Quantum polarization properties of two-mode energy eigenstates

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We show that any pure, two-mode, N-photon state with N odd or equal to two can be transformed into an orthogonal state using only linear optics. According to a recently suggested definition of polarization degree, this implies that all such states are fully polarized. This is also found to be true for any pure, two-mode, energy eigenstate belonging to a two-dimensional SU(2) orbit. Complete two- and three-photon bases whose basis states are related by only phase shifts or geometrical rotations are also derived.

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I. INTRODUCTION

The polarization of a propagating electromagnetic field is a robust characteristic, which is relatively simple to manipulate without inducing more than marginal losses. For this reason, many recent experiments in quantum optics, such as Bell tests [1,2], quantum tomography [3], entanglement witnesses [4], quantum cryptography [5,6], and quantum dense coding [7], have been performed using polarization bases.

As long as the polarization measurements involve only qubits, encoded in two orthogonal polarization states of a single photon, the classical theory of polarization [8] and the quantum theory [9] essentially coincide. However, for multiphoton states, there is a divergence between the classical and the quantum-mechanical concepts of polarization. States that have vanishing expectation values of all three Stokes parameters are unpolarized according to the conventional classical theory, but may result in full visibility in a quantum measurement [10–14]. Another example of a discrepancy between the classical and quantum notion of polarization is the existence of states that are transformed into orthogonal states by geometrical rotations of ±60 degrees around the propagation axis [15]. Classically, only linearly polarized states can evolve into an orthogonal polarization upon a rotation of ±90 degrees. The apparent "violations" of the classical concept have led to the notion of states with "hidden polarization" [16].

When discussing polarization properties of quantum states, it is instructive to look back on the early discussion of unpolarized light. In 1971, Prakash and Chandra [10] proposed that a reasonable definition of an unpolarized state of light was to require invariance of the state with respect to geometrical rotations and phase shifts, or any combination thereof. Restricting ourselves to the energy eigenstates, the corresponding transformations form the group SU(2) [17]. Following the thread of Prakash and Chandra, and later Agarwal [11] and Lehner *et al.* [12], we have proposed [18] that the degree of polarization of a quantum state should be given by the maximum observable generalized visibility [19] of the state under such SU(2) transformations. That is, if it is possible to transform a state to an orthogonal state by some combination of geometrical rotations and phase shifts, then the state has a unit degree of quantum polarization.

Other attempts have been made to quantify quantum polarization. One measure is due to Luis [20], where the degree of polarization is expressed by means of the dispersion of the SU(2) Q function over the Poincaré sphere. A quantity Σ that can be interpreted as the "effective area" of the sphere where the Q function is different from zero was defined, and the smaller this area is, the higher the degree of polarization. With this definition, SU(2) coherent states are fully polarized, while the vacuum state, having an isotropic Q function, has zero degree of polarization. In contrast to our measure, which only quantifies the smallest possible overlap between the state and any rotated and phase-shifted state, Luis' measure favors states that can become orthogonal (or almost orthogonal) under the least "action." That is, a state whose Q function occupies only a small effective area on the Poincaré sphere need not be rotated by much before the Q functions of the original and the rotated states no longer overlap. In Luis' theory, such states are assigned a large degree of polarization. This difference between Luis' measure and ours becomes poignant when we study the states $|2, 0\rangle$ (two horizontally linearly polarized photons) and $|1, 1\rangle$ (one horizontally and one vertically linearly polarized photon). According to Luis' theory, for which the degree of polarization of two-photon states can take on values between 0 and 4/9, these states have the degree of polarization 4/9 and 1/6, respectively. Our measure assigns the value unity for the degree of polarization for both states, as will be shown below. The reason is that by geometrical rotations only, both states may be transformed into orthogonal states.

