Cluster states from Heisenberg interactions

Massoud Borhani and Daniel Loss
Department of Physics and Astronomy, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland
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We show that a special type of entangled states, cluster states, can be created with Heisenberg interactions and local rotations in $2d$ steps where d is the dimension of the lattice. We find that, by tuning the coupling strengths, anisotropic exchange interactions can also be employed to create cluster states. Finally, we propose electron spins in quantum dots as a possible realization of a one-way quantum computer based on cluster states.

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I. INTRODUCTION

Entanglement plays a crucial role in quantum information processing [1]. Quantum algorithms (in particular, Shor's algorithm, to find the prime factors of an n-bit integer) exploit entanglement to speed up computation. In addition, quantum communication protocols use entangled states as a medium to send information through quantum channels. However, creating entangled states is a great challenge for both theoretical and experimental physicists. Recently, Briegel and Raussendorf [2] introduced a special kind of entangled states, the so-called cluster states, which can be created via an Ising Hamiltonian [3]. These states are eigenstates of certain correlation operators [see Eqs. (5) and (6) below]. It has been shown that via cluster states, one can implement a quantum computer on a lattice of qubits. In this proposal, which is known as "one way quantum computer," information is written onto the cluster, processed, and read out from the cluster by one-qubit measurements [4]. In other words, all types of quantum circuits and quantum gates can be implemented on the lattice of qubits by single-qubit measurements only. The entangled state of the cluster thereby serves as a universal resource for any quantum computation. However, in this model, cluster states are created with an Ising interaction, which may be difficult to realize, in particular in a solid-state system. Here, we propose an alternative way to create the same states with a Heisenberg interaction (isotropic exchange interaction), but in several steps, where the number of steps depends on the dimension of the lattice of cubic symmetry. Furthermore, we consider some deviations from the Heisenberg Hamiltonian due, for example, to lattice asymmetry, and obtain the same cluster state by tuning the exchange coupling strengths. It turns out that if these coupling strengths satisfy certain conditions, which can be tuned experimentally, we can obtain a cluster state, up to an overall phase. Following Ref. [7], we propose a lattice of electron spins in quantum dots as a possible realization of this scheme in solid-state systems. In this system, electron spins in nearest-neighbor quantum dots are coupled via a Heisenberg exchange interaction.

This paper is organized as follows: Sec. II is devoted to a brief introduction to cluster states. In Sec. III we introduce an alternative way to create cluster states. Section III C considers the anisotropic Heisenberg interaction between qubits on a lattice and how to get cluster states via this interaction. Finally, in Sec. IV, we propose electron spins in quantum dots as a physical realization of this proposal.

II. CLUSTER STATES

A cluster state [2] is an entangled state which has special features suitable for implementing a quantum computer on an array of qubits. According to this scheme, we can obtain a cluster state by applying an Ising Hamiltonian ($\hbar=1$),

$$H = g(t) \sum_{\langle a,a' \rangle} \frac{1-\sigma_z^{(a)}}{2} \frac{1-\sigma_z^{(a')}}{2}, \tag{1}$$

on a special kind of initial state. Here, $\sigma_i^{(a)}, i \in \{x,y,z\}$ are Pauli matrices at lattice site a and $\langle a,a' \rangle$ denotes that a' is the nearest neighbor of a. Furthermore, $g(t)$ allows for a possible overall time dependence. To be specific, consider a qubit chain [see Fig. 1(a)] prepared initially in a product state $|\phi_0\rangle = \otimes_a |+\rangle_a$, where index a refers to the sites of the qubits and $|+\rangle_a$ is eigenstate of $\sigma_x^{(a)}$ with eigenvalue 1. The time evolution operator for the qubit chain is then given by

$$U(\theta) = \exp\left(-i\ \theta \sum_a \frac{1-\sigma_z^{(a)}}{2} \frac{1-\sigma_z^{(a+1)}}{2} \right), \tag{2}$$

with $\theta = \int g(t)dt$. From now on we assume that $\theta = \pi$ [2]. Because the terms in the Ising Hamiltonian (1) mutually commute, we can decompose the evolution operator $U(\pi)$ into two-particle operators as follows:

$$U \equiv U(\pi) = \prod_a U^{(a,a+1)}, \tag{3}$$

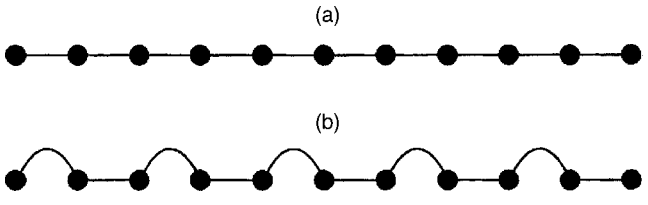

FIG. 1. (a) A one-dimensional cluster (a qubit chain). The connecting lines represent the interaction between nearest neighbors. (b) An alternative way to entangle a one-dimensional cluster. The qubits which are connected by straight lines are entangled in the first step and those connected by semicircles are entangled in the subsequent step.

