_j}$. The strength of the coupling is determined by the known switching function $\beta_j(t)$. Initially the system and the meters are prepared in the product state (15) and at the end of the measurement, at $t=T$, the system is described by the density operator

$$\hat{\rho} = \int d\vec{f} |\Phi(t|\vec{f})\rangle\langle\Phi(t|\vec{f})|. \tag{16}$$

An accurate determination of the meters' positions f_j at $t=T$ yields information about the system's past. We shall refer to such procedure as a measurement of the von Neumann type. Precise nature of the information obtained in a von Neumann–like measurement can be highlighted by interchanging the order of integration over $D\varphi$ and $\vec{\lambda}$ in Eqs. (10) and (13), which yields

$$|\Phi(t|\vec{f})\rangle = \int D\varphi \prod_{j=1}^{M} \delta(F_j[\varphi] - f_j) |\Phi(t|[\varphi])\rangle \tag{17}$$

with the functionals $F_j[\varphi]$ defined by

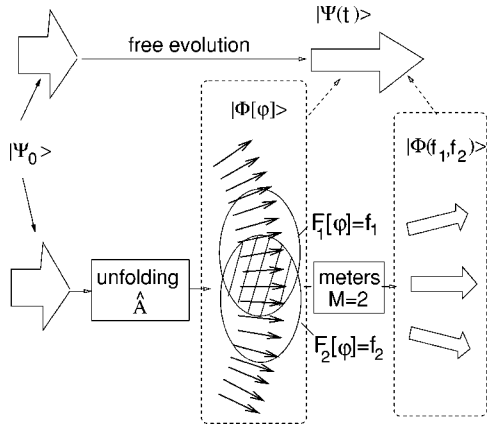

FIG. 2. A system starts in a state $\Psi_0\rangle$ which, without measurements, would evolve into $\Psi(t)\rangle$. $\Psi(t)\rangle$ can be decomposed into a fine set of substates $|\Psi[\varphi]\rangle$, each labeled by a particular history $\varphi(t')$. A meter decomposes $\Psi(t)\rangle$ into less informative substates, labeled by the value f of functional $F[\varphi]$, which are obtained by summing $|\Psi[\varphi]\rangle$ subject to the restriction $F[\varphi]=f$. For the two meters shown, the substates in the shaded area are such that $F_1[\varphi]=f_1$ and $F_2[\varphi]=f_2$, and add up coherently to the state $|\Phi(f_1\cdot f_2)\rangle$, whose norm determines the probability to register both f_1 and f_2.

$$F_j[\varphi] \equiv \int_0^T \beta_j(t')\varphi(t')dt'. \tag{18}$$

Thus, $|\Phi(t|\vec{f})\rangle$ is given by a path sum, restricted to those paths, for which the functionals F_j, $j=1,2,\ldots,M$ have the values f_j, and the set of paths $\{\varphi\}$ is divided into classes $\{\varphi\}_{\vec{f}}$,

$$\{\varphi\}_{\vec{f}} \equiv \{\varphi\colon F_j[\varphi]=f_j, \quad j=1,\ldots,M\}. \tag{19}$$

If a meter or meters are employed, $\{\phi\}_{\vec{f}}$ play the role of alternative "routes" along which the system may evolve from its initial state and a time-dependent probability amplitude can be assigned to each of them. Individual paths cannot be told apart within each class, and some of the detailed information contained in the substates $|\Phi(t|\vec{f})\rangle$ is lost through the residual interference (see Fig. 2). It is a remarkable property of the von Neumann interaction (14), which allows the measurement process to be analyzed either in terms of the dynamical interaction with the pointer degree(s) of freedom, or, which is conceptually simpler, in terms of restricting paths initially defined for an isolated system.

It is convenient to classify the most common of such measurements according to the type(s) of the switching functions and the number of meters involved.

A *von Neumann measurement* for which $M=1$, $\beta_1(t)=\delta(t-t_0)$, and which determines the instantaneous value of an operator $\hat{A}$ at some $t=t_0$ [19].

A *finite-time measurement*, $M=1$, $\beta(t)=\text{const}$, which determines the time average of an operator $\hat{A}$ over the time T. Measurements of this type were first discussed in [20] and extensively studied in connection with the tunneling time problem [13–17]. The spectral measurements [10] with $\beta_j(t)=(2/T)\sin\Omega t$, $j\geqslant 1$, also fall into this category.

A *continuous measurement*, where $M\to\infty$, $\beta_j(t)\sim\delta(t-t_j)$. In this limit, a sequence of values f_j, $j=1,2,\ldots,M$, is replaced by a continuous function $f(t)$, $\vec{f}\to f(t)$. Continuous measurements, where the meters form a "measuring medium," are analyzed in [10–12].

This list is not exhaustive, and one can envisage various sequences and combinations of von Neumann, finite-time, and continuous measurements.