Another suggested definition of the degree of polarization for multimode states was given by Karassiov [21]. This measure is only comparable to ours in the single-mode case

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(which treats two polarization modes). In this regime, Karassiov's measure coincides with the "classical" definition of polarization based on the expectation values of the Stokes operators. However, the objective of the present work is to avoid the known problems associated with the classical definition. For example, it has been experimentally demonstrated [22] that it is possible, by applying appropriately chosen SU(2) transformations, to transform pure two-photon states of various degree of classical polarization into an orthogonal state.

In this paper, we use the definition of polarization degree given in Ref. [18] and consider pure N-photon states. In order to speak about polarization at all, we have assumed that we are dealing with a propagating light field, for which we can define two orthogonal transverse modes. Far from the source, all electromagnetic fields propagating in isotropic media evolve towards transverse fields. Expressing the polarization state in terms of the excitation of these two modes is therefore justifiable. In the following, we shall take these modes to be plane-wave modes with the electric field directed in the horizontal and the vertical directions. If the horizontal and vertical modes have m and n photons, respectively, we shall denote this state as $|m,n\rangle$.

As phase shifts and geometrical rotations are lossless transformations, our treatment based on these operations naturally disintegrates into energy manifolds containing a different number of photons. Also from an experimental point of view it is natural to consider energy eigenstates, since photon counters are normally used as detectors in experiments involving quantized polarization states of light. Hence, the final projectors are photon number states and therefore the polarization properties of two-mode energy eigenstates are of significant current interest.

II. QUANTUM DEGREE OF POLARIZATION

Mathematically, differential phase shifts and geometrical rotations can be easily expressed using the Stokes operators. However, in order to make comparisons with work not related to polarization easier, we will instead use the Schwinger boson realization of the angular momentum operators

$$\hat{J}_x = \frac{\hat{S}_x}{2} = \frac{1}{2}(\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger),$$

$$\hat{J}_y = \frac{\hat{S}_y}{2} = \frac{1}{2i}(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger), \tag{1}$$

$$\hat{J}_z = \frac{\hat{S}_z}{2} = \frac{1}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}),$$

which equal the corresponding Stokes operators divided by 2, as indicated. Here $\hat{a}(\hat{b})$ is the annihilation operator of the horizontally (vertically) polarized mode [23]. The effects of a differential phase shift of α and a geometrical rotation by $\theta/2$ are given by the operators $e^{-i\alpha\hat{J}_z}$ and $e^{-i\theta\hat{J}_y}$, respectively. Using these two physical operations, we can construct unitary representations of the group SU(2) in the different energy manifolds as [17]

$$\hat{U}(\beta,\theta,\alpha) = e^{-i\beta\hat{J}_z}e^{-i\theta\hat{J}_y}e^{-i\alpha\hat{J}_z}. \tag{2}$$

Hence, any SU(2) transformation can be realized using differential phase shifts and geometrical rotations alone, and any such combination can be described by an operator of the form (2).

Since these transformations preserve the total number of photons N, we can treat the corresponding energy manifolds separately. We thus use the fact that the Hilbert space of the two harmonic oscillators can be expressed as a direct sum $\mathcal{H}=\oplus_{N=0}^{\infty}\mathcal{H}_N$, where $\mathcal{H}_N$ is the Hilbert space consisting of all two-mode N-photon states. Note that $\mathcal{H}_N$ has dimension $N+1$ and corresponds to a spin system of spin $S=N/2$.

For two-mode bosonic states in the $N=1$ manifold, there are three mutually complementary operators $\hat{J}_z$, $\hat{J}_x$, and $\hat{J}_y$, whose respective eigenstates have linear polarization in directions $+$ and $\times$, and circular polarization. Although the three Hermitian operators do not commute, having the commutation relations $[\hat{J}_x,\hat{J}_y]=i\hat{J}_z$, where x, y, and z can be cyclicly permuted, they are mutually complementary (or unbiased) [24] only in this manifold, which means that the overlap between the corresponding normalized eigenstates is $1/\sqrt{2}$ in the Hilbert space of dimension 2. The operators are not complementary in manifolds for which $N\geqslant 2$.