$$U^{(a,a+1)} = \tfrac{1}{2}(1 + \sigma_z^{(a)} + \sigma_z^{(a+1)} - \sigma_z^{(a)}\sigma_z^{(a+1)}). \quad (4)$$

Therefore U is a product of two-qubit conditional phase gates [1]. More generally we can define the cluster states as the eigenstates of the following correlation operators:

$$K^{(a)}\left|\phi_{\{\kappa\}}\right\rangle_{\mathcal{C}} = (-1)^{\kappa_a}\left|\phi_{\{\kappa\}}\right\rangle_{\mathcal{C}}, \quad (5)$$

$$K^{(a)} \equiv \sigma_x^{(a)} \underset{b\in nbgh(a)}{\otimes} \sigma_z^{(b)}, \quad (6)$$

with $\kappa \in \{0,1\}$ where "nbgh" means the set of all occupied nearest neighbor sites of site a. A cluster state is completely specified by the eigenvalue Eq. (5) and it can be shown [5] that all states $\left|\phi_{\{\kappa\}}\right\rangle_{\mathcal{C}}$ are equally suitable for computation. For simplicity we put $\kappa=0$ for all lattice sites. The detailed proof of the above assertions and properties of cluster states, especially their application in implementing a one-way quantum computer, have been given in Refs. [4] and [5]. We note that in one dimension a cluster is a qubit chain with nearest-neighbor interaction. However, in more than one dimension, the cluster does not have a regular shape. In the latter case, qubits can be arranged in a multidimensional square lattice such that only some of the lattice sites are occupied by qubits. A cluster is then defined as a set of qubits where any two qubits are connected by a sequence of neighboring sites that are occupied by a qubit.

III. CLUSTER STATES FROM HEISENBERG INTERACTION

Cluster states are produced through Ising interactions. However, an ideal Ising interaction is difficult to obtain in nature especially in a solid-state environment. So, how can such states be created? The usual spin-spin interaction is (nearly) isotropic in spin space and is described by the Heisenberg Hamiltonian [6]

$$H = -J\sum_{\langle ij\rangle} S_x^{(i)}S_x^{(j)} + S_y^{(i)}S_y^{(j)} + S_z^{(i)}S_z^{(j)}, \quad (7)$$

$$\vec{S}^{(i)} = (S_x^{(i)}, S_y^{(i)}, S_z^{(i)}) = \tfrac{1}{2}\vec{\sigma}^{(i)} \quad (\hbar = 1), \quad (8)$$

where $\vec{S}^{(i)}$ and $\vec{S}^{(j)}$ are spin-$\frac{1}{2}$ operators at lattice sites i and j, and J is the exchange coupling constant, which is assumed to be constant for all spin pairs and is positive (negative) for ferromagnetic (antiferromagnetic) coupling. Next we describe a method to create cluster states via Heisenberg instead of Ising interaction. We start with one dimension and then generalize to higher dimensions.

A. One-dimensional case

Recall that all operators $U^{(a,a+1)}$ in U [Eqs. (3) and (4) above] mutually commute and they can therefore be applied in arbitrary order, i.e., at the same or different times [see Fig. 1(a)]. Suppose we have a one-dimensional N-qubit chain where all qubits are prepared in the $|+\rangle$ state. The initial state of the cluster is then (as before) $\otimes_{a\in\mathcal{C}}|+\rangle_a$, and the index a refers to the lattice site. The idea is to apply first the sequence $U^{(1,2)}U^{(3,4)}U^{(5,6)}\cdots$, and then in a second step, the sequence $U^{(2,3)}U^{(4,5)}U^{(6,7)}\cdots$. In other words, first we let qubits 1-2, 3-4, 5-6, … interact with each other, and then qubits 2-3, 4-5, 6-7, … [Fig. 1(b)]. We obtain the same result (3), but now we have entangled the qubits in our chain *pairwise* in each step. This means that each qubit is entangled with only one of its nearest neighbors in each step. In one dimension, there are two nearest neighbors for each qubit, thus we entangle our chain in two steps.

We note that $U^{(a,a+1)}$, given by Eq. (4), describes a conditional phase shift. On the other hand, in Ref. [7] it was shown that this evolution operator can also be realized with a Heisenberg Hamiltonian (obtained, e.g., via a Hubbard model) and local one-qubit rotations (see also next section). Therefore the problem of generating a cluster state with a Heisenberg interaction has been solved provided that in each step each qubit interacts with only one of its nearest neighbors.