IV. FINITE ACCURACY AND COARSE GRAINING

The previous section describes an ideal, i.e., infinitely accurate, measurement in which the initial state of each pointer is an (unnormalizable) δ function. The initial state of a physical meter, $G(f)$, must be a square integrable (e.g., a Gaussian) wave packet,

$$\int |G(\vec{f})|^2 d\vec{f} < \infty, \tag{20}$$

which inevitably makes the initial pointer positions uncertain. Importantly, the effect of this additional uncertainty can also be described in terms of the restricted path sums. We note first that a solution of Eq. (7) can be multiplied by an arbitrary functional $\tilde{G}[\lambda]$. Taking the Fourier transform and using the convolution property (A11) shows that

$$|\Psi(t|[\varphi])\rangle \equiv \int D\varphi' G[\varphi-\varphi']|\Phi(t|[\varphi'])\rangle, \tag{21}$$

where $G[\varphi]$ is the Fourier transform of $\tilde{G}[\lambda]$, is a solution of Eq. (4) with the initial condition

$$|\Psi(t=0|[\varphi])\rangle = \int D\varphi' G[\varphi-\varphi']\delta[\varphi']|\Psi_0\rangle = G[\varphi]|\Psi_0\rangle. \tag{22}$$

Repeating the arguments of the previous section leading to Eq. (14) shows that the restricted path sum

$$|\Psi(t|\vec{f})\rangle \equiv \int D\varphi \prod_{j=1}^{M} \delta(F_j[\varphi]-f_j)|\Psi(t|[\varphi])\rangle \tag{23}$$

with $F_j[\varphi]$ defined in Eq. (18) also satisfies the meter equation (14) and the initial condition

$$|\Psi(t=0|\vec{f})\rangle = \int D\varphi \prod_{j=1}^{M} \delta(F_i[\varphi]-f_i)G[\varphi]|\Psi_0\rangle \equiv G(\vec{f})|\Psi_0\rangle. \tag{24}$$

Thus, choosing the functional G in Eq. (22) to be

$$G[\varphi] = G(F_1[\varphi],F_2[\varphi]\cdots F_M[\varphi]) \tag{25}$$

we can construct the wave function $|\Psi(t|\vec{f})\rangle$ describing a particle interacting with the meter(s), initially prepared in any desired product state,

$$|\Psi(t=0|\vec{f})\rangle = G(\vec{f})|\Psi_0\rangle. \tag{26}$$

Recalling that the functionals F_j are linear in φ, $F_j[\varphi+\varphi'] = F_j[\varphi]+F_j[\varphi']$, and using Eq. (23) we can rewrite $|\Psi(t|\vec{f})\rangle$ as a restricted path sum

$$|\Psi(t|\vec{f})\rangle = \int D\vec{f}'\, G(f_1 - F_1[\varphi]\ldots, f_M - F_M[\varphi])|\Phi(t|[\varphi]\rangle \tag{27}$$

in which the weight G for a path φ is determined by the initial state of the meters. Equations of this type have been used to analyze spectral measurements of position in [10] and in a more general context in [15]. Finally, writing $G(f_1 - F_1[\varphi],\ldots) = \int d\vec{f}'\, G(\vec{f}-\vec{f}')\Pi_{j=1}^{M}\delta(f_j - F_j[\varphi])$ and using Eq. (18) yields

$$|\Psi(t|\vec{f})\rangle = \int D\varphi\, G(\vec{f}-\vec{f}')|\Phi(t|\vec{f}')\rangle, \tag{28}$$

which can be verified by direct substitution into Eq. (14) and subsequent integration by parts. The physical meaning of Eq. (28) is best illustrated by choosing the shape of a rectangular window for a single meter, $M=1$,

$$G(f) = 1 \quad \text{for} \quad |f| < \Delta/2 \quad \text{and} \quad 0 \quad \text{otherwise}. \tag{29}$$

Taking the expectation value of the projection operator

$$\hat{P}(q,f) \equiv |f\rangle|q\rangle\langle q|\langle f|, \tag{30}$$

with the system-meter wave function (27) and (28) yields

$$|\langle q|\Psi(T|f)\rangle|^2 = \left| \int_{-\Delta/2}^{\Delta/2} df \int D\varphi\, \delta(f - F[\varphi])\, \eta[\varphi] \right|^2 \tag{31}$$

where $\eta[\varphi]$ is the probability amplitude for the path $\varphi(t')$ defined in Eq. (2). The left-hand side of Eq. (31) is the standard expression of the joint probability for finding the system in a state $|q\rangle$ and the pointer at f. Its right-hand side is the same probability expressed in terms of the amplitude assigned to the class of paths $\{\varphi\}_{f,\Delta}$, for which the value of the functional $F[\varphi]$ lies between $f-\Delta/2$ and $f+\Delta/2$,

$$\{\varphi\}_{f,\Delta} \equiv \cup\{\varphi\}_{f'}, \quad f-\Delta/2 < f' < f+\Delta/2, \tag{32}$$

where $\cup$ denotes the union of sets of paths. The probability amplitude for $\{\varphi\}_{f,\Delta}$ is found by adding those for its individual members. This assignment is justified, as the expression is readily linked to the frequencies observed if a meter is employed. Thus, the initial uncertainty in the meter's position result in broadening the set of paths, to which we shall refer as coarse graining [25], and gives an uncertainty in the value of the functional $F[\varphi]$. Note that such an interpretation is possible because both Eq. (4) and the von Neumann Hamiltonian (14) contain no explicit dependence on $\varphi(t)$ or the pointer position f, respectively. We note further that giving $G(f)$ a Gaussian rather than a rectangular shape would result in a restricted path sum with different weights for the paths with different values of F, and that for a $G(f)$ consisting of two disjointed "windows," the distribution of the meter's reading would exhibit an interference pattern similar to that observed in a double-slit experiment. If the final position of the meter is also determined with a finite accuracy or, more generally, the meters are observed in some basis $\phi_n(\vec{f})$, $n=1,2,\ldots$, one should replace in Eq. (28) the weight G by

$$G_n(\vec{f}) \equiv \int df'\, \phi_n^*(\vec{f})G(\vec{f}-\vec{f}'). \tag{33}$$