A sensible approach to avoid the weaknesses of the definition relying on the Stokes operators is to define a state that is not invariant under all possible linear polarization transformations to have a finite degree of quantum polarization. In an earlier paper [18], we have suggested a measure for the degree of quantum polarization of two-mode states, based on this approach. For pure states, the measure simplifies to

$$\eta_{\mathrm{q}} = \sqrt{1 - \min_{\beta,\theta,\alpha}|\langle\psi|\hat{U}(\beta,\theta,\alpha)|\psi\rangle|^2}, \tag{3}$$

where the overlap between the original state and the transformed state is a measure of distinguishability between the two states. According to this definition, any state that is invariant under the SU(2) transformations $\hat{U}(\beta,\theta,\alpha)$ is an unpolarized state and thus has zero degree of quantum polarization. A fully polarized pure state, on the other hand, satisfies

$$\min_{\beta,\theta,\alpha}|\langle\psi|\hat{U}(\beta,\theta,\alpha)|\psi\rangle| = 0 \Leftrightarrow \eta_{\mathrm{q}} = 1. \tag{4}$$

As we shall show below, for any pure N-photon state with N odd or $N=2$, there exists a transformation of the form $\hat{U}(\beta,\theta,\alpha)$ that transforms the state into an orthogonal one. That is, any pure state in these manifolds is fully polarized.

The general form of an unpolarized quantum state was derived already in the work of Prakash and Chandra [10] (see also [11–13]). The only unpolarized N-photon state is

$$\hat{\rho} = \frac{1}{N+1}\sum_{n=0}^{N} |n,N-n\rangle\langle n,N-n|, \tag{5}$$

which is a maximally mixed state. In other words, it is the N-photon state with the largest von Neumann entropy.

III. SU(2) ORBITS

From our definition of degree of quantum polarization, it is clear that all states that can be transformed into each other by an operator of the form (2) have the same degree of quantum polarization. Such a set of N-photon states form an *orbit* [of the SU(2) group], and $\mathcal{H}_N$ is a union of disjoint orbits.

The case of $N=1$ (a polarization qubit) is trivial: there is only a single orbit. It is characterized by two real parameters, which cover the whole two-dimensional space, i.e., the Poincaré sphere. As discussed above, the classical theory of polarization coincides with the quantum theory for this case. We know from the classical theory that we can always find a combination of geometrical rotations and phase shifts that transforms a point on the Poincaré sphere to a diametrically opposite point, implying an orthogonal polarization. Hence, all pure polarization qubit states have a unit degree of quantum polarization.

For $N>1$, there are two types of orbits [25].

(i) Type 1. Orbits of states with nontrivial stability group U(1) and any additional group of discrete symmetry. These orbits are two-dimensional and are generated from the bare basis states by applying the operator of the group representation. However, due to the relation [26]

$$\hat{U}(0,\pi,0)|n,N-n\rangle = (-1)^n|N-n,n\rangle, \tag{6}$$

there are only $\lfloor N/2\rfloor+1$ orbits of this type, where $\lfloor x\rfloor$ denotes the largest integer smaller than or equal to x. These orbits are isomorphic to $\mathcal{S}_2/H$, where $\mathcal{S}_2$ denotes the two-dimensional sphere and H is the discrete group of symmetry.

(ii) Type 2. Orbits which allow only groups of discrete symmetries. These orbits are three-dimensional and isomorphic to $\mathcal{S}_3/H$. The orbit space is defined as the quotient $\mathcal{H}_N/\mathrm{SU}(2)$ and is $(2N-3)$-dimensional for $N>1$. For example, for $N=2$, we have $\dim[\mathcal{H}_2\mathrm{SU}(2)]=1$, which means that a single (real) parameter is needed to separate the orbits. However, for $N=3$, we have $\dim[\mathcal{H}_3/\mathrm{SU}(2)]=3$ and one hence needs three parameters.

