

Entanglement and bifurcation in the integrable dimer

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In this Brief Report the properties of both dynamical and static entanglement in the integrable quantum dimer are studied in terms of the reduced-density linear entropy and von Neumann entropy with various coupling parameters, total boson numbers, and initial states. The mean entanglement, which is defined to be averaged over time, is used to describe the influence of the classical separatrix on the behavior of entanglement. It is shown that the mean entanglement exhibits a maximum near the position of the corresponding classical separatrix energy and that the static entanglement of the state with the largest eigenvalue of the quantum spectrum displays a maximum near the bifurcation point. For weak coupling and larger total boson number the maximum entanglement state is exactly at the position of the classical separatrix and bifurcation. In strong coupling all initial states have nearly the same mean entanglement.

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Quantum entanglement has rightly been the subject of much study in recent years as a potential resource for communication and information processing [1]. Lots of effects have been devoted to characterize the impact of quantum phase transitions on entanglement [2–8]. It is shown that the entanglement measured by the von Neumann entropy or the pairwise concurrence is largest near quantum critical points in some models [2–5]. But in other models the entanglement measured by the concurrence or the linear entropy displays a jump at the transition point [6–8]. These mean that the entanglement behavior near and at the quantum phase transition depends on the measure of entanglement and the studied models. For the Dicke model the quantum phase transition is just at the classical bifurcation point in the corresponding analog [8,9]. Thus systematic analysis of the classical bifurcation behavior in the corresponding classical model helps one to understand the entanglement properties near the transition point or bifurcation point [8,9]. Recently, such an investigation has also been applied to Jahn-Teller models [10]. It is demonstrated that the static entanglement described by von Neumann entropy in the ground state of Jahn-Teller models has an extinct change near the classical bifurcation [10]. In this Brief Report, we study the integrable quantum dimer [11], whose classical version exhibits exactly one bifurcation and separatrix manifold. Therefore, the influence of the bifurcation and separatrix on both dynamical and static entanglement is investigated. It should be mentioned that the correspondence between classical and quantum properties of the dimer system was studied in Refs. [11,12]. It is shown that a drastic change in the splittings of quantum energy levels occurs at the position of the classical separatrix [11].

The integrable quantum dimer is of physical important in many subjects such as in a two-species Bose-Einstein condensate, two modes of electromagnetic fields interacting in Kerr media, and excitons in two coupled semiconductor microcrystallites. For the former two cases, the entanglement dynamics has been studied in detail in terms of both the reduced von Neumann entropy and the reduced linear en-

ropy [13]. For the latter case, exciton entanglement is quantified in terms of the reduced von Neumann entropy [14], where it is observed that the nonlinear interaction between excitons increases the maximum values of the entropy of entanglement more than that of the linear coupling model. However, the entanglement near or at the classical separatrix and bifurcation has not yet been considered in Refs. [13,14]. Here we will explore the behavior of the entanglement near or at the classical bifurcation and separatrix from various coupling parameters, total boson numbers, and initial states in the integrable quantum dimer.

We study the integrable quantum dimer model with the same Hamiltonian as in Ref. [11] ($\hbar=1$ hereafter):

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 + \frac{1}{2}[(a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2] + C(a_1^\dagger a_2 + a_2^\dagger a_1), \quad (1)$$

where a_i and a_i^\dagger ($i=1,2$) are the annihilation and creation operators on site i and C is the linear coupling parameter between site 1 and site 2. The quantum Hamiltonian, Eq. (1), conserves the total boson number $N=n_1+n_2$, where $n_i=a_i^\dagger a_i$ is the boson number on site i , so that together with the energy there are two integrals of the motion and the quantum system is integrable. Diagonalization of the quantum Hamiltonian, Eq. (1), in a basis of number states $|n_1, n_2\rangle$ gives the quantum spectrum with the corresponding eigenvectors. The extension of Eq. (1) becomes a useful model for the description of the highly excited vibrational spectra of polyatomic molecules [15].

