Stationary vortex clusters in nonrotating Bose-Einstein condensates

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We investigate the recently found stationary vortex cluster states in dilute atomic Bose-Einstein condensates confined by a nonrotating trap, and also present a stationary three-vortex cluster. We find the stationary states by minimizing directly an error norm for the stationary Gross-Pitaevskii equation, and study the dynamic and energetic stability of the resulting states by solving the corresponding Bogoliubov equations for the elementary excitations. The results are verified by integrating the time-dependent Gross-Pitaevskii equation. Contrary to previously reported results, the stationary states were observed to be both energetically and dynamically unstable. The dynamical decay rate of the clusters is typically very slow, but it should be experimentally observable. The most promising circumstances to experimentally generate and observe these structures and their dynamics is in weakly dissipative condensate systems, using phase-imprinting techniques.

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The quantum phase coherence of the alkali-metal atom Bose-Einstein condensates (BECs) implies that they will have superfluid properties when rotated. Since the condensate flow is irrotational, these systems respond to external rotation by creating vortex lines with quantized circulation. Condensate states containing a single vortex line were first created using Raman transition phase-imprinting methods [1] and by rotating the system with a laser spoon [2,3]. Later, vortex lattices containing more than 100 vortices have been created by the latter technique [4,5]. Recently, also multiquantum vortices were created using topological phase engineering methods [6,7]. Based especially on the development of phase-imprinting methods [8–11] it is to be expected that even more complicated vortex clusters can be created in the future.

Atomic BECs are interacting systems, and their nonlinearity implies the dynamics to be in general quite complicated. Especially interesting is the dynamics of states containing several vortex lines. Vortex dynamics is still under investigation even in noninteracting systems, in which the motion of vortex lines is essentially determined by four factors: the shape of the vortex line, the shape of the background condensate wave function, the interaction between vortex lines, and possible external forces [12]. In interacting systems, the nonlinearity adds more to the complexity of the problem. However, when all these factors balance each other in a specific way, it is possible to find stationary vortex cluster states. Recently, it was shown that there exist a multitude of stationary clusters, so-called H clusters for noninteracting trapped wave fields [13]. One of these clusters, the quadrupole cluster shown in Fig.1(c), was also shown to have a stationary counterpart for interacting BECs, but in general there does not exist a simple correspondence between stationary clusters in the interacting and noninteracting cases. For example, the recently found stationary vortex dipole state shown in Fig. 1(a) exists only in sufficiently strongly interacting systems and, hence, can be viewed as solitonic state [14]; see also Ref. [15]. By observing the dynamics of the dipole and quadrupole states after imposing initial perturbations on them, it has been argued that these states are dynamically stable [13,14]. Another important issue is the energetic stability of these states—at finite temperatures the thermal cloud provides the condensate with a dissipative mechanism, and the fate of stationary states is determined by energetics.

In this paper we study in detail the structure and both the energetic and dynamic stability of the above-mentioned stationary dipole and quadrupole vortex clusters, and also present a stationary cluster consisting of three vortices. We search for stationary vortex cluster states using a gradient method to directly minimize an L^2 error norm for the Gross-Pitaevskii equation. The advantage of our method compared to energy-minimization methods is that it finds the nearest stationary state even if it is not a local energy minimum, as turns out to be the case for the vortex cluster states. In fact,