IV. ORBITS OF TYPE 1

Let us first consider the orbits of type 1. These orbits can be labeled by the value $n\in\{0,1,\ldots,\lfloor N/2\rfloor\}$. From Eq. (6), it is clear that for every state belonging to an orbit with $n\neq N/2$, there exists an orthogonal state within the same orbit, because the states $|n,N-n\rangle$ and $|N-n,n\rangle$ are then orthogonal. Therefore, all states belonging to orbits with $n\neq N/2$ are fully polarized.

States belonging to orbits for which N is an even number and $n=N/2$ also have unit quantum polarization. To prove this, we note that

$$\langle N/2,N/2|\hat{U}(0,\theta,0)|N/2,N/2\rangle = P_{N/2}(\cos\theta), \tag{7}$$

where $P_n(x)$ is the Legendre polynomial of order n. Such a polynomial has n zeros in the interval $-1<x<1$, so it is always possible to find a solution to the equation $\langle N/2,N/2|\hat{U}(0,\theta,0)|N/2,N/2\rangle=0$.

In conclusion, we have shown that all pure states belonging to orbits of type 1 have unit degree of quantum polarization irrespective of their excitation.

V. TWO-PHOTON ORBITS OF TYPE 2

It is fairly easy to show that all pure two-photon states ($N=2$) have unit degree of quantum polarization. Defining the step operators $\hat{J}_\pm=\hat{J}_x\pm i\hat{J}_y$, transitions between any pair of orbits can be realized by unitary operators of the form

$$e^{\vartheta(\hat{J}_-^2-\hat{J}_+^2)/2} = \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix}, \tag{8}$$

which correspond to rotation matrices. Here and throughout the paper, we have used the basis $(|0,N\rangle,|1,N-1\rangle,\ldots,|N,0\rangle)$ when writing vectors and matrices. All SU(2) orbits can now be generated by applying the operators (8) to the state $|0,2\rangle$ or $|2,0\rangle$ (but not $|1,1\rangle$). Defining the states

$$|\psi(\vartheta)\rangle = e^{\vartheta(\hat{J}_-^2-\hat{J}_+^2)/2}|2,0\rangle = \sin\vartheta|0,2\rangle + \cos\vartheta|2,0\rangle, \tag{9}$$

the orbits can be identified by a single parameter, which is in agreement with our findings in Sec. III. Due to the symmetry expressed in Eq. (6), it suffices [25] to consider $0\leqslant\vartheta\leqslant\pi/4$. We note that the end points $\vartheta=0$ and $\vartheta=\pi/4$ correspond to the states $|2,0\rangle$ and $(|0,2\rangle+|2,0\rangle)/\sqrt{2}$, which belong to the two orbits of type 1 characterized by the values $n=0$ and $n=1$ in Eq. (6), respectively. The latter can be seen from the equality

$$\hat{U}(\beta,\pm\pi/2,\mp\pi/2)|\psi(\pi/4)\rangle = i|1,1\rangle. \tag{10}$$

In order for the state $|\psi(\vartheta)\rangle$ to have unit degree of polarization (4), we must have

$$\begin{aligned}\langle\psi(\vartheta)|\hat{U}(\beta,\theta,\alpha)|\psi(\vartheta)\rangle &= \cos(\alpha-\beta)\sin 2\vartheta\sin^2(\theta/2) \\ &\quad + [\cos(\alpha+\beta) - i\sin(\alpha+\beta)\cos 2\vartheta]\cos^2(\theta/2) = 0.\end{aligned} \tag{11}$$

Eight solutions that are independent of ϑ are easily found. They are given by $\theta=\pi$, $\beta=\alpha\pm\pi/2$, and $\alpha=\pm\pi/4$ or $\alpha=\pm3\pi/4$. All these solutions give the same physical final state, described by $\cos\vartheta|0,2\rangle-\sin\vartheta|2,0\rangle$.

Since any pure two-photon state can be obtained by applying an SU(2) operator to some orbit-generating state $|\psi(\vartheta)\rangle$, we conclude that all pure two-photon states can be mapped onto an orthogonal one using only linear optics. According to our definition (3), these states thus have unit degree of polarization.