Finally, we recall the relation between the accuracy of a measurement and the strength of the coupling between the measured system and the meter(s) [17]. Indeed, the resolution of the meters can be improved, by replacing the initial state $G(\vec{f})$ by $G(\alpha\vec{f})$, $\alpha>1$. A change of variables $\vec{f}\to\alpha\vec{f}$ shows that the resulting finer set of substates satisfies Eq. (14) with the old initial condition $|\Psi(t=0|\vec{f})\rangle = G(\vec{f})|\Psi_0\rangle$, but with the coupling term increased α-fold, $i\alpha\Sigma_{j=1}^{M}\partial_{f_j}\beta_j(t)\hat{A}$. The same can be observed by writing $|\Psi(T|\vec{f})\rangle$ as

$$|\Psi(T|\vec{f})\rangle = \int d\vec{\lambda}\, G(\vec{\lambda})\exp\left\{ -i\int_0^T \left[\hat{H} + \hat{A}\sum_{j=1}^{M} \lambda_i\beta_i(t)dt \right] \right\} \times|\Psi_0\rangle, \tag{34}$$

where $G(\vec{\lambda})$ is the Fourier transform of $G(\vec{f})$, which shows that the substate is obtained by evolving the initial state of the system with the Hamiltonians involving all possible magnitudes of the coupling. Among these, only the $\vec{\lambda}=\vec{0}$ term corresponds to the unperturbed evolution, while the rest contain the effects of the meter. As the coarse graining becomes finer, $G(\vec{f})\to G(\alpha\vec{f})$, the Fourier transform becomes broader, $G(\vec{\lambda})\to\alpha^{-1}G(\alpha\vec{\lambda}/\alpha)$, and the number of $\vec{\lambda}\neq\vec{0}$ which contribute to the formation of the substate $|\Psi(T|\vec{f})\rangle$ therefore increases.

V. EIGENPATHS AND EXAMPLES

So far our approach has been general and relied neither on the properties of the Hermitian operator $\hat{A}$ in Eq. (4) nor on the dimension of the Hilbert space of the measured system. To complete our analysis, it remains to demonstrate that Eq. (4) generates a nonzero substate $|\Phi(t|[\varphi])\rangle$ only for such paths that at any given time t' the value of $\varphi(t')$ coincides with one of the eigenvalues a_i of the $\hat{A}$, which we will presently assume to be nondegenerate. The set of such eigenpaths, $\{a\}$, may coincide with $\{\varphi\}$ or form a smaller subset embedded in the latter. Consider the time-discretized version of Eq. (9), whereby we slice the time interval $[0,T]$ into N subintervals ϵ, so that

$$t_j \equiv (j-1)\epsilon, \quad z_j \equiv z(t_j), \quad K \equiv \text{int}\{\min(t,T)/\epsilon\},$$

where int$\{x\}$ denotes the integer part of x. Thus the operator in the right-hand side (RHS) of Eq. (9) takes the form

$$\exp\left\{-i\int_0^t [\hat{H}+\lambda(t')\hat{A}]dt'\right\} = \lim_{N\to\infty}\prod_{j=1}^{K}\exp[-i\lambda_j\hat{A}\epsilon] \times\exp[-i\hat{H}\epsilon], \tag{35}$$

where we have made use of the Trotter product formula [2] (see also Appendix B) to factorize the exponentials containing $\hat{H}$ and $\lambda\hat{A}$. Using the eigenstates of $\hat{A}$, $|a_k\rangle$, to write

$$\exp[-i\lambda_j\hat{A}\epsilon] = \sum_k \exp(-i\lambda_j a_k)|a_k\rangle\langle a_k|, \tag{36}$$

and performing integrations over λ_j, $j=1,2,\ldots,N-1$ yields

$$|\Phi(t|[\varphi])\rangle = \sum_{[a]} \delta[\varphi-\theta_t a]|\Phi_t[a]\rangle, \tag{37}$$

$$|\Phi_t[a]\rangle \equiv \hat{U}_t[a]|\Psi_0\rangle, \tag{38}$$

where

$$\hat{U}_t[a] \equiv \lim_{N\to\infty}\prod_{j=1}^{K}|a_{k_j}\rangle\langle a_{k_j}|\exp(-i\hat{H}\epsilon), \tag{39}$$

$a_{k_j}\to a(t')$, and for an arbitrary functional Z we have introduced the notation

$$\sum_{[a]} Z[a] \equiv \lim_{N\to\infty}\sum_{k_1\ldots k_N} Z(a_{k_1},a_{k_2},\ldots,a_N). \tag{40}$$

Therefore, a path $\varphi(t')$ contributes to the state of the system $|\Psi(T)\rangle$ if, and only if, at any time $t'\leqslant t$, $\varphi(t')=a_k$, in which case the substate itself is the eigenstate corresponding to the eigenvalue $\varphi(t)$. Note that we could have used Eqs. (37) and (39) to define the decomposition of the system's state and then establish its relation to von Neumann–like measurements, but chose instead an equivalent approach, based on Eq. (4). It is easy to check that in Eq. (37) each term in the sum over the eigenpaths $a(t')$ satisfies Eq. (4) with the initial condition (5) (see Appendix C).