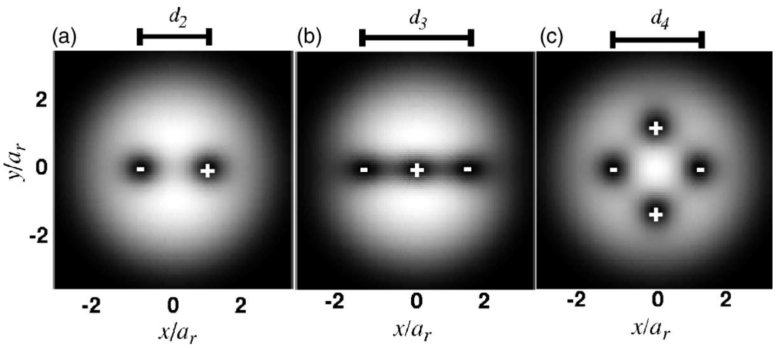

FIG. 1. Density profiles of a stationary vortex dipole (a), tripole (b), and quadrupole (c). The separations of the outermost vortices are marked by d_2, d_3, and d_4 for the dipole, tripole, and quadrupole, respectively. The plus signs in the vortex core denote vortices and minus signs antivortices. The strength of the interactions $\tilde{g}$ is 170 for the vortex dipole and 160 for the tripole and quadrupole.

they are local *maxima* of energy with respect to cluster size. The local dynamic and energetic stability of the resulting stationary vortex cluster states is investigated by solving the Bogoliubov equations for them. Excitations with negative energies but positive norm were found for all the clusters, implying their local energetic instability. In addition, and contrary to previously published results [13,14] stating that the vortex dipole and quadrupole clusters are dynamically stable, we find also excitations with imaginary frequencies for all the cluster states. The existence of the imaginary modes implies that infinitesimal perturbations may grow exponentially in time, e.g., in the case of a doubly quantized vortex these excitations are responsible for the splitting of the vortex state into two singly quantized vortices [16,17]. The dynamic instability of the vortex cluster states was confirmed by integrating the Gross-Pitaevskii equation in real time for states in which a small amount of unstable excitation was added to the stationary state wave function. In addition, we have used energy-minimization methods to investigate the total energy of cluster states as functions of their size, and in this way cast light on their global energetic stability properties.

At sufficiently low temperatures, the thermal gas component can be neglected and, hence, the dynamics of a harmonically trapped dilute atomic BEC is determined by the time-dependent Gross-Pitaevskii (GP) equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \hat{H}\psi(\mathbf{r},t), \qquad (1)$$

where the nonlinear Hamiltonian $\hat{H}=\hat{H}[\psi]$ containing the condensate wave function $\psi(\mathbf{r},t)$ itself is given by

$$\hat{H}[\psi] = -\frac{\hbar^2}{2m}\nabla^2 + V_{\text{tr}}(\mathbf{r}) + g|\psi(\mathbf{r},t)|^2. \qquad (2)$$

Above, the external potential $V_{\text{tr}}(\mathbf{r})=\frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)$ is used to trap the atoms having mass m and the strength of the interactions is governed by the parameter $g=4\pi\hbar^2a/m$ written in terms of the s-wave scattering length a. The condensate wave function is normalized according to $\int|\psi|^2d\mathbf{r}=N$, where N is the total number of atoms in the condensate.

In the following we are interested in the energy of various vortex cluster states, and their stability. The total energy of the condensate can be calculated as

$$E[\psi] = \int\left[\psi^*(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{tr}}(\mathbf{r})\right)\psi(\mathbf{r}) + \frac{g}{2}|\psi(\mathbf{r})|^4\right]d\mathbf{r}. \qquad (3)$$

On the other hand, the local stability of the stationary solutions $\psi(\mathbf{r},t)=\psi(\mathbf{r})e^{-i\mu t}$ of the GP equation can be determined from the quasiparticle spectrum given by the Bogoliubov equations

$$\mathcal{L}u_q(\mathbf{r}) + g\psi^2(\mathbf{r})v_q(\mathbf{r}) = \epsilon_q u_q(\mathbf{r}),$$

$$\mathcal{L}v_q(\mathbf{r}) + g\psi^{*2}(\mathbf{r})u_q(\mathbf{r}) = -\epsilon_q v_q(\mathbf{r}), \qquad (4)$$