VI. COMPLETE TWO-PHOTON BASES

As we have already noted, the state

$$|\psi(\pi/4)\rangle = \frac{1}{\sqrt{2}}(|0,2\rangle + |2,0\rangle) \tag{12}$$

belongs to one of the two orbits of type 1 for $N=2$. It can also be seen as a peculiar circularly polarized state [15]. As a consequence of its circular nature, it can be transformed into an orthogonal state according to

$$\hat{U}(\pm\pi/2,\theta,0)|\psi(\pi/4)\rangle = \pm\frac{i}{\sqrt{2}}(|0,2\rangle - |2,0\rangle) \tag{13}$$

for any value of θ. According to Eq. (10), SU(2) operators can also transform $|\psi(\pi/4)\rangle$ into a state that is orthogonal to both Eqs. (12) and (13). Hence, in this orbit of type 1, the SU(2) transformations generate the whole basis set. We have already used this fact to experimentally generate basis states that differ only by phase shifts [27] or geometrical rotations [15]. That the states (10), (12), and (13) are orthogonal was recently pointed out by Chekhova *et al.* [22], who denoted them $|HV\rangle$, $|RL\rangle$, and $|D\overline{D}\rangle$, respectively, reflecting the fact that they represent one photon in each of a horizontal-vertical linear basis, the right- and left-hand circularly polarized basis, and the linear basis ±45 degrees from the vertical. That is, they are the eigenstates with eigenvalue zero of the operators $\hat{J}_z$, $\hat{J}_y$, and $\hat{J}_x$, respectively.

States that can generate whole basis sets using only linear transformations are very useful, since these transformations are easily realized experimentally. In particular, it is desirable to find states that can generate a whole basis set by using only phase shifts or geometrical rotations. As phase shifts and geometrical rotations are described by the operators $e^{-i\alpha\hat{J}_z}$ and $e^{-i\theta\hat{J}_y}$, such bases can be generated from equipartition states in the $\hat{J}_z$ and $\hat{J}_y$ basis [28], respectively.

An equipartition state in the $\hat{J}_z$ basis, i.e., the horizontal-vertical basis, can be realized by an SU(2) transformation acting on the state $|\psi(\pi/4)\rangle$ since

$$|\xi_1\rangle = \frac{\hat{U}(0,\theta_z,\pi/2)}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{i}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}, \tag{14}$$

where $\theta_z=\pi/2-\arccos(1/\sqrt{3})$. This state then forms a complete basis together with the states generated by subsequent phase shifts of $2\pi/3$ and $-2\pi/3$,

$$|\xi_{2,3}\rangle = \hat{U}(0,0,\pm 2\pi/3)|\xi_1\rangle = \frac{i}{\sqrt{3}}\begin{bmatrix}e^{\pm i2\pi/3}\\-1\\-e^{\mp i2\pi/3}\end{bmatrix}. \tag{15}$$

Starting with the same state $|\psi(\pi/4)\rangle$, an equipartition state in the $\hat{J}_y$ basis, i.e., the circularly polarized basis, can be obtained by application of the phase shift $\alpha_y=\arctan(1/\sqrt{2})-\pi/2$. Expressed in the horizontal-vertical basis, we then have

$$|\psi_1\rangle = \frac{\hat{U}(0,0,\alpha_y)}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1-i\sqrt{2}\\0\\1+i\sqrt{2}\end{bmatrix}. \tag{16}$$

This state can be transformed into two other orthogonal two-mode states under geometrical rotations of ±60 degrees,

$$|\psi_{2,3}\rangle = \hat{U}(0,\pm 2\pi/3,0)|\psi_1\rangle = \frac{1}{2\sqrt{3}}\begin{bmatrix}\sqrt{2}+i\\\pm i\sqrt{6}\\\sqrt{2}-i\end{bmatrix}, \tag{17}$$

which together with $|\psi_1\rangle$ form a complete orthonormal basis.

