Affine maps of density matrices

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For quantum systems described by finite matrices, linear and affine maps of matrices are shown to provide equivalent descriptions of evolution of density matrices for a subsystem caused by unitary Hamiltonian evolution in a larger system; an affine map can be replaced by a linear map, and a linear map can be replaced by an affine map. There may be significant advantage in using an affine map. The linear map is generally not completely positive, but the linear part of an equivalent affine map can be chosen to be completely positive and related in the simplest possible way to the unitary Hamiltonian evolution in the larger system.

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We are accustomed to the use of linear maps of matrices to describe evolution of density matrices in the dynamics of open quantum systems [1–9]. It was pointed out recently [10] that affine maps might be used as well. I will show here that the linear and affine methods of description are equivalent for systems described by finite matrices: an affine map can be replaced by a linear map, and a linear map can be replaced by an affine map. There may be significant advantage in using an affine map. A linear map that describes evolution of density matrices for a subsystem caused by unitary Hamiltonian evolution in a larger system is generally not completely positive [11], but the linear part of an equivalent affine map can be chosen to be completely positive and related in the simplest possible way to the unitary Hamiltonian evolution in the larger system. The equivalence demonstrated here is for a finite-dimensional Hilbert space. It may be that one kind of map will work where the other does not when the Hilbert space is infinite-dimensional. There too it may be advantageous to have two alternatives.

Consider a quantum system described by $N\times N$ matrices. Let L be a linear map of $N\times N$ matrices to $N\times N$ matrices; it takes each $N\times N$ matrix Q to an $N\times N$ matrix $L(Q)$. Let K be an $N\times N$ matrix. The map M that takes each $N\times N$ matrix Q to the matrix

$$M(Q) = L(Q) + K \tag{1}$$

is called *affine*. Let ρ and σ be $N\times N$ density matrices and let

$$\tau = q\rho + (1-q)\sigma \tag{2}$$

with $0<q<1$. Then τ is a density matrix and

$$\begin{aligned} M(\tau) &= qL(\rho) + (1-q)L(\sigma) + K \\ &= q[L(\rho)+K] + (1-q)[L(\sigma)+K] \\ &= qM(\rho) + (1-q)M(\sigma). \end{aligned} \tag{3}$$

If $M(\rho)$ and $M(\sigma)$ are density matrices, then $M(\tau)$ is a density matrix and it is related to $M(\rho)$ and $M(\sigma)$ the same as it would be if M were linear. The property of linear maps that has physical meaning for density matrices is provided by affine maps as well.

In the N^2-dimensional linear space of $N\times N$ matrices we can find a basis of N^2 Hermitian matrices F_μ for $\mu=0,1,\ldots,N^2-1$ such that F_0 is 1 and

$$\mathrm{Tr}[F_\mu F_\nu] = N\delta_{\mu\nu}. \tag{4}$$

If ρ is a density matrix, then

$$\rho = \frac{1}{N}\left(1 + \sum_{\alpha=1}^{N^2-1} \langle F_\alpha\rangle F_\alpha\right) \tag{5}$$

with $\langle F_\alpha\rangle = \mathrm{Tr}[F_\alpha\rho]$, and

$$\begin{aligned} M(\rho) &= \frac{1}{N}\left(L(1) + \sum_{\alpha=1}^{N^2-1} \langle F_\alpha\rangle L(F_\alpha)\right) + K \\ &= \frac{1}{N}\left(L(1) + NK + \sum_{\alpha=1}^{N^2-1} \langle F_\alpha\rangle L(F_\alpha)\right). \end{aligned} \tag{6}$$

Compare this with the result of a linear map that takes each $N\times N$ matrix Q to an $N\times N$ matrix Q'. It gives

$$\rho' = \frac{1}{N}\left(1' + \sum_{\alpha=1}^{N^2-1} \langle F_\alpha\rangle F'_\alpha\right). \tag{7}$$

The affine map and the linear map give the same result for all density matrices if

$$1' = L(1) + NK, \quad F'_\alpha = L(F_\alpha). \tag{8}$$

Specification of a linear map means independent specification of its action on each basis matrix 1 and F_α for $\alpha=1,2,\ldots,N^2-1$. We can choose $1'$ and F'_α to match any affine map. Conversely, we can choose $L(1)$, K, and $L(F_\alpha)$ to match any linear map, but the choice of $L(1)$ and K is not unique.

Specifically, suppose the density matrices are for a subsystem of a larger system. Evolution of these density matrices caused by unitary Hamiltonian evolution in the larger system can almost always be described by a linear map of matrices [11]. This linear map is generally not completely positive, but the linear part L of the equivalent affine map is

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completely positive when $L(1)$ and K are chosen so that $L(1)$ is 1. We can see this from the paper of Jordan, Shaji, and Sudarshan [11]: if $L(1)$ is 1, then L is their linear map with zeros for the parameters that involve mean values of quantities for the larger system; and when these parameters are all zero, their linear map is completely positive. For the two-qubit example that they work out in detail, if $L(1)$ is 1 then L is the linear map with a_1 and a_2 both zero; they observe explicitly that if a_1 and a_2 are zero, the map is completely positive for all t. In general, if $L(1)$ is 1 then L is the linear map with the parameters d_μ all zero. For any initial state of the subsystem, there is a density matrix for an initial state of the larger system that gives zeros for the parameters d_μ and is a product of the density matrix for the subsystem and a density matrix for the other part of the larger system; all that is required is zeros for mean values of quantities for the other part of the larger system. When the parameters d_μ are zero, the map can be obtained from unitary Hamiltonian evolution starting from an initial product state for the larger system. This implies that the map is completely positive. In fact, if $L(1)$ is 1, then for each matrix ρ for the subsystem, density matrix or not,

$$L(\rho) = \frac{1}{M}\mathrm{Tr}_R[e^{-iHt}\rho \otimes 1_R e^{iHt}] \tag{9}$$

where $e^{-iHt}\cdots e^{iHt}$ gives the unitary Hamiltonian evolution of density matrices in the larger system, R denotes the other part of the larger system (the remainder or rest of the larger system, which could be a reservoir), so 1_R is the unit matrix for R and Tr_R is the trace over the states of R, and M is the dimension of the Hilbert space for R; this holds for all the basis matrices except 1 regardless of how $L(1)$ and K are chosen, and it holds for 1 when $L(1)$ is 1. Thus L is related in the simplest possible way to the unitary Hamiltonian evolution in the larger system.

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