where $\mathcal{L}\equiv-(\hbar^2/2m)\nabla^2+V_{\text{tr}}(\mathbf{r})-\mu+2g|\psi(\mathbf{r})|^2$, $u_q(\mathbf{r})$ and $v_q(\mathbf{r})$ are the quasiparticle amplitudes, and ϵ_q is the energy of the mode specified by quantum numbers q. If the spectrum for a stationary state contains an anomalous excitation with positive norm $\int[|u_q(\mathbf{r})|^2-|v_q(\mathbf{r})|^2]d\mathbf{r}$ but negative energy $\epsilon_q<0$, the state is locally energetically unstable, and decays in the presence of dissipation and quantum fluctuations. Vice versa, if the positive-norm spectrum is strictly positive, the state is locally energetically stable. Also dynamic stability can be inferred from the Bogoliubov equations: If there exists an excitation for which the energy ϵ_q has nonvanishing imaginary part, the state is dynamically unstable and small initial perturbations begin to grow exponentially in time.

The Hamiltonian (2) of the system is obtained by functional differentiation of the energy functional (3)

$$\hat{H}[\psi]\psi(\mathbf{r},t) = \frac{\delta E[\psi]}{\delta\psi^*}, \qquad (5)$$

where the conjugate ψ^* of the order parameter may be regarded as independent of ψ. Thus, the local energy minima may be found by minimizing the energy functional with the steepest-descent method using $\hat{H}[\psi]\psi(\mathbf{r},t)$ as the gradient. However, since the stationary vortex cluster states turn out to be not local minima of the energy, this method cannot be used to find them. Instead, we minimize a functional $F[\psi,\mu]=\int|(\hat{H}[\psi]-\mu)\psi(\mathbf{r})|^2d\mathbf{r}$, which clearly has global minima only at the stationary solutions of the GP equation, for which the functional vanishes. Thus, we searched for stationary states using the gradient

$$\frac{\delta F[\psi,\mu]}{\delta\psi^*} = ((\hat{H}[\psi]-\mu)^2 + 2g\,\text{Re}\{\psi^*(\mathbf{r})(\hat{H}[\psi] - \mu)\psi(\mathbf{r})\})\psi(\mathbf{r}). \qquad (6)$$

On the other hand, the chemical potential that minimizes the functional is obtained as $N\mu=\int\psi^*(\mathbf{r})\hat{H}[\psi]\psi(\mathbf{r})d\mathbf{r}$. As in the case of the energy minimization, the total number of particles N is to be held constant during the minimization procedure.

For simplicity and computational convenience, we consider only the effectively two-dimensional pancake geometry with $\omega_z\gg\omega_r:=\omega_x=\omega_y$, such that the z dependence of the condensate wave function and the lowest-energy quasiparticle amplitudes can be taken to be of the simple factorized form $e^{-z^2/(2a_z^2)}$, where $a=\sqrt{\hbar/m\omega_z}$. This simplifies the GP and Bogoliubov equations and, actually, they become essentially independent of the trapping frequency ω_r. The resulting dimensionless GP equation is

$$i\frac{\partial}{\partial\tilde{t}}\tilde{\psi}(\tilde{r},\tilde{t}) = \frac{1}{2}\left[\tilde{\nabla}^2 + \tilde{r}^2 + \tilde{g}|\tilde{\psi}(\tilde{r},\tilde{t})|^2 + \frac{\omega_z}{\omega_r}\right]\tilde{\psi}(\tilde{r},\tilde{t}), \qquad (7)$$

where the dimensionless quantities denoted by the tilde are obtained from the original ones by scaling the length by $a_r=\sqrt{\hbar/(m\omega_r)}$, the time by ω_r^{-1}, and the energy by $\hbar\omega_r$. Here we choose the normalization condition $\int|\tilde{\psi}(\tilde{r})|^2d\tilde{r}=1$, which implies the strength of the interaction to be $\tilde{g}=4\sqrt{2\pi}Na/a_r$. Equation (7) shows that the only physical parameter we need to fix is $\tilde{g}$, which we take to be 160 for the vortex tripole and quadrupole and 170 for the vortex dipole, except for the results presented in Fig. 2, where the value 160 is used for all