B. Higher dimensions

In two dimensions, the minimum number of steps increases to 4 in a two-dimensional square lattice. In general for a d-dimensional cubic lattice, the minimum number of steps required is $2d$. (Note that cluster states are only defined on lattices with cubic symmetry. See also the last paragraph in Sec. II).

However, in dimensions higher than 1, there is no regular shape for an arbitrary cluster. How then can we obtain cluster states with just $2d$ steps? There may be several optimal ways to do this but we mention only one. For simplicity, consider a two-dimensional cluster and suppose that this cluster can be contained within a rectangle of n rows and m columns. Now, entangle all qubits in the cluster within each of these n rows independently (recall that each row requires two steps to be entangled). Then, do the same for the m columns. There is no need to worry about the qubits which are within the rectangle but not part of the cluster, since they are excluded automatically if we do not entangle them with their nearest neighbors. The idea is the same for $d=3$ cubic lattice, except that we would need six steps to entangle the cluster.

C. Anisotropic Heisenberg Hamiltonian

We do not consider the most general form of an anisotropic Heisenberg model since it is beyond the scope of this work. Here we introduce a special case, known as *symmetric* anisotropic Heisenberg model (SAH) which does not include the cross-spin terms. It has the following form in one dimension:

$$H_{\mathrm{SAH}} = \sum_a H_{\mathrm{SAH}}^{(a,a+1)}, \quad (9)$$

$$H_{\mathrm{SAH}}^{(a,a+1)} = \alpha(t)S_x^{(a)}S_x^{(a+1)} + \beta(t)S_y^{(a)}S_y^{(a+1)} + \gamma(t)S_z^{(a)}S_z^{(a+1)}. \quad (10)$$

This situation occurs for example, when our lattice does not have enough symmetry to use the isotropic interaction. However,

$$\left[S_p^{(a)}S_p^{(a+1)}, S_q^{(a)}S_q^{(a+1)}\right] = 0,$$

$$\forall\, p,q = x,y,z. \tag{11}$$

Therefore these three terms in the Hamiltonian mutually commute and consequently when we write the unitary evolution operator for two adjacent qubits, $U_{\mathrm{SAH}}^{(a,a+1)}$, it can be decomposed into three unitary operators. The order of application of these three operators does not matter,

$$U_{\mathrm{SAH}}^{(a,a+1)} = U_{xx}^{(a,a+1)}U_{yy}^{(a,a+1)}U_{zz}^{(a,a+1)}, \tag{12}$$

$$U_{xx}^{(a,a+1)} = \exp(-i\, J_{xx}S_x^{(a)}S_x^{(a+1)}), \tag{13}$$

$$U_{yy}^{(a,a+1)} = \exp(-i\, J_{yy}S_y^{(a)}S_y^{(a+1)}), \tag{14}$$

$$U_{zz}^{(a,a+1)} = \exp(-i\, J_{zz}S_z^{(a)}S_z^{(a+1)}). \tag{15}$$

Now, according to our alternative method to create cluster states, if the coefficients α,β, and γ satisfy the following conditions:

$$J_{xx} = \int \alpha(t)dt = 4n\pi, \tag{16}$$

$$J_{yy} = \int \beta(t)dt = 4m\pi, \tag{17}$$

$$J_{zz} = \int \gamma(t)dt = (2k+1)\pi, \tag{18}$$

where n,m, and k are arbitrary integers, then $U_{xx}^{(a,a+1)}$ and $U_{yy}^{(a,a+1)}$ are just *unity* operators (up to a minus sign) and do not affect the initial state [8]. If we could tune these coefficients properly in our lattice, we would get the same cluster states, up to some local (single-qubit) operations. The crucial point is that $U_{\mathrm{SAH}}^{(a,a+1)}$ and $U_{\mathrm{SAH}}^{(a+1,a+2)}$ do not commute and thereby we *cannot* decompose U_{SAH}, the total evolution operator of the cluster with a SAH interaction, into two-qubit evolution operators. This is why we need at least two steps to entangle the chain.

In general, when the Hamiltonian includes cross-spin terms [the asymmetric anisotropic Heisenberg model (AAH)], the problem cannot be solved exactly because the terms in the AAH Hamiltonian do not mutually commute. There is still a hope of solving this problem if we have the following interaction between spins [9]:

$$H_{\mathrm{AAH}}^{(a,a+1)} = \alpha'(t)S_x^{(a)}S_y^{(a+1)} + \beta'(t)S_y^{(a)}S_x^{(a+1)} + \gamma'(t)S_z^{(a)}S_z^{(a+1)}. \tag{19}$$

Again, the terms in this Hamiltonian mutually commute and we can decompose the two-qubit evolution operator like above. However, this Hamiltonian is related to the previous Hamiltonian (10) via a single-qubit unitary transformation (through $\pi/2$ rotation of one of the spins about the z axis) and therefore both have the same structure. In the end, we emphasize that the basic cornerstone of this method is that in each step, each qubit can interact with *only* one of its nearest neighbors. Generalizing the above method to higher dimensions is straightforward (see previous section). Therefore we have shown that the problem of creating cluster states with more realistic interaction models other than Ising can be solved exactly.