The classical counterpart of the quantum Hamiltonian, Eq. (1), is obtained using the well-known correspondence between creation-annihilation operators and action-angle variables,

$$a_i^\dagger \rightarrow J_i^{1/2} e^{i\phi_i}, \quad a_i \rightarrow J_i^{1/2} e^{-i\phi_i}, \quad (2)$$

so the classical Hamiltonian in terms of canonical variables $(J_1, J_2, \phi_1, \phi_2)$ has the form

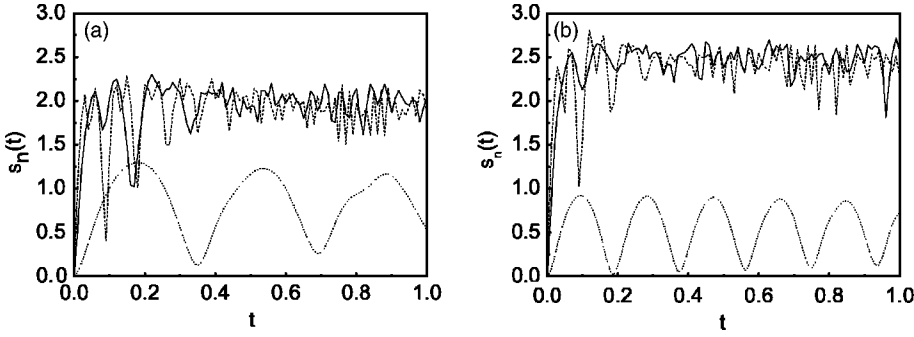


FIG. 1. von Neumann entropy $s_n(t)$ for the initial state $|0, N\rangle$ or $|N, 0\rangle$ with two total boson numbers $N=20$ (a) and $N=35$ (b). Here three different coupling parameters are used: $C=2$ (dotted line), $C=10$ (solid line), and $C=18$ (dashed line).

$$H_c = J_1 + J_2 + \frac{1}{2}(J_1^2 + J_2^2) + 2C\sqrt{J_1 J_2} \cos(\phi_1 - \phi_2). \quad (3)$$

The generating function $F = (\phi_1 - \phi_2)J_1 + \frac{1}{2}(\phi_1 + \phi_2)J_2$ defines a canonical changes of variables $(I_1, I_2, \phi_1, \phi_2) \rightarrow (J_1, J_2, \psi_1, \psi_2)$. The Hamiltonian, Eq. (3), in terms of new variables becomes

$$H_c = I_2 + \frac{1}{2}\left(2I_1^2 + \frac{1}{2}I_2^2\right) + 2C\sqrt{\frac{1}{4}I_2^2 - I_1^2} \cos(\psi_1). \quad (4)$$

The new angle $\psi_2 = \frac{1}{2}(\phi_1 + \phi_2)$ is ignorable so that the conjugate variable $I_2 = J_1 + J_2$ is a constant of the motion, which corresponds to the total boson number N —that is, $I_2 = N$.

The fixed points in the (ψ_1, I_1) plane can be obtained analytically by finding the root of the set of equations

$$\dot{\psi}_1 = \frac{\partial H_c}{\partial I_1} = 0, \quad \dot{I}_1 = -\frac{\partial H_c}{\partial \psi_1} = 0. \quad (5)$$

We find such four fixed points, labeled by P_1, P_2, P_3 , and P_4 , where the fixed point P_4 is just the reflection through the $I_1=0$ line of P_3 . The positions of those fixed points in the (ψ_1, I_1) are

$$P_1: \psi_1 = 0, \quad I_1 = 0,$$

$$P_2: \psi_1 = \pi, \quad I_1 = 0,$$

$$P_3, P_4: \psi_1 = 0, \quad I_1^2 = \frac{1}{4}I_2^2 - C^2. \quad (6)$$

Computation of the eigenvalues of the fixed-point stability matrices shows that the fixed point P_1 is stable for all C and given energy E and the fixed point P_2 remains stable for $I_2 \leq 2C$ and becomes unstable for $I_2 \geq 2C$. The fixed points P_3 and P_4 exist and are stable for $I_2 \geq 2C$ and occur through a bifurcation from P_1 . The bifurcation is at the position of coupling $C_b = I_2/2$ for given I_2 . The corresponding separatrix manifold is uniquely defined by the energy of P_1 at a given values of $I_2 \geq 2C$. It is straightforward to show that the fixed points, Eq. (6), correspond to maxima, minima, or saddle point of the energy in the allowed energy interval for a given value of I_2 . The energies of those fixed points are given by