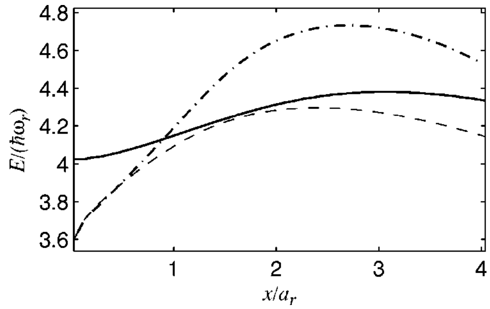

FIG. 2. Energy of the vortex dipole (dashed line), tripole (solid line), and quadrupole (dash-dotted line) as functions of the distances d_2, d_3, and d_4, respectively. The strength of the interactions is $\tilde{g}=160$ for all the configurations.

the cases. These parameters correspond, for example, to a BEC of $N \approx 9000$ ^{23}Na atoms with s-wave scattering length $a=2.75$ nm trapped using the frequency $\omega_z = 2\pi \times 200$ Hz.

The vortex dipole shown in Fig. 1(a) is a pair of a vortex and an antivortex, whereas there is one vortex in the center of the condensate and two antivortices next to it in the vortex tripole [see Fig. 1(b)]. As shown in Fig. 1(c), the vortex quadrupole consists of two vortices and two antivortices opposite to each other. The total angular momentum and topological charge of the vortex dipole and quadrupole vanishes. However, the vortex tripole holds, in general, a nonzero angular momentum—even when stationary.

Let us now consider a minimal-energy vortex tripole configuration with the cluster size d_3 [see Fig. 1(b)]. Due to symmetry and energy conservation, the vortex separation must remain constant in temporal evolution, and the three-vortex chain can only rotate clockwise or counterclockwise if it is not stationary. For $d_3 \to 0$, the vortices coalesce into one antivortex in the center, and the rotation of the chain must be in the clockwise direction seen from the positive z direction. On the other hand, for increasing d_3 such that the antivortices disappear in the boundary region of the condensate, one is left with one vortex in the center, and counterclockwise rotation is expected. Between these extremes, there should be a critical separation d_3^c for which the tripole cluster is stationary. This indeed turned out to be the case.

To find good initial values for the method of the steepest descent for the functional $F[\psi,\mu]$, we first minimize the total energy functional using as initial *Ansätze* wave functions in which the vortex phases are printed by hand. In this initial minimization process, we fix the condensate phase. The locations of the phase singularities, the vortices, are thus conserved in the minimization process. The energies of the BECs obtained by this method are shown as functions of the cluster sizes d_2, d_3, and d_4 in Fig. 2 for the vortex dipole, tripole and quadrupole, respectively. From these energy curves we can immediately make two important observations. First of all, we note that all the clusters have a critical vortex separation d_i^c for which the configuration energy is stationary with respect to the vortex separation parameter d_i. Thus for all the clusters we have a candidate ψ_i^c for a stationary state. On the other hand, we see that all the energy curves are essentially concave, with the stationary points corresponding to maxima of the constrained energy. This suggests that none of the possible stationary clusters is energetically stable.

TABLE I. Energies of the lowest elementary excitations in units of $\hbar\omega_r$ for the vortex dipole, tripole, and quadrupole are denoted as ϵ_2, ϵ_3, and ϵ_4, respectively. The strength of the interactions $\tilde{g}$ is 160 for the vortex dipole and 170 for the tripole and quadrupole.

Re(ϵ_2)	Im(ϵ_2)	Re(ϵ_3)	Im(ϵ_3)	Re(ϵ_4)	Im(ϵ_4)
−0.56	0	−0.69	0	−1.1	±0.31
0	±0.017	0	±0.046	−0.48	0
1.0	0	0	±0.60	−0.48	0
1.0	0	0.97	0	0	±0.010
1.2	0	1.0	0	1.0	0

The stationary states $\{\psi_i^s\}$ are found by minimizing the functional $F[\psi,\mu]$ with the method of the steepest descent by using $\{\psi_i^c\}$ as initial wave functions. Relative errors $F[\psi^s,\mu]/(N\mu^2) < 10^{-21}$ were found for all the cluster configurations, which justifies that the states $\{\psi_i^s\}$ are, in fact, very accurate stationary states. The configurations shown in Fig. 1 were obtained by this method.