For $t=T$ and $\theta_t(t')\equiv 1$, and inserting Eq. (37) into Eq. (17) gives the expression of the measurement amplitude as a restricted sum over eigenpaths,

$$|\Phi(T|\vec{f}\rangle = \sum_{[a]}\prod_{j=1}^{M}\delta(f_j-F_j[a])|\Phi_T|[a]\rangle, \tag{41}$$

where $F_j[a]\equiv\int_0^T\beta_j(t')a(t')dt'$. Integrating Eq. (37) over $D\varphi$ gives the path expansion of the propagator,

$$|\Psi(T)\rangle = \sum_{[a']}\hat{U}_T[a']|\Psi_0\rangle, \tag{42}$$

equivalent to the identity

$$\sum_{[a']}\hat{U}_T[a'] = \exp(-i\hat{H}T). \tag{43}$$

The nature of the summation over a' depends on the spectrum of $\hat{A}$ and next we consider two examples which illustrate the connection between Eq. (4) and some well-known path summation techniques.

(1) *The Feynman path integral*. Let the system be a one-dimensional particle of mass m in a potential $V(x)$ and the variable of interest its position x, $\hat{A}\equiv\hat{x}$. The coordinate histories generated by the equation

$$i\partial_t|\Phi(t|[\varphi])\rangle = \left\{-\partial_x^2/2m + V(x) - ix\frac{\delta}{\delta\varphi(t)}\right\}|\Phi(t|[\varphi])\rangle \tag{44}$$

are the celebrated Feynman paths [1]. If the position operator $\hat{x}=x$ has a continuous spectrum extending from $-\infty$ to $+\infty$, the set of paths $\{x(t)\}$ in Eq. (37) coincides with $\{\varphi(t)\}$ in Eq. (3), the sum $\Sigma_{[a]}$ becomes $\int dx_j$, and the path sums (37) and (3) are essentially the same. Further, the standard derivation shows (see, for example, Ref. [2])

$$\langle x|\hat{U}_T[x]|x'\rangle = \lim_{N\to\infty}(m/2\pi\epsilon)^{N/2}\exp(iS[x]), \tag{45}$$

where $S[x]=\int_0^T[m\dot{x}^2/2-V(x)]dt'$ is the classical action, and Eq. (42) becomes the familiar expression for the Feynman propagator [1]. Measurement amplitudes obtained by restricting the Feynman path integral (43) have been often studied in literature (see, for instance Refs. [13–17,10]).

(2) *Path sum for a two-level system (qubit)*. Another example is a two-level system in a two-dimensional Hilbert space. A two-dimensional version of Eq. (4) has the form

$$i\partial_t|\Phi(t|[\varphi])\rangle = \begin{pmatrix}\epsilon_1 & V\\ V & \epsilon_2\end{pmatrix}|\Phi\rangle - i\frac{\delta}{\delta\varphi(t)}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}|\Phi\rangle \tag{46}$$

where $|\Phi(t|[\varphi])\rangle$ is a two-component vector in the representation in which the "position operator," given by the second matrix on the right, is diagonal, and, without loss of generality, we have ascribed "coordinates" 1 and 2 to the first and second states, respectively. Now the eigenpaths $a(t')$ in Eq. (39) can take only the values 1 or 2 at any given time, which they can change at any t' [Fig. 1(b)]. Each such jump is facilitated by the off-diagonal part of the Hamiltonian, proportional to V. Rearranging in Eq. (43) the paths according to the number of jumps and summing over all paths yields the expansion of the evolution operator in powers of V,

$$\hat{U}(T) = 1+\sum_{n=1}^{\infty}(-i)^n\int_0^T dt_n\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_2}dt_1 \times\exp[-i\hat{H}(T-t_n)]V\exp[-i\hat{H}(t_n-t_{n-1})V\cdots V \times\exp[-i\hat{H}t_1], \tag{47}$$

which is the standard decomposition of the perturbation theory [26].

VI. COMMUTING OPERATORS AND FEYNMAN'S FUNCTIONALS

Once a path decomposition has been obtained for an operator $\hat{A}$ with nondegenerate eigenvalues a_k, the method can be extended to also describe von Neumann–like measurements of the quantities that commute with $\hat{A}$. Such operators

share with $\hat{A}$ its complete set of eigenstates $|a_j\rangle$, $j=1,\ldots,n$, and can be written in the form

$$\hat{A}' = \mathcal{F}(\hat{A}) \equiv \sum_{j=1}^{n} |a_j\rangle \mathcal{F}(a_j) \langle a_j|, \tag{48}$$

where the eigenvalues $\mathcal{F}(a_j)$, $j=1,2,\ldots$, may or may not be all different. The fine grained decomposition $|\Phi^{\mathcal{F}(A)}[\varphi]\rangle$ for the operator $\mathcal{F}(\hat{A})$ can then be written as

$$|\Phi^{\mathcal{F}(A)}(T,[\varphi])\rangle = \int D\varphi' \, \delta[\varphi - \mathcal{F}(\varphi')] |\Phi^{A}(T,[\varphi])\rangle. \tag{49}$$