IV. PHYSICAL REALIZATION OF THE MODEL

In Refs. [7,10,11], a detailed scenario has been proposed for how quantum computation may be achieved in a coupled quantum dots system. In this proposal, a qubit is realized as the spin of the excess electron on a single-electron quantum dot. A mechanism has been proposed there for two-qubit quantum-gate operation that operates by a purely electrical gating of the tunneling barrier between neighboring quantum dots, rather than by spectroscopic manipulation as in other models. Consider two quantum dots which are labeled by "1" and "2" and coupled to each other via exchange interaction (see below). If the barrier potential is "high," tunneling is forbidden between dots, and the qubit states are held stably without evolution in time (t). If the barrier is pulsed to a "low" voltage, the usual physics of the Hubbard model [6] says that the spins will be subject to a transient Heisenberg coupling,

$$H = J(t)\vec{S}^{(1)}\cdot\vec{S}^{(2)}, \tag{20}$$

where $J(t)$ is the time-dependent exchange constant which is produced by the turning on and off of the tunneling matrix element [7,10].

For instance, a quantum XOR gate is obtained by a simple sequence of operations [7]:

$$U_{XOR} = e^{i(\pi/2)S_z^{(1)}}e^{-i(\pi/2)S_z^{(2)}}U_{sw}^{1/2}e^{i\pi S_z^{(1)}}U_{sw}^{1/2}, \tag{21}$$

where U_{sw} is a swap gate, created in this model via Heisenberg interaction, and $e^{i\pi S_z^{(1)}}$, etc., are single-qubit operations only, which can be realized by applying local Zeeman interaction. (It has been established that XOR along with single-qubit operations may be assembled to do any quantum computation [12].) Note that the XOR of Eq. (21) is given in the basis where it has the form of a conditional phase-shift operation; the standard XOR is obtained by a simple basis change for qubit "2." According to Eq. (21), we need five steps to realize an XOR gate. However, in Ref. [13] it has been shown that for a certain choice of system parameters (for example, opposite direction of the local B fields), we can generate an XOR gate in one step. The crucial observation now is that the XOR operation can be written as [7] $U_{XOR} = \frac{1}{2} + S_z^{(1)} + S_z^{(2)} - 2S_z^{(1)}S_z^{(2)}$, which has exactly the same form as $U^{(a,a+1)}$ in Eq. (4). In other words, we can generate the operation $U^{(a,a+1)}$ (and thus the cluster states) with the Heisenberg interaction as described, e.g., by the sequence in Eq. (21). We finally note that an alternative way to achieve the XOR operation is given by [7] $U_{XOR} = e^{i\pi S_z^{(1)}}U_{sw}^{-1/2}e^{-i(\pi/2)S_z^{(1)}}U_{sw}e^{i(\pi/2)S_z^{(1)}}U_{sw}^{1/2}$. This form has the potential advantage that the single qubit operations involve only spin 1.

The mechanisms described above for performing gate operations with spin qubits are independent of the details of the pulse shape $P(t)$, where P stands for the exchange coupling J or the Zeeman interaction. It is only the value of the integral $\int_0^\tau P(t)dt (\mathrm{mod}\, 2\pi)$ which determines the quantum gate action. This is true provided that the parameters $P(t)$ are switched adiabatically, guaranteeing the validity of the effective Hamiltonian Eq. (20). The unwanted admixture of a state with double occupation of a dot in the final state is found to be tiny if a suitable pulse is used and the adiabaticity criterion is fulfilled [14,15].

We note that as long as an XOR (or CNOT) gate is realized, cluster states (and consequently, a one-way quantum computer) can be generated. This result does not depend on the type of interaction in the system. Therefore other proposals such as trapped ion [16] and superconducting qubits [17] can be used as well to create cluster states.

V. CONCLUDING REMARKS

In summary, an alternative way, using Heisenberg interaction between qubits, was introduced to create cluster states which is useful for solid-state systems. In this method the qubits in the cluster are entangled pairwise, leading to $2d$ steps in d-dimensional cubic lattices. Furthermore, by tuning the coupling strengths of the interaction, it is possible to create cluster states via anisotropic Heisenberg exchange interaction. Experimentally, these cluster states can be generated in coupled quantum dots or similar systems.

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