Since any unitary transformation $\hat{V}$ preserves the inner products, new bases can be created by applying $\hat{V}$ to the original basis states. If the original basis states belong to the same orbit and we use an SU(2) transformation, all basis states remain within the orbit. Any state belonging to such an orbit can thus be made a basis state of a complete basis by an appropriately chosen SU(2) transformation.

The orbit considered above, which is characterized by $\vartheta=\pi/4$, is easily reached experimentally using a photon pair generated by spontaneous parametric down-conversion. The fact that this orbit spans the whole Hilbert space $\mathcal{H}_2$ has been exploited in the experimental realization of relative-phase states [27], three mutually orthogonal polarization states [15], and two-mode, two-photon qutrits [29,30].

VII. PURE STATES WITH AN ODD NUMBER OF PHOTONS

Let us denote an arbitrary pure N-photon state as

$$|\chi\rangle = \sum_{n=0}^{N} r_n e^{i\varphi_n}|n,N-n\rangle. \tag{18}$$

For N odd, we then find

$$\langle\chi|\hat{U}(0,\pi,\alpha)|\chi\rangle = -i2\sum_{k=0}^{(N+1)/2}(-1)^k r_k r_{N-k} \times \sin\left[\varphi_k - \varphi_{N-k} + \frac{(N-2k)\alpha}{2}\right]. \tag{19}$$

We thus have $\langle\chi|\hat{U}(0,\pi,0)|\chi\rangle=-\langle\chi|\hat{U}(0,\pi,2\pi)|\chi\rangle$. Since the expression (19) is purely imaginary, there exists at least one value of α for which a state orthogonal to $|\chi\rangle$ is obtained. Hence, all pure states with a given odd number of photons have unit degree of quantum polarization.

We note that for any N-photon state, a differential phase shift of 2π does not change the physical state. However, for odd N, it introduces an overall phase factor that equals -1, which we made use of in the proof above.

VIII. COMPLETE THREE-PHOTON BASES

Noting that the state $|\psi(\pi/4)\rangle$, which we used to generate complete two-photon bases in Sec. VI is invariant under interchange of the horizontally and vertically polarized modes, let us now consider the three-photon states

$$|\zeta_1\rangle = \frac{1}{\sqrt{2}}(|0,3\rangle + |3,0\rangle), \qquad (20)$$

$$|\zeta_2\rangle = \frac{1}{\sqrt{2}}(|1,2\rangle + |2,1\rangle). \qquad (21)$$

Like $|\psi(\pi/4)\rangle$, these states are symmetric with respect to the horizontal and vertical modes, however they belong to orbits of type 2. Application of the transformation $\hat{U}(0,\theta,\pi/2)$ to $|\zeta_1\rangle$ gives

$$\frac{\hat{U}(0,\theta,\pi/2)}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \frac{e^{i3\pi/4}}{\sqrt{2}}\begin{bmatrix}u_{11}+iu_{14}\\-u_{12}+iu_{13}\\u_{13}+iu_{12}\\-u_{14}+iu_{11}\end{bmatrix}, \qquad (22)$$

where u_{ij} are the real matrix elements of the unitary operator $e^{-i\theta\hat{J}_y}$. In order to generate a complete basis in manifold $N=3$, we now look for an equipartition state in analogy with the treatment in Sec. VI. That is, we want the magnitude of all the probability amplitudes to equal $1/\sqrt{N+1}=1/2$, which implies $u_{11}^2+u_{14}^2=u_{12}^2+u_{13}^2=1/2$. This requirement can be fulfilled by choosing one of the rotation angles $\theta_\pm=\arccos(\pm 1/\sqrt{3})$, which give

$$\frac{\hat{U}(0,\theta_\pm,\pi/2)}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \frac{\pm ie^{\pm i(3/4)\arccos(1/3)}}{2}\begin{bmatrix}1\\ \frac{i\mp\sqrt{2}}{\sqrt{3}}\\ \frac{1\pm i\sqrt{2}}{\sqrt{3}}\\ i\end{bmatrix}. \qquad (23)$$