To verify the stationarity and to investigate stability of the vortex clusters, we solved the Bogoliubov equations for the states $\{\psi_i^s\}$. All the quasiparticle spectra contain a condensate mode with energy having magnitude smaller than $10^{-6}\hbar\omega_r$, confirming that the wave functions satisfy the stationary Gross-Pitaevskii equation and thus that the clusters are stationary states. Furthermore, all the spectra contain at least one negative-energy anomalous mode, implying the states to be energetically unstable—note that this was already suggested by the fact that the stationary clusters correspond to local maxima of the constrained energy curves presented in Fig. 2. Finally, the computations revealed that the Bogoliubov equations had solutions with nonreal eigenvalues, suggesting that all the clusters are dynamically unstable. The computed Bogoliubov energies of the lowest modes are shown in Table I.

The Bogoliubov equations yield the response of the system to infinitesimal perturbations. Therefore, it is convenient to test the dynamical stability of the system also in the regime where the perturbations are allowed to grow to be macroscopic, which is accomplished by adding a small perturbation with imaginary energy to the stationary state and by solving the dynamics from the time-dependent GP equation. For all three different clusters, the vortices in the perturbed states were first observed not to move noticeably in time when the small perturbation had not yet grown to be macroscopic. After a time interval denoted as T, the vortex dipole was observed to rotate clockwise or counterclockwise depending on the initial perturbation. The interval T was also observed to be larger for smaller initial perturbations. For example, the interval length for a vortex dipole in the present calculations was about $200/\omega_r$ when the population of the excited mode was approximately ten particles. The small imaginary part of the energy of the excitation was also observed to yield a large time interval T.

The vortex dipole with different types of random noise was reported to be robust under temporal evolution in Ref.

[14] up to maximum times of the order of $1000/\omega_r$. However, it is not clear whether an asymmetric trap was used, what was the particle number, and what was the exact form of the perturbations. It is possible that the perturbations imposed did not excite the rotational mode or that the parameters of the system were such that the vortex dipole was, indeed, dynamically stable. Our preliminary calculations show that complex modes exist in a wide range of the total particle number. However, the detailed study of the excitation spectra as a function of particle number and trap asymmetry is beyond the scope of the recent studies and is left for future research.

From all three types of vortex clusters in question, only the perturbed vortex dipole state preserves its shape in the temporal evolution. The vortices in the quadrupole tend to drift to the center of the condensate and annihilate each other. Depending on the initial perturbation, one of the antivortices in the vortex tripole drifts to the edge of the BEC and a leftover vortex dipole remains in the condensate. These inspections suggest that the vortex dipole is in some sense the most stable structure of these configurations.

In conclusion, we have studied several stationary vortex clusters in pancake-shaped nonrotating BECs. The stationary states were found in very high precision using the method of the steepest descent to directly minimize the GP error norm. As far as the authors are aware this method has not been previously used to search for stationary states of BECs. The stability of the vortex dipole, tripole, and quadrupole was studied in terms of the elementary excitations using the Bogoliubov equations and macroscopic perturbations using the time-dependent GP equation. Both methods showed that the cluster configurations are both energetically and dynamically unstable. The detailed dependence of the stability of the stationary vortex clusters on the system parameters is left for future research which could solve the apparent contradiction of the results with the ones presented in Refs. [13,14]. It is also an interesting question whether the recently discovered [18] genuinely three-dimensional structures, namely, vortex stars, parallel vortex rings, and perpendicular vortex rings, have counterparts in interacting BECs.

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