$$E_1 = I_2 + \frac{1}{4}I_2^2 + CI_2,$$

$$E_2 = I_2 + \frac{1}{4}I_2^2 - CI_2,$$

$$E_{3,4} = I_2 + \frac{1}{2}I_2^2 + C^2. \quad (7)$$

For $I_2 \leq 2C$, we have $E_1 \geq E_2$ (P_1 maximum, P_2 minimum). For $I_2 \geq 2C$ it follows that $E_{3,4} \geq E_1 \geq E_2$ (P_3 and P_4 maxima, P_1 saddle, P_2 minimum).

Now, we turn to study the dynamics of entanglement in the quantum model. Several measures of quantum entanglement have been introduced, which include entanglement of formation [16], entanglement of distillation [17], relative entropy of entanglement [18], linear entropy of entanglement [19], concurrence [20], negativity [21], and so on. In our case the entanglement is described by the linear entropy $s_l(t)$ and the von Neumann entropy $s_n(t)$, defined by [19]

$$s_l(t) = 1 - \text{Tr}_1 \rho_1(t)^2, \quad s_n(t) = -\text{Tr}_1[\rho_1(t) \ln \rho_1(t)], \quad (8)$$

where Tr_1 denotes a trace over the first subsystem and $\rho_1(t)$ is the reduced-density matrices, $\rho_1(t) = \text{Tr}_2 |\psi(t)\rangle\langle\psi(t)|$, where indices 1 and 2 stand for the degree of freedom on sites 1 and 2, respectively, and $|\psi(t)\rangle$ is the quantum state of the full system, which evolves in time under the action of Hamiltonian, Eq. (1).

The static entanglement of a pure state can be measured by the linear entropy E_l and the von Neumann entropy E_n of reduced density of matrices [19],

$$E_l = 1 - \text{Tr}_i \rho_i^2, \quad E_n = -\text{Tr}_i(\rho_i \ln \rho_i), \quad i \in \{1, 2\}, \quad (9)$$

where ρ_i is the reduced density of subsystem i . The entropies defined by Eq. (9) do not depend on time because they are for a pure state.

For studying the dynamical properties of quantum entanglement, we take state $|0, N\rangle$ or $|N, 0\rangle$ to be the initial state since its quantum eigenenergy is $N + \frac{1}{2}N^2$, which is the maximum energy without the coupling between two sites in both the classical and quantum cases. With such an initial state, Fig. 1 shows the entanglement dynamics described by von Neumann entropy for the coupling $C=2, 10$, and 18 with the total boson number $N=20$ in Fig. 1(a) and with $N=35$ in Fig. 1(b). Similar behaviors are also found for the calculations of the linear entropy. A number of observations on Fig. 1 are in order. First, the increasing rate of the entropy is more rapidly as the coupling C increases in the early-time evolution. Second, for a weak coupling $C=2$, the entropy is quasiperiodic