Indeed, repeating the derivation of the previous section for $\mathcal{F}(\hat{A})$ in place of $\hat{A}$ yields Eq. (37) with $\theta_t a$ replaced by $\theta_t \mathcal{F}(a)$, which for $t=T$ becomes Eq. (49). If none of the eigenvalues of $\mathcal{F}(\hat{A})$, $\mathcal{F}(a_j)$, are degenerate, the two sets of substates are identical, and there is a one-to-one correspondence between the sets of eigenpaths, i.e., the same substate corresponds to the eigenpath $\varphi(t')=a(t')$ for the operator $\hat{A}$ and the eigenpath $\varphi'(t')=\mathcal{F}(a(t'))$ for the operator $\mathcal{F}(\hat{A})$. If, on the other hand, some of the eigenvalues are degenerate, e.g., $\mathcal{F}(a_m)=\mathcal{F}(a_n)$, several paths $a(t)$ become indistinguishable and cannot be told apart even by a detailed measurement of $\mathcal{F}(\hat{A})$, $\varphi(t)\equiv a_m$ and $\varphi'(t)\equiv a_n$ being two obvious examples. In the extreme case $\mathcal{F}(a_1)=\mathcal{F}(a_2)=\cdots=\mathcal{F}(a_n)=\mathcal{F}_0$, i.e., $\mathcal{F}(\hat{A})=\mathcal{F}_0=\text{const}$, the set of eigenpaths collapses to a single constant path $\varphi=\mathcal{F}_0$ and the solution of Eq. (14) takes the form

$$|\Phi^{\mathcal{F}_0}(T|[\varphi])\rangle = \delta[\varphi - \mathcal{F}_0] |\Psi(T)\rangle. \tag{50}$$

In this case, no meaningful decomposition of the Schrödinger state $|\Psi(T)\rangle$ is obtained and no information about the system may be gained.

For $\mathcal{F}(\hat{A})$, the measurement amplitude is still given by the expression in the RHS of Eq. (31), where the functionals $F_j[\varphi]$ are no longer required to be linear in the path variable, and are given by

$$F_j[a] \equiv \int_0^T \beta_j(t') \mathcal{F}(a(t')) dt'. \tag{51}$$

Thus, using a set of von Neumann–like meters one is able to distinguish between the classes of eigenpaths

$$\{\varphi\}_{\vec{f}} = \left\{ a: \int_0^T \beta_j(t') \mathcal{F}(a(t')) dt' = f_j, \quad j=1,2,\ldots,M \right\} \tag{52}$$

where, as before, M is the number of meters employed and $\beta_j(t)$ are the known switching functions. The functionals similar to (51) defined on the Feynman paths $a(t')=x(t')$ were introduced in Ref. [1] and extensively used thereafter. With the help of Eqs. (51) and (45) one can apply the path integral analysis to the measurement of any observable $\mathcal{F}(x)$, which commutes with the particle's coordinate x. With the particular choice $\beta(t)=T^{-1}=\text{const}$, Eq. (51) gives the time average of $\mathcal{F}(x)$,

$$\langle \mathcal{F}(x) \rangle_T \equiv T^{-1} \int_0^T \mathcal{F}(x(t)) dt. \tag{53}$$

The scalar product $\langle x|\Phi^{\mathcal{F}(x)}(T|f)\rangle$ yields the amplitude to observe, at a location x, a particle with a definite average value $\mathcal{F}(x)$. It follows from Eqs. (10) and (45) that the amplitudes $\langle x|\Phi^{\mathcal{F}(x)}(T|f)\rangle$ can be found by solving the Schrödinger equation corresponding to the modified classical action

$$S_\lambda[x(t)] = S[x(t)] + \lambda \int_0^T \mathcal{F}(x) dt$$

containing an extra potential $-\lambda\mathcal{F}(x)$ and then taking the Fourier transform with respect to λ. The possibility of using Eq. (53) as a starting point for formulating measurement theory in the coordinate space has been studied in Refs. [13–17]. The case of the mean coordinate $\mathcal{F}(x)=x$ was analyzed in detail [15]. For $\mathcal{F}(x)=\theta_\Omega(x)$ [$\theta_\Omega(z)=1$ for z inside Ω and 0 otherwise], Eq. (53) yields the fraction of time a Feynman path spends inside a spatial region Ω. This expression was used to define the quantum traversal time studied in [17] and in the references therein. A similar approach was extended to two-level systems [18], in order to analyze the amount of time a qubit spends in given quantum state. Finally, a brief discussion of the continuous measurements is given in Appendix D.

VII. CONCLUSIONS AND DISCUSSION

In summary, any measurement of the von Neumann type can be described as destruction of coherence between classes of histories (paths) defined for the measured variable A. Such paths provide a decomposition of the state of an isolated quantum system into a set of substates, each labeled by a virtual sequence (path) of the values taken by A throughout a time interval in the past. The substates contain the most detailed information about the values of A and can be used as the building blocks for constructing probabilities for the outcomes of a measurement. Such probabilities, given by the norms of the coherent superpositions of the substates whose paths share the required property (2), can be assigned directly to the corresponding sets of the system's paths. The properties of the paths, which can be ascertained in this way, are the values of the functionals given by the integrals (51), and include, as particular cases, the value of A at a given time and its time average. Although this approach most readily applies to Feynman paths, it is equally valid for a variety of variables describing quantum systems of an arbitrary dimensionality.

Analysis in terms of the paths restores, as much as possible, the logic implied in classical measurements; namely, that a measurement should reveal certain properties pertaining to the system in isolation. A classical von Neumann–like measurement yields the value of the corresponding functional on the unique classical path without perturbing the motion of the system [13]. Quantally, not just one but many

paths are available, and an accurate quantum meter ensures instead that the system is restricted to those that have the measured properties. This inevitably perturbs the system's evolution, yet this perturbation results from and is synonymous with the destruction of coherence between the classes of paths. Thus, while one defines virtual properties for an isolated system, an external meter is required if these properties are to be observed.