In fact, an equipartition state in the horizontal-vertical basis can be created by applying any of the eight operators $\hat{U}(0,\pm\theta_\pm,\pm\pi/2)$ to the state $|\zeta_1\rangle$ or $|\zeta_2\rangle$. For example, we have

$$\frac{\hat{U}(0,\theta_\pm,\pi/2)}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \frac{e^{\pm i(1/4)\arccos(1/3)}}{2}\begin{bmatrix}1\\ \frac{-i\pm\sqrt{2}}{\sqrt{3}}\\ \frac{-1\mp i\sqrt{2}}{\sqrt{3}}\\ i\end{bmatrix}. \qquad (24)$$

Subsequent phase shifts of $\pi/2$, π, and $3\pi/2$ to the equipartition states will then generate complete bases. That is, for any equipartition state $|\epsilon_0\rangle$, the states

$$|\epsilon_k\rangle = \hat{U}(0,0,k\pi/2)|\epsilon_0\rangle, \quad k=0,1,2,3, \qquad (25)$$

are mutually orthonormal.

By applying a geometrical rotation of 45 degrees followed by a phase shift of $\beta_y=\arccos(-\sqrt{2/3})$ to the two symmetrical states $|\zeta_1\rangle$ and $|\zeta_2\rangle$, one can also obtain equipartition states in the circularly polarized basis. In the horizontal-vertical basis, we then have

$$\frac{\hat{U}(\beta_y,\pi/2,0)}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \frac{-ie^{-i(3/4)\arccos(1/3)}}{2}\begin{bmatrix}1\\0\\ \frac{1+i2\sqrt{2}}{\sqrt{3}}\\0\end{bmatrix} \qquad (26)$$

and

$$\frac{\hat{U}(\beta_y,\pi/2,0)}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \frac{-ie^{-i(3/4)\arccos(1/3)}}{2}\begin{bmatrix}\sqrt{3}\\0\\ -\frac{1+i2\sqrt{2}}{3}\\0\end{bmatrix}. \qquad (27)$$

Each of these states forms a complete basis together with the states obtained by geometrically rotating the respective state by 45, 90, and 135 degrees. The corresponding transformations are given by the operators $\hat{U}(0,k\pi/2,0)$, where $k=1,2,3$.

Also in the case of three photons, the orbits to which some particular symmetrical states belong thus span the whole Hilbert space, and allow complete bases to be generated by applying only phase shifts or geometrical rotations.

IX. DISCUSSION AND CONCLUSIONS

Polarization properties of quantum states deviate from the properties one may extrapolate from classical physics. The fundamental reason is the concept of orthogonality, which, for classical states, takes on a direct geometrical meaning (in the plane perpendicular to the propagation direction), whereas orthogonality in Hilbert spaces of dimension greater than 2 instead implies distinguishability. This means that in Hilbert spaces of dimension greater than 2, there are more than two orthogonal states of polarization, as shown above.

The spaces spanned by two- and three-photon states are still tractable, and we have shown that all pure states in these spaces have unit degree of quantum polarization. This was also found to be true for all pure states with any given odd number of photons. That is, by using only geometrical rotations and phase shifts it is always possible to transform any such state into an orthogonal one. Equivalently, using the proper observable, the transformation will result in a unit-visibility projection probability. We have also shown that there exist complete two- and three-photon bases whose basis states are related by transformations that can be realized using linear optics. In particular, we derived such bases with basis states related by only phase shifts or geometrical rotations. For two-photon states, these properties have already been exploited in various applications of quantum optics.

Finally, we note that it may be possible to generalize several of the results presented here. For example, it is natural to ask if all pure, two-mode, energy eigenstates can be made orthogonal using linear optics. It would also be interesting to know which SU(2) orbits can be used to generate complete bases.

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