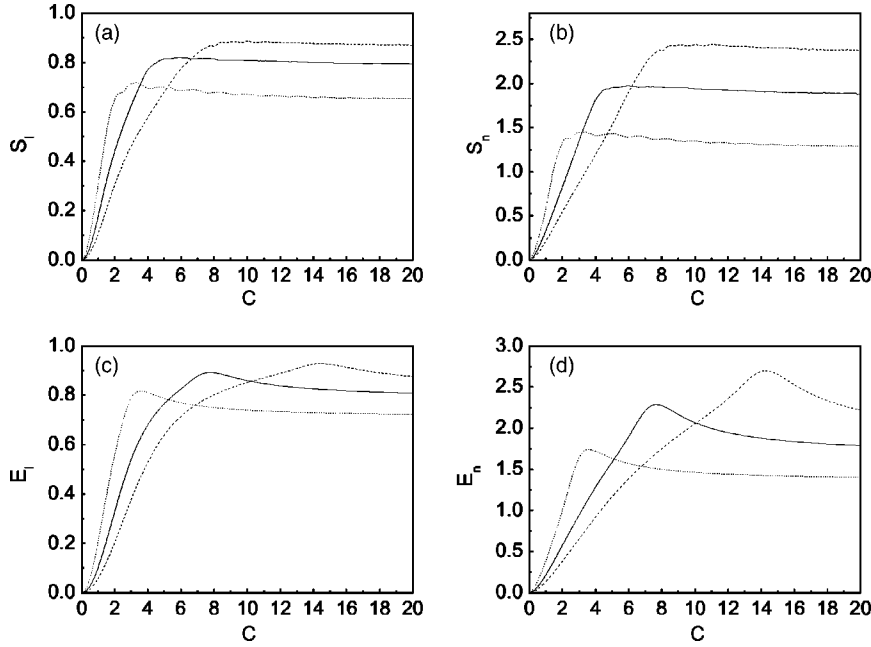


FIG. 2. The dependence of the entropy S_I (a), S_n (b), E_I (c), and E_n (d) on the coupling parameter C with three different total boson numbers $N=10$ (dotted line), $N=20$ (solid line), and $N=35$ (dashed line). The initial state of (a) and (b) is $|0, N\rangle$ or $|N, 0\rangle$, and the state of (c) and (d) is the eigenstate of the maximum quantum eigenvalue.

with its period decreasing as N increases. However, for the larger coupling $C=10$ and 18 , the maxima of the entropy increase with N . Third, for a fixed N , the maximum entropy increases up to a saturation value as the coupling increases. The larger the coupling, the larger the oscillatory frequency of the entropy is. A similar behavior is also observed in an integrable Jaynes-Cummings model [22]. We noted that the small-amplitude oscillations of entropy may be regarded as fingerprints of the underlying classical chaos in smooth Hamiltonian systems [23]. Thus such behavior appearing in integrable models should be a border effect. Finally, the influence of the bifurcation at $C_b=10$ for $N=20$ and $C_b=17.5$ for $N=35$ on the entanglement cannot be recognized in terms of the dynamical property of both entropies. Therefore, we turn to our attention to study the mean entanglement defined to be averaged over time. It is shown that such a mean entanglement is a good quantity in describing the influences of quantum phase transitions and classical or quantum chaos on the behavior of the entanglement in some models [6,8].

The mean entanglements S_I and S_n are defined by

$$S_I = \frac{1}{T} \int_0^T s_I(t) dt,$$

$$S_n = \frac{1}{T} \int_0^T s_n(t) dt, \quad (10)$$

where T is the total time of evolution, which is taken to be 1 in scaled units throughout the paper. We also used the total time of evolution in the calculation of the mean entanglement. It was found that that did not change the results.

Figures 2(a) and 2(b) show the mean entanglements S_I and S_n varying with the coupling parameter C , where the N is taken to be 10, 20, and 35 as an example and the initial state is also taken to be $|0, N\rangle$ or $|N, 0\rangle$. The static entanglements

E_I and E_n calculated by Eq. (9) for the state with the largest eigenvalues of the quantum spectrum are also plotted in Figs. 2(c) and 2(d) for comparison.