It must be stressed that we have only established the equivalence between the von Neumann–like measurement amplitudes and the restricted path sums and that a further generalization of the approach is far from trivial. For example, the paths can be classified according to the time τ at which some value of A is reached for the first time (the first passage time) [28] and the probabilities may formally be constructed as discussed above. Yet to ensure that the given value is reached for the first time the integrand in Eq. (51) must contain the dependence on all times preceding t, so that the functional is not of the type considered in this work. As a result, no meter equation similar to Eq. (14) can be derived for the first passage time and no obvious physical meaning can be ascribed to probabilities obtained.

To conclude, we note that the path decomposition has been constructed for a particular choice of the variable A. The important question of whether such a decomposition can be used to describe a measurement of variables which do not commute with A will be discussed in our future work.

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APPENDIX A: THE FUNCTIONAL FOURIER TRANSFORM

On a set $\{\varphi(t)\}$ of all continuous, but not necessarily smooth real functions defined on an interval $0 \leqslant t \leqslant T$ with arbitrary boundary values $\varphi(0)$ and $\varphi(T)$, a functional $F[\varphi]$ is defined as the limit [1]

$$F[\varphi] = \lim_{N\to\infty} F(\vec{\varphi}) \tag{A1}$$

where $\vec{\varphi} \equiv (\varphi_1, \varphi_2, \ldots, \varphi_{N+1})$, $\varphi_i = \varphi(\epsilon(i-1))$ and $\epsilon \equiv T/N$.

$$I[\varphi] \equiv \int_0^T \varphi(t)\lambda(t)dt = \lim_{N\to\infty} \sum_{i=1}^{N+1} \varphi_i \lambda_i \epsilon \tag{A2}$$

where λ is some function of time, is defined as the limit of its Riemann sum. The functional derivative is defined as

$$\delta F[\varphi]/\delta\varphi(t) = \lim_{N\to\infty} \epsilon^{-1}\, \partial F(\vec{\varphi})/\partial\varphi_m, \quad m = t/\epsilon, \tag{A3}$$

so that for $I[\varphi]$ in Eq. (A2) $\delta I[\varphi]/\delta\varphi(t) = \lambda(t)$, as it should. For the sum over the functions φ we have

$$\int_{\{\varphi\}} D\varphi\, F[\varphi] \equiv \lim_{N\to\infty} K(N) \int_{-\infty}^{\infty} d\varphi_1 d\varphi_2 \cdots d\varphi_{N+1} F(\vec{\varphi}) \tag{A4}$$

where $K(N)$ is some normalization constant. It is readily seen that if $\lim_{\varphi(t)\to\pm\infty} F[\varphi] = 0$, then

$$\int D\varphi\, \delta F[\varphi]/\delta\varphi(t) = 0. \tag{A5}$$

Since the path sum will be applied to both sides of the homogeneous equation (4), the choice of $K(N)$ is arbitrary and we put $K(N) \equiv 1$. With this choice, the functional Fourier transform for $F[\varphi]$ is defined as

$$\tilde{F}[\lambda] \equiv \lim_{N\to\infty} (\epsilon/2\pi)^{N+1} \tilde{F}(\epsilon\vec{\lambda}) \tag{A6}$$

for $\tilde{F}(\vec{\lambda}) \equiv \int F(\vec{\varphi}) \exp(-i\vec{\lambda}\cdot\vec{\varphi})$ and $\vec{\lambda}\cdot\vec{\varphi} \equiv \Sigma_{j=1}^{N+1} \lambda_j \varphi_j$.

Thus, $F[\varphi]$ can be written as a Fourier functional integral

$$F[\varphi] = \int D\lambda\, \tilde{F}[\lambda] \exp\left(i \int_0^T \lambda(t)\varphi(t)dt \right). \tag{A7}$$

A particular choice

$$\tilde{F}[\lambda] \equiv \tilde{\delta} = \lim_{N\to\infty} (\epsilon/2\pi)^{N+1} \tag{A8}$$

yields the δ functional required to define the solution of Eq. (4) [$\delta(z)$ is the Dirac δ function]

$$\delta[\varphi] = \lim_{N\to\infty} \prod_{j=1}^{N+1} \delta(\varphi_j) \tag{A9}$$

with the obvious property

$$\int D\varphi\, F[\varphi]\delta[\varphi] = F[\varphi \equiv 0]. \tag{A10}$$

Finally, we will also require the convolution property

$$\int D\lambda\, \tilde{F}[\lambda]\tilde{G}[\lambda] \exp\left(i \int_0^T \lambda(t)\varphi(t)dt \right)$$

$$= (\epsilon/2\pi)^N \int D\varphi' F[\varphi - \varphi']G[\varphi], \tag{A11}$$

which can be obtained by considering the time-discretized Fourier transform.

APPENDIX B: LIE-TROTTER AND BAKER-CAMPBELL-HAUSDORFF FORMULAS

The generalized Lie-Trotter formula reads [27]

$$\lim_{N\to\infty} [\hat{F}(t/N)]^N = \exp[t\partial_t \hat{F}(0)] \tag{B1}$$

where $\hat{F}(t)$ is an operator function of t such that

$$\hat{F}(0) = \hat{1}. \tag{B2}$$

Choosing

$$\hat{F}(t) = \exp[t\hat{A}]\exp[t\hat{B}]$$

yields the Trotter product formula

$$\lim_{N\to\infty}\{\exp[t/N\hat{A}]\exp[t/N\hat{B}]\}^N = \exp[t(\hat{A}+\hat{B})]. \quad \text{(B3)}$$

The Baker-Campbell-Haussdorff identity states that for two operators $\hat{A}$ and $\hat{B}$, such that

$$[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{B},\hat{A}]] \quad \text{(B4)}$$

where the square brackets denote the commutator,

$$\exp[\hat{A}+\hat{B}] = \exp[\hat{A}]\exp[\hat{B}]\exp\{-[\hat{A},\hat{B}]/2\}. \quad \text{(B5)}$$

APPENDIX C: THE EIGENPATH EXPANSION AS A SOLUTION OF THE "RECORDER" EQUATION

In order to verify that the eigenpath expansion (37) satisfies Eq. (4), consider a more general form

$$|\Psi(t|[\varphi])\rangle = \sum_{[a]} G[\varphi - u_t a]|\Phi_t[a]\rangle \quad \text{(C1)}$$

where $G[\varphi]$ is an arbitrary functional, $u_t(t')$ is a function of t', which also depends on the time t, and $|\Phi_t[a]\rangle$ is defined in Eq. (38). Then

$$\partial_t|\Psi(t|[\varphi])\rangle = -\sum_{[a]}\int_0^T \frac{\delta G}{\delta\varphi(t')}\partial_t u_t(t')a(t')dt'|\Phi_t[a]\rangle + \sum_{[a]} G[\varphi - u_t a]\partial_t|\Phi_t[a]\rangle. \quad \text{(C2)}$$

Using Eq. (39) we have

$$\partial_t|\Phi_t[a]\rangle = -i\hat{H}|\Phi_t[a]\rangle, \quad \text{(C3)}$$

and choosing

$$u_t = \theta_t(t'), \quad \partial_t u_t = \delta(t-t') \quad \text{(C4)}$$

yields

$$\partial_t|\Psi(t|[\varphi])\rangle = -\frac{\delta}{\delta\varphi(t)}\sum_{[a]} G[\varphi - \theta_t a]a(t)|\Phi_t[a]\rangle - i\hat{H}\sum_{[a]} G[\varphi-\theta_t a]|\Phi_t[a]\rangle. \quad \text{(C5)}$$

For $G[\varphi]=\delta[\varphi]$, with the help of the relation

$$\hat{A}|\Phi_t[a]\rangle = a(t)|\Phi_t[a]\rangle, \quad \text{(C6)}$$

Eq. (C5) reduces to Eq. (4).

APPENDIX D: CONTINUOUS MEASUREMENTS AND MENSKY'S FORMULA

Consider a special case of the transformation (21), with

$$G[\varphi] = \exp\left(-i\int_0^T g(t',\varphi)dt'\right). \quad \text{(D1)}$$

With the help of Eq. (37) we obtain

$$|\Psi(t|[\varphi])\rangle = \sum_{[a]}\exp[-ig(t',\varphi-\theta_t a)dt']\hat{U}_t[a]|\Psi_0\rangle, \quad \text{(D2)}$$

where the last operator is given by the discretization

$$\exp[-ig(t',\varphi-\theta_t a)dt']\hat{U}_t[a] = \lim_{N\to\infty}\prod_{j=1}^{N}|a_{k_j}\rangle\langle a_{k_j}|\exp[-i\theta_t(t_j)\hat{H}\epsilon] \times\exp\{-ig[t_j,\varphi-\theta_t(t_j)a_{k_j}]\epsilon\}. \quad \text{(D3)}$$

Applying the Trotter formula (B3) to recombine the two exponentials and summing over the eigenpaths yields a compact expression for $|\Psi(t|[\varphi])\rangle$:

$$|\Psi(t|[\varphi])\rangle = \exp\left\{-i\int_0^T[\theta_t\hat{H}+g(t',\varphi-\theta_t\hat{A})]dt'\right\}|\Psi_0\rangle \quad \text{(D4)}$$

which for $0<t<T$ satisfies the "recorder" equation (4) with the initial condition Eq. (22). It follows from Eq. (D4) that

$$|\Psi(t|[\varphi])\rangle = \exp\left(-i\int_t^T g(t',\varphi)dt'\right)|\Psi_\varphi(t)\rangle \quad \text{(D5)}$$

where $|\Psi_\varphi(t)\rangle$ satisfies the effective Schrödinger equation

$$i\partial_t|\Psi_\varphi(t)\rangle = \{\hat{H}+g(t,\varphi-\hat{A})\}|\Psi_\varphi(t)\rangle, \quad \text{(D6)}$$

$$|\Psi_\varphi(0)\rangle = |\Psi_0\rangle. \quad \text{(D7)}$$

The problem of evaluating the restricted path sum for $|\Psi(T|[\varphi])\rangle$ in Eq. (21) reduces, therefore, to solving the time-dependent Schrödinger equation (D6) with the time dependence determined by $\varphi(t)$. Equations (D4) and (D6) were first suggested by Mensky [10] for the case when the functional $G[\varphi]$ reaches its maximum value for $\varphi(t')\equiv 0$ and rapidly falls off as φ deviates from zero. One such choice is

$$g(\varphi) = -i\varphi^2/\sigma^2 \quad \text{(D8)}$$

which ensures that only the eigenpaths φ' in a tube of width σ around φ contribute to $|\Psi(t|[\varphi])\rangle$ in Eq. (21) [see Fig. 1(a)], and the effective Schrödinger equation Eq. (D6) contains a non-Hermitian imaginary term

$$-i(\varphi-\hat{A})^2/\sigma^2|\Psi_0\rangle.$$

Application of Eqs. (D4) and (D6) to two-level systems can be found in [29].