Some important properties are observed in Fig. 2. First, the mean dynamical entropies S_I and S_n and the static entropies E_I and E_n in the strong coupling regime increase as the total boson number N increases. The slope of curves in the small coupling regime decreases as the total boson number N increases. Second, the entropies S_I and S_n fluctuate with small amplitude near their maximum value for $N=10$. This amplitude decreases as N increases further, while the mean linear entropy above the critical point fluctuates in small amplitude in nonintegrable Dicke models [8]. Third, both dynamical and static entropies increase as the coupling parameter C increases up to a value and then decrease by a little as C increases further. For the mean dynamical entropy, such a value of C obtained numerically is 3.3, 5.6, and 7.8 for $N=10$, 20, and 35, respectively, which is near the separatrix coupling $C_s=N/4$ that is determined by setting the quantum eigenenergy $N+\frac{1}{2}N^2$ of the initial state $|0, N\rangle$ or $|N, 0\rangle$ be equal to the classical separatrix energy E_1 given by Eq. (7). The difference between the obtained coupling and the separatrix coupling decreases as N increases. For the static entropies E_I and E_n of the state with maximum eigenvalues, the coupling with its maximum entropy is 3.6, 7.8, and 14.5 for $N=10$, 20, and 35, respectively, which can be compared with the bifurcation coupling $C_b=N/2$. It is also noted that the difference between the obtained value and the bifurcation coupling decreases as N increases. Finally, both quantities S_I and S_n give the same coupling with the mean maximum entropy, and so do the quantities E_I and E_n . However, the von Neumann entropy E_n exhibits a more extinct maximum value as N increases.

Figure 3 shows the mean dynamical von Neumann entropy S_n with varying initial states, where the total boson number N is taken to be 20 and the coupling parameter C is set to be 0.05, 0.8, 5.0, and 18.0, as an example. The initial

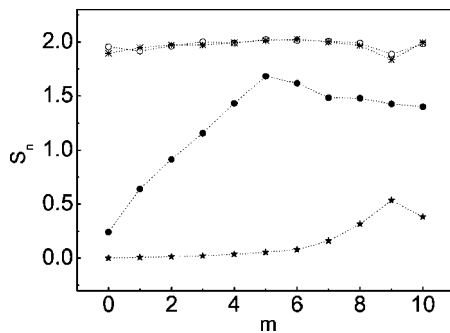


FIG. 3. The mean von Neumann entropy S_n for various initial states $|m, N-m\rangle$ or $|N-m, m\rangle$, $m=0, \dots, N/2$, with total boson number $N=20$ and four different coupling parameters $C=0.05$ (with \star), $C=0.8$ (with \bullet), $C=5$ (with \circ), and $C=18$ (with \ast).

states are $|m, N-m\rangle$ or equivalently $|N-m, m\rangle$ with $m=1, \dots, 10$. Some observations in Fig. 3 are in the following. The state $|1, 19\rangle$ or $|19, 1\rangle$ with the larger entropy changes to the state $|0, 20\rangle$ or $|20, 0\rangle$ as the coupling parameter C increases from small coupling to the coupling $C=5$. This suggests that the larger entanglement state can be prepared with the nearly the same boson number on each site in weak coupling. For strong coupling that is larger than 5, all states have nearly the same mean entropy as indicated for $C=5$ and $C=18$. For the coupling $C=0.05$, the state with the maximum entropy is $|1, 19\rangle$ or $|19, 1\rangle$ whose quantum eigenenergy is exactly equal to the classical separatrix energy, while for the coupling $C=0.8$, that is $|5, 15\rangle$ or $|15, 5\rangle$, which is not $|6, 14\rangle$

or $|14, 6\rangle$ corresponding to its classical separatrix energy. This implies that the maximum entanglement state is exactly at the state whose quantum eigenenergy is the same as the classical separatrix energy only in weak coupling.

In summary, we have investigated both dynamical and static entanglements in the integrable quantum dimer with various coupling parameters, total boson numbers, and initial states, where the entanglement is measured by the linear entropy and von Neumann entropy. The dependence of the mean entanglement averaged over time and the static entanglement on the coupling parameter shows that a maximum entanglement is near the position of the corresponding classical separatrix and bifurcation. In the strong coupling regime all initial states have nearly the same mean entanglement. It is demonstrated that a maximum entanglement for finite total boson number is exactly at the position of the classical separatrix and bifurcation only in weak coupling. Therefore, the larger entanglement state can be prepared by a quantum state with the total boson number being made to be equal on each site in the model in weak coupling since the capability of extremely long coherence time in Bose-Einstein condensates [24] might lead to engineering many-particle entanglement to be feasible.

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