One notes that Eq. (D4) for $|\Psi(T|[\varphi])\rangle$ can, with the help of Eq. (A1), also be written as the limit

$$|\Psi(T|[\varphi])\rangle = \lim_{M\to\infty}\int \prod_{j=1}^{M} df'_j \exp[-(\varphi_j - f_j)^2\epsilon/\sigma^2]\int D\varphi' \times\delta\left(f'_j - \int_0^T \delta(t'-t_j)\varphi'(t')dt'\right)|\Phi(T|[\varphi'])\rangle. \quad \text{(D9)}$$

Comparing Eq. (D9) with Eq. (14) shows that the second integral in (D9) is the fine grained amplitude for an array of M von Neumann meters, each firing at $t_j = j\epsilon$, $1 \leqslant j \leqslant M$. Upon integration over $df'_1, df'_2, \ldots, df'_M$, this amplitude is coarse grained with a product of Gaussians whose widths increase as $\sigma/\epsilon^{1/2}$ for $\epsilon = T/M \to 0$. Thus, this is a sequence of very inaccurate "weak" [22] measurement, by a set of meters weakly coupled to the system. Taking the limit $\sigma \to 0$ in expression (D8) yields the solution $|\Phi_W[\varphi)\rangle$, which satisfies the initial condition

$$|\Phi_W(t=0|[\varphi])\rangle \approx \delta\left(\int_0^T \varphi^2(t')dt'\right)|\Psi_0\rangle. \quad \text{(D10)}$$

The condition (D10), which requires that the mean-square deviation of φ from zero must vanish, is similar to Eq. (5), which needs $\varphi(t')$ to vanish pointwise, and one can show that either $|\Phi_W(t|[\varphi])\rangle$ or $|\Phi(T|[\varphi])\rangle$ can be used for calculating the finite-time measurement amplitude (17). The set $|\Phi_W[\varphi]\rangle$, which corresponds to scattering by the "measuring medium," was first suggested in [10]. Equations (D6) and (D9) can be applied to various coarse graining and unitary transformations. No similar formulas exist, in general, for less informative finite-time measurements, which yield information about certain global properties of the paths, e.g., the value of $\int_0^T \beta(t')\varphi(t')dt'$, while precise values of $\varphi(t')$ remain indeterminate. In that case, $|\Psi(t|\vec{f})\rangle$ cannot be obtained from $|\Psi_0\rangle$ by evolution with a generalized Hamiltonian, containing $\vec{f}$ as a parameter, and one should use Eq. (14) instead.

[1] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York 1965).
[2] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).
[3] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets: Techniques and Applications of Path Integration* (Wiley, New York, 1981).
[4] M. Büttiker, Phys. Rev. B **27**, 6178 (1983).
[5] R. Sala Mayato, D. Alonso, and I. L. Egusquiza, in *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza (Springer, Berlin, 2002).
[6] N. Yamada, Prog. Theor. Phys. **83**, 985 (1991).
[7] N. Yamada, Phys. Rev. A **54**, 182 (1996).
[8] M. B. Mensky, Phys. Rev. D **20**, 384 (1979).
[9] A. Aharonov and M. Vardi, Phys. Rev. D **21**, 2235 (1980).
[10] M. B. Mensky, *Quantum Measurements and Path Integrals* (IOP, Bristol, 1993).
[11] M. B. Mensky, Phys. Lett. A **196**, 159 (1994); **219**, 137 (1996); **231**, 1 (1997); Int. J. Theor. Phys. **37**, 273 (1998); **37**, 215 (1998); Phys. Lett. A **307**, 85 (2003); **308**, 243 (2003).
[12] M. B. Mensky, *Quantum Measurements and Decoherence: Models and Phenomenology* (Kluwer Academic Publishers, Dordrecht, 2000).
[13] D. Sokolovski, Phys. Rev. A **52**, R5 (1995); **57**, R1469 (1998); **59**, 1003 (1999).
[14] D. Sokolovski and Y. Liu, Phys. Lett. A **281**, 207 (2001).
[15] Y. Liu and D. Sokolovski, Phys. Rev. A **63**, 014102 (2001).
[16] D. Sokolovski, Phys. Rev. A **66**, 032107 (2002).
[17] D. Sokolovski, in *Time in Quantum Mechanics* (Ref. [5]).
[18] D. Sokolovski, Proc. R. Soc. London, Ser. A **460**, 1505 (2004).
[19] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955), pp. 183–217.
[20] A. Peres and W. K. Wooters, Phys. Rev. D **32**, 1968 (1985).
[21] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[22] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics* (Ref. [5], pp. 369–413; Phys. Rev. A **41**, 11 (1990); D. Sokolovski, Proc. R. Soc. London, Ser. A **460**, 499 (2004); D. Sokolovski, A. Z. Msezane, and V. Shaginyan, e-print quant-ph/0401159; C. R. Leavens, Found. Phys. (to be published).
[23] R. Omnes, *The Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1994).
[24] To ensure that φ has the same units as the variable $\hat{A}$ one requires to multiply the second term in Eq. (4) by a constant K with dimensions of energy. For simplicity, in the following we put $K=1$.
[25] "Coarse graining" may not be the most appropriate term as it usually implies separation of a set into a number of well-defined nonoverlapping subsets. In our case, the subsets of paths overlap. Yet, in the absence of a better word and to preserve the continuity with the previous work, we will continue to use it in this paper.
[26] Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961).
[27] B. Bodmann, H. Leschke, and S. Warzel, e-print quant-ph/9703031.
[28] N. Yamada (unpublished).
[29] R. Onofrio, C. Presilla, and U. Tambini, Phys. Lett. A **183**, 135 (1993); Phys. Rev. A **51**, 967 (1995).